Modelling error propagation in vector-based overlay analysis

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Abstract

This paper proposes two methods for analyzing error propagation in overlay analysis of vector polygons, an analytical and simulation method. First, for the analytical method, the error is derived based on the error propagation law in statistics. In the second method, error propagation of overlay is simulated based on the positional error of the original polygon vertices, assuming a given error distribution of the vertices. For both methods, several geometrical error measures for the derived polygons are defined: (i) error measures for the perimeter, area and center of gravity of a polygon, and (ii) the error intervals for the polygon vertices. A test is performed to compare the differences between the two methods and their applicability. The results indicate that there is no significant difference in estimating the propagated error between the two methods. However, these two methods are suitable for different cases. The analytical method is applicable when the error ellipse of any intersection point of the original polygons does not intersect with error ellipses of all the vertices of the original polygons, if the intersection point exists; or, in case of disjoint polygon boundaries, the error ellipse of any point on the boundary of one original polygon does not intersect with the error ellipse of any point on the boundary of the other original polygons. On the other hand, the simulation method is more generic, but more time consuming.

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1. Introduction

Overlay is one of the most fundamental spatial analysis functions in GIS. Some studies have been conducted on errors in raster-based overlay (McAlpine and Cook, 1971; MacDougall, 1975; Chrisman, 1987; Veregin, 1995). Boolean operators are often employed for an overlay analysis when two or more attribute layers overlay and form a new attribute layer (Newcomer and Szaigin, 1984; Walsh et al., 1987; Veregin, 1989; Lanter and Veregin, 1992; Veregin, 1995; Arbia et al., 1998).

In error analysis for vector-based overlay, two major issues are involved. One concerns the spurious or sliver polygons in an overlaid coverage (Good-
child, 1978; Chrisman, 1989; Veregin, 1989). An error tolerance band, which is called usually the epsilon error band, is applied to identify sliver polygons in a vector map overlay (Chrisman, 1983; Veregin, 1989). Another issue is the estimation of positional error in the derived polygons based on error propagation analysis (Kraus and Kager, 1994). Error propagation analysis covers two areas: (a) a discussion of the stochastic properties of dependent variables given the characteristics of another set of random variables, and (b) a study of the functional relationships (linear or nonlinear) relating to the two sets of variables with errors (Mikhail and Ackermann, 1976).

In vector-based GIS, the perimeter, area, and center of gravity are the major measures of a polygon. Research has been conducted on the nature of errors of these measures (Neumyvakin and Panfilovich, 1982; Chrisman and Yandell, 1988; Griffith, 1989; Shi, 1998). Prisley et al. (1989) presented a study on estimating the error of the polygon area based on the works of Neumyvakin and Panfilovich (1982) and Chrisman and Yandell (1988). Caspary and Scheuring (1992) proposed different measures of positional error in a polygon based on the assumption that all coordinate errors were random and independent with known variances. For positional error in overlay analysis, Kraus and Kager (1994) proposed a model for describing the accuracy of an area computed from an overlay of two networks of polygons. In the above error models for the polygon and the overlay analysis, the correlation among the points of the original polygons, and the correlation among the intersection and original points have been considered.

In this paper, we propose two methods to study the propagation of geometric errors in vector-based overlay analysis: (a) the analytical method based on the error propagation law, given the variance–covariance matrix of all original polygon vertices, and (b) the simulation method, assuming a known error distribution of the original polygon vertices. We also provide a relatively comprehensive set of error measures in overlay analysis, i.e., (i) positional error at the vertices of derived polygons, (ii) error measures for the perimeter, area, and center of gravity of the derived polygons, and (iii) maximum and minimum error intervals for the vertices of the derived polygons.

The rest of this paper is organized as follows. Section 2 describes the overlay analysis and the algorithms for computing the vertices of derived polygons. The analytical error propagation model and the simulation model are given in Sections 3 and 4, respectively. In Section 5, a test to demonstrate the practical use of the two solutions is described. Finally, an analysis of the analytical and simulation methods is provided in Section 6.

2. Overlay analysis and derived polygons

The various cases of overlay analysis differ in the problems that need to be considered. For point overlay, the pertinent issue is whether or not the locations of two points are the same. For the line overlay case, if two line segments intersect, they will be divided into two, three, or four line segments. The polygon overlay case is the most complex among the three, because in deriving derived polygons, it involves both the line segment intersection problem and the point-in-polygon problem.

2.1. Line segment intersection

Suppose two straight lines $L_1$ and $L_2$ intersect at $P$. The line segment $P_iP_j$ of two endpoints $(x_i, y_i)$ and $(x_j, y_j)$ lies on $L_1$, while line segment $P_kP_l$ of two endpoints $(x_k, y_k)$ and $(x_l, y_l)$ lies on $L_2$.

The equations of lines $L_1$ and $L_2$ are:

\[
\begin{align*}
L_1: \quad y - y_i &= \frac{y_j - y_i}{x_j - x_i} (x - x_i) \\
L_2: \quad y - y_k &= \frac{y_k - y_j}{x_k - x_j} (x - x_k)
\end{align*}
\] (1)

Let $(x_p, y_p)$ denote the coordinates of $P$. Solving the above system of equations, the coordinate of the intersection point is obtained.

Case (a): When $\Delta x_{kl} \Delta y_{ij} - \Delta y_{kl} \Delta x_{ij} \neq 0$,

\[
\begin{align*}
x_p &= x_i + \frac{\Delta x_j \Delta y_{kl} - \Delta y_j \Delta x_{kl}}{\Delta x_{kl} \Delta y_{ij} - \Delta y_{kl} \Delta x_{ij}} \Delta x_j \\
y_p &= y_i + \frac{\Delta y_j \Delta x_{kl} - \Delta x_j \Delta y_{kl}}{\Delta x_{kl} \Delta y_{ij} - \Delta y_{kl} \Delta x_{ij}} \Delta y_j
\end{align*}
\] (2)

where $\Delta x_{st} = x_i - x_s$, $\Delta y_{st} = y_i - y_s$ and $t,s \in \{i, j, k, l\}$. 

\[
\begin{align*}
\Delta x_{st} &= x_i - x_s, \quad \Delta y_{st} = y_i - y_s \quad \text{and} \quad t,s \in \{i, j, k, l\}.
\end{align*}
\]
Although $P$ is the intersection point of the lines $L_1$ and $L_2$, this does not imply that $P$ is an intersection point of the line segments $P_iP_j$ and $P_kP_l$ because these line segments may not intersect. To ensure the intersection of these two line segments, $P$ must satisfy the following conditions:

$$0 \leq r_1 \leq 1 \text{ and } 0 \leq r_2 \leq 1$$ (3)

where

$$r_1 = \frac{\sqrt{(x_p - x_i)^2 + (y_p - y_i)^2}}{\sqrt{(x_p - x_j)^2 + (y_p - y_j)^2}}$$

$$r_2 = \frac{\sqrt{(x_p - x_k)^2 + (y_p - y_k)^2}}{\sqrt{(x_p - x_l)^2 + (y_p - y_l)^2}}$$

If $P$ is the intersection point of the line segments, there will be three cases. If one endpoint of one segment coincides with an endpoint of the other segment, the intersection point of $P_iP_j$ and $P_kP_l$ is the common endpoint through which the two line segments are linked. The second case occurs when only one endpoint of one segment lies on the other segment and does not coincide with any of the endpoints of the other segment. Then, line segments $P_iP_j$ and $P_kP_l$ will intersect at this endpoint and split into three (e.g., $P_iP_k$, $P_jP_l$ and $P_iP_l$). When neither of the above two cases occurs, segments $P_iP_j$ and $P_kP_l$ will split into four segments: $P_iP$, $P_jP$, $P_kP$ and $P_lP$.

Case (b): When $\Delta x_{ij} \Delta y_{ij} - \Delta y_{kl} \Delta x_{kl} = 0$, $L_1$ is parallel to or coincident with $L_2$. There are also three cases. The first one is when segments $P_iP_j$ and $P_kP_l$ overlap entirely. This occurs when the endpoints of the two line segments coincide; only one segment will be obtained in this situation. If one endpoint of one segment lies on an endpoint of the other segment, segments $P_iP_j$ and $P_kP_l$ will be divided into two new segments. Moreover, if none of the endpoints of one segment lies on the other segment and vice versa, segment $P_iP_j$ and $P_kP_l$ will have no intersection. When none of the above cases occurs, segments $P_iP_j$ and $P_kP_l$ will partially overlap and be divided into three segments.

2.2. Polygon intersection

When two arbitrary polygons $A$ and $B$ overlap, derived polygons are generated. The derived polygons include the difference between $A$ and the intersection of $A$ and $B$, the difference between $B$ and the intersection of $A$ and $B$, and the intersection of $A$ and $B$. In Fig. 1, an overlay analysis is applied to two triangles. The derived polygons are $A \cap B$ in white, $A - A \cap B$ in dark grey, and $B - A \cap B$ in light grey. The vertices of the original polygons and their newly generated intersection points, if any, determine the vertices of the derived polygons. As a result, the error in the derived polygons is caused by both the error propagated from the original polygons and the error in the intersection points. There exist various overlay cases as illustrated in Fig. 1, which shows some possible overlay cases between two triangles. Therefore, it is important to derive generic solutions for deriving the vertices of the derived polygons and to assess the positional error in the derived polygons based on their vertices.

The first step in finding vertices of derived polygons from two original polygons is to find all intersection points between the original polygons, and identify these vertices of one original polygon which lie on the boundary of the other original polygon and vice versa. These vertices and the intersection points will be stored in a basis set which is a collection of some vertices of the derived polygons.

When one original polygon is either inside or outside the other original polygon (including the case where they touch), the basis set is empty or all of its elements are the vertices of the original polygons. In the case of one original polygon ($A$) inside the other original polygon ($B$) (including the case where polygon $A$ touches the boundary of $B$), all vertices of polygon $A$ should be inside polygon $B$ (see Fig. 1(h) and (i)); two derived polygons are then polygon $B$ minus polygon $A$ (i.e., polygon $B - A$) and the polygon $A$. In the second case, if the two original polygons are disjoint (see Fig. 1(m)), all vertices of either original polygon will be outside the other original polygon. Furthermore, there is a case when one original polygon is outside the other one but their boundaries touch (see Fig. 1(n)). In the latter two cases, two derived polygons are the two original polygons.

If the basis set is non-empty and at least one of its elements is not a vertex of the original polygons (i.e., it is an intersection point), at least one derived polygon will be the intersection of the original polygons and at least two derived polygons will be the difference between the original polygons and the
intersection of the two original polygons. Vertices of these derived polygons are estimated as listed in pseudo-code given in Appendix A, which is included as supplementary material in the electronic, online version of this paper, available by the Publisher. The proposed method for computing the vertices of derived polygons can handle not only original polygons that are convex but also concave. Therefore, it is a generic solution. The details of this algorithm can be found in the work of Cheung (2003).

In this study, positional error of a derived polygon is described by a set of error measures. Here, we propose to use the following three error measures: (a) variance–covariance matrix of any intersection point of original polygons, which can be further used to derive the variance–covariance matrix of all polygon vertices—a relatively comprehensive description but with large data volume, (b) an error of particular characteristics, such as the area and perimeter of a polygon, each of which reflects a particular feature of the polygon and require a smaller volume of data for description, and (c) a radial error interval for the polygon vertices.

3. Analytical method and error measures for derived polygons

In this paper, the error propagation law in statistics is used to derive the propagated error in the derived polygon based on the error at the original polygon vertices. This method is used to provide a series of analytical expressions for each of the error measures and the analytical relationships between the derived polygons and the original polygons.

3.1. Variance–covariance matrix of an intersection point of two polygons

In the previous section, we illustrated how an intersection point of two line segments can be determined. We now discuss the error at an intersection point. To obtain the variance–covariance matrix of the intersection point $P$, Eq. (2) is differentiated and we have:

$$
\begin{bmatrix}
dx_p \\
dy_p
\end{bmatrix} = A_p \cdot d\zeta
$$

(4)
where \( d \varepsilon = [dx_i, dy_i, dx_j, dy_j, dx_k, dy_k, dx_l, dy_l]^T \), and

\[
A_p = \begin{bmatrix}
\frac{\partial x_p}{\partial x_i} & \frac{\partial x_p}{\partial y_i} & \frac{\partial x_p}{\partial x_j} & \frac{\partial x_p}{\partial y_j} & \frac{\partial x_p}{\partial x_k} & \frac{\partial x_p}{\partial y_k} & \frac{\partial x_p}{\partial x_l} & \frac{\partial x_p}{\partial y_l}
\end{bmatrix}
\]

Let \( D \) denote the variance–covariance matrix of the vertices of \( x_i, y_i, x_j, y_j, x_k, y_k, x_l, y_l \). It is an \( 8 \times 8 \) symmetric matrix, as shown in Eq. (5).

\[
D = \begin{bmatrix}
\sigma_{x_i}^2 & \sigma_{x_i y_i} & \cdots & \sigma_{x_i y_l} \\
\sigma_{x_i y_i} & \sigma_{x_i y_i} & \cdots & \sigma_{x_i y_l} \\
\sigma_{x_i y_l} & \sigma_{x_i y_l} & \cdots & \sigma_{x_i y_l} \\
\sigma_{y_i}^2 & \sigma_{y_i} & \cdots & \sigma_{y_i} \\
\sigma_{y_i} & \sigma_{y_i} & \cdots & \sigma_{y_l} \\
\sigma_{y_l} & \sigma_{y_l} & \cdots & \sigma_{y_l} \\
\sigma_{x_i}^2 & \sigma_{x_i} & \cdots & \sigma_{x_i} \\
\sigma_{x_i} & \sigma_{x_i} & \cdots & \sigma_{x_i}
\end{bmatrix}
\]

where \( \sigma_{sx} \) is the covariance between variables \( s \) and \( t \); and \( \sigma_{s}^2 \) is the variance of variable \( s \).

According to the error propagation law, the variance–covariance matrix of the intersection point \( P \) is:

\[
D_p = \begin{bmatrix}
\sigma_{x_p}^2 & \sigma_{x_p y_p} \\
\sigma_{x_p y_p} & \sigma_{y_p}^2
\end{bmatrix} = A_p D A_p^T
\]

3.2. Variance of the perimeter, area and center of gravity of the derived polygons

Perimeter, area and center of gravity are measures used to describe the geometric characteristics of a polygon. The errors of these measures can be represented by their variance. For example, a derived polygon \( E \) has vertices \((x_1, y_1), (x_2, y_2), \ldots (x_m, y_m)\), where \((x_0, y_0) = (x_m, y_m)\) and \((x_1, y_1) = (x_{m+1}, y_{m+1})\). Perimeter, area and center of gravity are then expressed as a function of the vertices of \( E \):

\[
P_E = \sum_{i=1}^{m} \sqrt{(x_i - x_{i+1})^2 + (y_i - y_{i+1})^2}
\]

\[
S_E = \frac{1}{2} \sum_{i=1}^{m} x_i(y_{i+1} - y_{i-1})
\]

\[
M_{Ex} = \frac{1}{m} \sum_{i=1}^{m} x_i
\]

\[
M_{Ey} = \frac{1}{m} \sum_{i=1}^{m} y_i
\]

Differentiating the above three equations, we have

\[
dP_E = \left[ \frac{\partial P_E}{\partial x_1}, \frac{\partial P_E}{\partial y_1}, \frac{\partial P_E}{\partial x_2}, \frac{\partial P_E}{\partial y_2}, \ldots, \frac{\partial P_E}{\partial x_m}, \frac{\partial P_E}{\partial y_m} \right]^T = \frac{\partial P_E}{\partial E} dE
\]

\[
dS_E = \left[ \frac{\partial S_E}{\partial x_1}, \frac{\partial S_E}{\partial y_1}, \frac{\partial S_E}{\partial x_2}, \frac{\partial S_E}{\partial y_2}, \ldots, \frac{\partial S_E}{\partial x_m}, \frac{\partial S_E}{\partial y_m} \right]^T = \frac{\partial S_E}{\partial E} dE
\]

\[
\begin{bmatrix}
\frac{dM_{Ex}}{dE} \\
\frac{dM_{Ey}}{dE}
\end{bmatrix} = \begin{bmatrix}
\frac{1}{m} & 0 & \frac{1}{m} & \cdots & \frac{1}{m} & 0 \\
0 & \frac{1}{m} & 0 & \cdots & 0 & \frac{1}{m}
\end{bmatrix} \times \left[ dx_1, dy_1, dx_2, dy_2, \ldots, dx_m, dy_m \right]^T
\]

By the error propagation law, the variance of the perimeter and area as well as the variance–covariance matrix of the center of gravity are given as:

\[
\sigma_{P_E}^2 = \frac{\partial P_E}{\partial E} D_E \left( \frac{\partial P_E}{\partial E} \right)^T
\]

\[
\sigma_{S_E}^2 = \frac{\partial S_E}{\partial E} D_E \left( \frac{\partial S_E}{\partial E} \right)^T
\]

\[
\sum_{M_{Ex}, M_{Ey}} \begin{bmatrix}
\sigma_{M_{Ex}}^2 & \sigma_{M_{Ex} M_{Ey}} \\
\sigma_{M_{Ex} M_{Ey}} & \sigma_{M_{Ey}}^2
\end{bmatrix} = \frac{\partial M_{Ex}}{\partial E} D_E \left( \frac{\partial M_{Ex}}{\partial E} \right)^T + \frac{\partial M_{Ey}}{\partial E} D_E \left( \frac{\partial M_{Ey}}{\partial E} \right)^T
\]

where \( D_E \) is the variance–covariance matrix of the vertices of \( E \).
Replacing $E$ in Eqs. (13)–(15) by $A - A \cap B$, $B - A \cap B$, or $A \cap B$, we can obtain the variance of perimeter, area, and center of gravity of $A - A \cap B$, $B - A \cap B$, or $A \cap B$.

3.3. Error interval for polygon vertices

By using the variance–covariance matrix of the vertices of a derived polygon, we can have a full description of a polygon. However, this method may lead to a large volume of data. An alternative is to describe the error at the polygon vertices by three intervals for describing radial positional error at all vertices, which considers both the $x$- and $y$-directional errors at individual polygon vertices simultaneously. Although from a practical point of view, the maximum value for these error measures is the most important (in the sense of a tolerance and upper bound), the minimum value for these error measures is also used to differentiate between cases when the maximum value is the same, but the minimum value differs, thus cases with different error properties. Since the error at a point between two vertices will be smaller than the error at the vertices (as shown, e.g., in the work of Shi, 1998 and Shi and Liu, 2000), the error intervals for all vertices will also include the error intervals for all points of the polygon.

This analytical method assumes that (a) when the given locations of two edges of original polygons intersect, their true locations intersect, and (b) when the given locations of the boundaries of the original polygons are disjoint, their true locations are disjoint. Such an assumption may not be valid when the intersection points of the original polygons and the vertices of the original polygons are close in case (a), or the boundaries of the original polygons are close in case (b). Therefore, this analytical method will be only applicable, if the above assumptions are valid.

4. Simulation method for modelling error propagation

A simulation method for analyzing error propagation in overlay analysis is the second approach. Similar error measures as proposed for the analytical method are computed. Fig. 2 gives a flowchart of the simulation method. In order to study positional error in a vector-based overlay, one realization of the “true” location of each vertex of the original polygons is simulated based on an assumption for its error distribution. Combining the simulated “true” locations of the vertices of one original polygon forms a realization of the “true” location of this original polygon. We then derive vertices of the corresponding derived polygons based on this realization of the “true” locations of the original polygons. This is the first simulation run. If we repeat the simulation several times (say, NP iterations), sample values of the error measures will be estimated.
We have developed a program in Visual Basic, based on the algorithms for the vertices of derived polygons given in Section 2.2 and the simulation flowchart shown in Fig. 2. This program can automatically and on-line perform an overlay analysis of two polygons, compute the vertices of the derived polygons, and estimate the error measures of an original polygon or derived polygon.

5. A case study

Here, we will carry out a case study with an example of the error measures for each of the two methods and identify properties of the error measures for the original and derived polygons. In practice, the properties of the covariance matrices of the used polygons will depend on the data capture method.
and its characteristics. The matrices used here are simple and for demonstration purposes only.

Fig. 3 shows two overlaying polygons $A$ and $B$. The coordinates of the vertices of $A$ and $B$ are shown in Table 1.

Suppose the error in each vertex of the two original polygons is correlated in the $x$- and the $y$-directions, and the error between the vertices of the two polygons is also correlated, e.g., it is supposed that for $i,j=1, 2, \ldots, 4$ or 5 and $i \neq j$,

\[
\sum_A = \begin{bmatrix}
3.000 \times 10^{-2} \text{ mm}^2 \\
3.000 \times 10^{-2} \text{ mm}^2 & 2.000 \times 10^{-3} \text{ mm}^2 \\
3.000 \times 10^{-2} \text{ mm}^2 & \cdot \\
2.000 \times 10^{-3} \text{ mm}^2 & 3.000 \times 10^{-2} \text{ mm}^2
\end{bmatrix}
\]

i.e., $\sigma_{x_i}^2 = \sigma_{y_i}^2 = 3.000 \times 10^{-2} \text{ mm}^2$, $\sigma_{x_iy_i} = \sigma_{x_jy_j} = \sigma_{x_iy_j} = 2.000 \times 10^{-3} \text{ mm}^2$;

\[
\sum_B = \begin{bmatrix}
2.000 \times 10^{-2} \text{ mm}^2 \\
2.000 \times 10^{-2} \text{ mm}^2 & -2.000 \times 10^{-3} \text{ mm}^2 \\
2.000 \times 10^{-2} \text{ mm}^2 & \cdot \\
-2.000 \times 10^{-3} \text{ mm}^2 & 2.000 \times 10^{-2} \text{ mm}^2
\end{bmatrix}
\]

i.e., $\sigma_{x_i}^2 = \sigma_{y_i}^2 = 2.000 \times 10^{-2} \text{ mm}^2$, $\sigma_{x_iy_i} = \sigma_{x_jy_j} = \sigma_{x_iy_j} = -2.000 \times 10^{-3} \text{ mm}^2$;

and

\[
\sum_{AB} = \sum_{BA} = \begin{bmatrix}
-1.000 \times 10^{-3} \text{ mm}^2 \\
-1.000 \times 10^{-3} \text{ mm}^2 & -1.000 \times 10^{-3} \text{ mm}^2 \\
-1.000 \times 10^{-3} \text{ mm}^2 & \cdot \\
-1.000 \times 10^{-3} \text{ mm}^2 & -1.000 \times 10^{-3} \text{ mm}^2
\end{bmatrix}
\]

i.e., $\sigma_{x_i}^2 = \sigma_{y_i}^2 = \sigma_{x_iy_i} = \sigma_{x_jy_j} = \sigma_{x_iy_j} = -1.000 \times 10^{-3} \text{ mm}^2$. 
According to these variance–covariance matrices, the coordinates of the intersection points \( P \) and \( Q \) are computed, and the analytical error at the intersection points are estimated from Eq. (6). The coordinates obtained are \( P = (4.544 \text{ mm}, 6.591 \text{ mm}) \) and \( Q = (2.339 \text{ mm}, 6.336 \text{ mm}) \), while their variance–covariance matrices are

\[
\begin{bmatrix}
1.528 \times 10^{-2} \text{ mm}^2 & 3.607 \times 10^{-3} \text{ mm}^2 \\
3.607 \times 10^{-3} \text{ mm}^2 & 1.354 \times 10^{-2} \text{ mm}^2
\end{bmatrix}
\]

and

\[
\begin{bmatrix}
1.863 \times 10^{-2} \text{ mm}^2 & -5.718 \times 10^{3} \text{ mm}^2 \\
-5.718 \times 10^{-3} \text{ mm}^2 & 1.078 \times 10^{-2} \text{ mm}^2
\end{bmatrix}
\].

The values of the analytical error measures are shown in Table 2, in which the statistical properties for the perimeter, area and center of gravity are derived based on Eqs. (13)–(15) and the overall radial positional error for the vertices of the original and derived polygons are derived based on Eq. (17).

In addition, the relative error is defined as variance divided by the mean value. The analytical relative error in the perimeter and area of the original and derived polygons are also shown in Table 2.

The simulation model is based on the variance–covariance matrices of \( A \) and \( B \) given above and the assumption that the error at the vertices of the original polygons follows normal distribution. The sample variance–covariance matrix of the center of gravity and the radial positional error interval are given in Table 4, where \( NP = 100 \).

6. Analysis of the analytical and simulated method results

In the following, the analytical and simulation methods are analyzed regarding two aspects: (a) comparison between the analytical and simulated method results, and (b) an analysis of the results of the case study.

6.1. A comparison between the analytical and simulated methods

Here, we compare the analytical and simulated methods results for analyzing error propagation in overlay analysis, in combination with the results from the case study. Firstly, the analytical method assumes that intersection points of the original polygons exist (or do not exist) in the “true” and given locations of the original polygons simultaneously. Practically, the analytical method can assess the error in polygon overlay when the error ellipse of any intersection point of the original polygons does not intersect with error ellipses of all the vertices of the original polygons, if the intersection point exists; or, in case of disjoint polygon boundaries, the error ellipse of any point on the boundary of one original polygon does not intersect with the error ellipse of any point on the boundary of the other original polygons.

The computational load in the simulation method is higher than that in the analytical method, as the derived polygons are generated and their vertices are computed in each simulation iteration.

In summary, the analytical method is able to assess the error in an overlay analysis under the assumption

![Fig. 3. An example of a polygon overlay.](image-url)

Table 1

<table>
<thead>
<tr>
<th>Coordinates of the vertices of polygons ( A ) and ( B ) (in mm—in a 1:500 scale map)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Polygon ( A )</td>
</tr>
<tr>
<td>( P_1 )</td>
</tr>
<tr>
<td>( x )</td>
</tr>
</tbody>
</table>

The sample variance of the intersection points is given in Table 3.
that (a) uncertain intersection points of the original polygons are farther away than the uncertain vertices of the original polygons, or (b) the uncertain boundaries of the original polygons do not intersect, if the original polygons are disjoint. The simulation method can model the error in the overlay analysis in a more generic case, when the error distributions of the vertices of the original polygons are given.

The results obtained from the analytical and simulation models are then compared directly. The 95% confidence interval for the simulated mean and the simulated variance of the perimeter, area and center of gravity are estimated (Table 5). Using this confidence interval, we can compare the means and variances of the two methods. If the results of the analytical method are within the confidence interval of the simulation method results, then we accept the hypothesis that these results are equal.

From Table 5, it can be noted that some of the analytical mean and variance values of the error measures are within the confidence intervals of the corresponding measures of the simulation method (e.g., the analytical variance of the perimeter of \( A \), see Tables 2 and 5). Some of the analytical mean or variance values of the error measures are slightly outside the corresponding simulated confidence regions (e.g., the analytical mean of the perimeter of \( A \)). This may be due to round-off errors in the simulation method and the truncation error in the analytical method that is introduced, when Eqs. (7) and (8) are approximated by the first order Taylor series expansion to calculate the variance of the error measures based on the error propagation law. However, the simulation results are very close to the analytical results—the absolute difference between the mean values is less than 1 mm, and that of the variances less than 0.1 mm². Therefore, to a certain extent the analytical and the simulated results in our example are identical.

### 6.2. An analysis of the results of the case study

The experimental study in both the analytical and simulation models indicates that the variance of the intersection points is always smaller than that of the vertices of the original polygons sides that intersect.

The error of perimeter, area and center of gravity of a polygon relates to (a) the number of the polygon

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**Table 2**

<table>
<thead>
<tr>
<th>Analytical method error measures, of original and derived polygons (in mm and for the variances mm²)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Polygon ((E))</td>
</tr>
<tr>
<td>----------------</td>
</tr>
<tr>
<td>(A)</td>
</tr>
<tr>
<td>(B)</td>
</tr>
<tr>
<td>(C)</td>
</tr>
<tr>
<td>(D)</td>
</tr>
<tr>
<td>(E)</td>
</tr>
</tbody>
</table>
vertices, and (b) the error at the polygon vertices. This can be explained from Eqs. (13)–(15), by which the variance of the perimeter, area and center of gravity of a polygon can be computed. In these three equations, the variances of these polygon measures are estimated by multiplying a vector, such as \( (\partial P/\partial E) \) for the perimeter (or a matrix for the center of gravity) with the variance–covariance matrix \( D_E \) of the vertices of polygon \( E \) and the transpose of the vector (or the matrix). The variance–covariance matrix \( D_E \) is computed by multiplying \( dE \) with its transpose \( dE^T \), where \( dE = [dx_1, dy_1, dx_2, dy_2, \ldots, dx_m, dy_m] \) and \( m \) is the number of the vertices of \( E \). Therefore, increasing the number of the vertices in \( E \) will increase the variance of the perimeter, area and center of gravity, if the error at the vertices of \( E \) remains constant. Furthermore, increasing the error at the vertices of \( E \) will also increase the variance of the perimeter, area and center of gravity, if the number of the vertices of \( E \) remains the same.

When the number of the vertices of a derived polygon and the error of its vertices both increase (or decrease), the error of the perimeter, area and center of gravity of this polygon will change similarly. For example, in Tables 2 and 4, both the analytical and simulated results show that the variances for these error measures of \( A \) are larger than those for \( B \), where \( A \) and \( B \) have 5 and 4 vertices, respectively, and the error at the vertices of \( A \) is also larger than the error at the vertices of \( B \).

However, when the number of the vertices of the derived polygon increases (or decreases) and the error at the vertices decreases (or increases), there is no consistent trend for the variance of the perimeter, area and center of gravity. For example, the derived polygons \( A - A \cap B \) and \( B - A \cap B \) have six and seven vertices, respectively, and the average error at the vertices of \( A - A \cap B \) is larger than the average error at the vertices of \( B - A \cap B \). Both analytical and simulated results show that the variance of the perimeter of \( A - A \cap B \) is smaller than that of \( B - A \cap B \), while the variances of the area and center of gravity of \( A - A \cap B \) are larger than those of \( B - A \cap B \).

In addition, the relative error is influenced strongly by the mean of the perimeter and area, as these values are used in the denominator for the computation of the relative error.

Moreover, Tables 2 and 4 show that the radial error intervals for the original polygons obtained from both methods are almost identical. The radial error intervals of polygons \( A \) and \( B \) have each equal minimum and maximum values, because the error in all vertices of each original polygon is identical. However, the radial error intervals for the derived polygons have different minimum values in both methods, and the intervals for both of them are not zero.

7. Conclusions

We have proposed two methods to estimate error propagation in vector overlay. An analytical error model derived based on the error propagation law, and a simulation error model. For each of these two error models, the positional error in the original or derived polygons is proposed to be assessed by three error measures: (a) the variance–covariance matrices of the polygon vertices, (b) the radial error interval for all vertices of the original or derived polygons, and (c) the variance of the perimeter, area and center of gravity of the original or derived polygons. Whereas, within the framework of the analytical method, the area, perimeter and center of gravity of the derived polygons was introduced in previous studies, the radial error interval for the vertices of the derived polygons are newly proposed.

The variance–covariance matrix of the vertices of an original or derived polygon is a relatively comprehensive error description. However, such a matrix is not practical and easy to use as an error measure and
Table 4
Simulation method error measures of original and derived polygons with NP = 100 (in mm and for the variances mm²)

<table>
<thead>
<tr>
<th>Polygon (E)</th>
<th>Perimeter (P_E)</th>
<th>Area (S_E)</th>
<th>Center of gravity (M_{Ex}, M_{Ey})</th>
<th>Radial error interval</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Mean of P_E</td>
<td>Variance of P_E (relative error)</td>
<td>Mean of S_E</td>
<td>Variance of S_E (relative error)</td>
</tr>
<tr>
<td>A</td>
<td>9.654</td>
<td>0.190 (0.020)</td>
<td>6.231</td>
<td>0.401 (0.064)</td>
</tr>
<tr>
<td>B</td>
<td>9.961</td>
<td>0.171 (0.017)</td>
<td>5.610</td>
<td>0.242 (0.043)</td>
</tr>
<tr>
<td>A - A \cap B</td>
<td>8.887</td>
<td>0.250 (0.028)</td>
<td>4.335</td>
<td>0.248 (0.057)</td>
</tr>
<tr>
<td>B - A \cap B</td>
<td>10.870</td>
<td>0.282 (0.026)</td>
<td>3.714</td>
<td>0.230 (0.062)</td>
</tr>
<tr>
<td>A \cap B</td>
<td>5.669</td>
<td>0.210 (0.037)</td>
<td>1.896</td>
<td>0.133 (0.070)</td>
</tr>
</tbody>
</table>

Table 5
95% confidence intervals for the mean and variance of error measures of the original and derived polygons for the simulation model with NP = 100 (in mm and for the variances mm²)

<table>
<thead>
<tr>
<th>Polygon (E)</th>
<th>Perimeter (P_E)</th>
<th>Area (S_E)</th>
<th>Center of gravity (M_{Ex}, M_{Ey})</th>
<th>\sigma M_{Ex}^2</th>
<th>\sigma M_{Ey}^2</th>
<th>\sigma_{M_{Ex,M Ey}}^2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Mean of P_E</td>
<td>Variance of P_E (relative error)</td>
<td>Mean of S_E</td>
<td>Variance of S_E (relative error)</td>
<td>Mean of M_{Ex}</td>
<td>Mean of M_{Ey}</td>
</tr>
<tr>
<td>A</td>
<td>[9.645, 9.663]</td>
<td>[0.17, 0.21]</td>
<td>[6.219, 6.243]</td>
<td>[0.37, 0.44]</td>
<td>[3.509, 3.513]</td>
<td>[5.724, 5.728]</td>
</tr>
<tr>
<td>B</td>
<td>[9.953, 9.969]</td>
<td>[0.16, 0.19]</td>
<td>[5.600, 5.620]</td>
<td>[0.22, 0.26]</td>
<td>[3.568, 3.570]</td>
<td>[7.142, 7.144]</td>
</tr>
<tr>
<td>A - A \cap B</td>
<td>[8.877, 8.897]</td>
<td>[0.23, 0.27]</td>
<td>[4.325, 4.345]</td>
<td>[0.23, 0.27]</td>
<td>[3.559, 3.563]</td>
<td>[5.556, 5.558]</td>
</tr>
<tr>
<td>B - A \cap B</td>
<td>[10.860, 10.881]</td>
<td>[0.26, 0.31]</td>
<td>[3.705, 3.723]</td>
<td>[0.21, 0.25]</td>
<td>[3.509, 3.511]</td>
<td>[7.110, 7.112]</td>
</tr>
<tr>
<td>A \cap B</td>
<td>[5.660, 5.678]</td>
<td>[0.19, 0.23]</td>
<td>[1.889, 1.903]</td>
<td>[0.12, 0.14]</td>
<td>[3.453, 3.457]</td>
<td>[6.559, 6.563]</td>
</tr>
</tbody>
</table>
may also have large size. Therefore, the radial positional error interval is proposed as more practical for describing the error at the polygon vertices.

A case study was carried out based on the proposed methods. The results indicated that the intersection points of the original polygon sides were of higher accuracy than the vertices of the intersecting original polygon sides. Furthermore, it is also shown that the number of the vertices and the error at the polygon vertices are two major factors that may affect the accuracy of the polygon parameters like perimeter, area and center of gravity. Increasing (or decreasing) both the number and the error of the polygon vertices will influence the error of these polygon parameters similarly. However, there is no consistent trend on the change of the error of these parameters, if the number of the vertices and the error at the vertices are not decreased or increased simultaneously.

Although the simulation technique has been used to assess attribute and positional errors of objects in GIS, this technique is further extended to assess errors in spatial analysis, vector overlay in this paper.

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