REGULARITY OF LAGRANGE MULTIPLIERS FOR OPTIMAL CONTROL PROBLEMS WITH PDEs AND MIXED CONTROL STATE CONSTRAINTS

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1. INTRODUCTION

Lagrange multipliers for distributed parameter systems with mixed control-state constraints may exhibit better regularity properties than those for problems with pure pointwise state constraints, (1), (2), (4). Under natural assumptions, they are functions of certain $L^p$-spaces, while Lagrange multipliers for pointwise state constraints are, in general, measures. Following an approach suggested in (3) for ODEs, a new and simplified technique is applied to prove $L^1$-regularity in the case of elliptic PDEs. Moreover, an idea of (5) is extended to derive $L^\infty$-estimates for the Lagrange multipliers, along with the proof of Lipschitz regularity of optimal controls.

2. OPTIMAL CONTROL PROBLEM

We consider first the following elliptic optimal control problem:

$$\min_{y,u} J(y,u) = \int_\Omega \varphi(x,y,u) \, dx + \int_\Gamma \psi(x,y) \, ds$$

subject to

$$Ay + d(x,y) = u \quad \text{in } \Omega$$
$$\frac{\partial y}{\partial \nu_A} + b(x,y) = 0 \quad \text{on } \Gamma$$

and to

$$g_i(x,y(x),u(x)) \leq 0 \quad \text{a.e. on } \Omega, \ i = 1, \ldots, k.$$  

The inequalities (3) are the mixed control-state constraints.

In this setting, $\Omega \subset \mathbb{R}^N$, $N \in \mathbb{N}$, is a bounded Lipschitz domain and $A$ is a uniformly elliptic differential operator of the form

$$Ay = -\sum_{i,j=1}^N \frac{\partial}{\partial x_i} \left( a_{ij}(\frac{\partial}{\partial x_j} y) \right) + c_0 y$$

with coefficients $a_{ij} \in C_0^1(\bar{\Omega})$, $i, j = 1, \ldots, N$, where $c_0(x) \geq 0$ belongs to $L^\infty(\Omega)$ and satisfies $c_0(x) > 0$ on a set of positive measure.

The functions $\varphi = \varphi(x,y,u) : \Omega \times \mathbb{R}^2 \to \mathbb{R}$, $g_i = g_i(x,y,u) : \Omega \times \mathbb{R}^2 \to \mathbb{R}$, $\psi = \psi(x,y) : \Gamma \times \mathbb{R} \to \mathbb{R}$, $d = d(x,y) : \Omega \times \mathbb{R} \to \mathbb{R}$, and $b = b(x,y) : \Gamma \times \mathbb{R} \to \mathbb{R}$, are assumed to enjoy the following properties (consider all functions formally as depending on $(x,y,u)$):

For all fixed $(y,u)$, they are measurable with respect to $x \in \Omega$ or $x \in \Gamma$, respectively. They are partially differentiable with respect to $(y,u)$ for all fixed $x \in \Omega$ or $x \in \Gamma$. These functions and their derivatives are locally Lipschitz with respect to $(y,u)$ in the sense that the associated Lipschitz constant depends only on $|y| + |u|$ but not on $x$.

Moreover, we require that these functions and their partial derivatives are essentially bounded with respect to $x$ in $\Omega$ or $x \in \Gamma$, respectively, at $(y,u) = (0,0)$. The derivatives $\frac{\partial d}{\partial y}(x,y)$ and $\frac{\partial b}{\partial y}(x,y)$ are assumed to be nonnegative for almost all $x \in \Omega$ or $x \in \Gamma$ to guarantee existence and uniqueness of the solution $y$ to (2).

3. REGULARITY OF LAGRANGE MULTIPLIERS

The existence of Lagrange multipliers is obtained first in $(L^\infty(\Omega))^*$, the dual space to $L^\infty(\Omega)$. The elements of $(L^\infty(\Omega))^*$ can be represented
by finitely additive set functions on \( \hat{\Omega} \) that are also called finitely additive measures.

To derive necessary optimality conditions, a standard linearized Slater condition is assumed as constraint qualification: There exist \( \tilde{u} \in L^\infty(\Omega) \) and \( \sigma > 0 \) such that

\[
g_i(x, \tilde{y}(x), \tilde{u}(x)) + \frac{\partial g_i}{\partial y}(x, \tilde{y}(x), \tilde{u}(x)) \tilde{y}(x) + \frac{\partial g_i}{\partial u}(x, \tilde{y}(x), \tilde{u}(x)) \tilde{u}(x) \leq -\sigma \quad \text{a.e. in } \Omega,
\]

where \( \tilde{y} = G'(\tilde{u})\tilde{u} \) is the directional derivative of the control-to-state mapping \( G : u \to y, G : L^\infty(\Omega) \to H^1(\Omega) \cap C(\hat{\Omega}) \).

**Theorem 1:** Suppose that \( \tilde{u} \) with associated state \( \tilde{y} \) is locally optimal for (1)–(3) and the condition (4) is satisfied at \((\tilde{y}, \tilde{u})\). Then there exist non-negative finitely additive measures \( \mu_i \in L^\infty(\Omega)^*, \ i = 1, \ldots, k \), and an adjoint state \( p \in W^{1,1}(\Omega) \) for all \( 1 \leq s < \frac{N}{N-1} \), such that the conditions

\[
\int_\Omega \left( \frac{\partial \varphi}{\partial u} + p \right) h dx + \sum_{i=1}^k \frac{\partial g_i}{\partial u} h d\mu_i = 0
\]

\[
\forall h \in L^\infty(\Omega),
\]

\[
\int_\Omega g_i(\cdot, \tilde{y}, \tilde{u}) d\mu_i = 0, \quad i = 1, \ldots, k,
\]

and the adjoint equation

\[
A^*p + \frac{\partial d}{\partial y} p = \frac{\partial \varphi}{\partial y} + \sum_{i=1}^k \frac{\partial g_i}{\partial y} \mu_i |_{\Omega},
\]

\[
\frac{\partial p}{\partial \nu_A^*} + \frac{\partial b}{\partial y} p = \frac{\partial \psi}{\partial y} + \sum_{i=1}^k \frac{\partial g_i}{\partial y} \mu_i |_{\Gamma}
\]

are satisfied, if the derivatives of \( \varphi, \psi, g_i, d, b \) in the expressions above are taken at \((x, \tilde{y}, \tilde{u})\).

As linear functionals on \( L^\infty(\Omega) \), the finitely additive measures \( \mu_i \) must vanish on sets of Lebesgue measure zero. Thanks to a theorem by Yosida and Hewitt (6), each \( \mu \in L^\infty(\Omega)^* \) can be uniquely written in the form \( \mu = \mu_c + \mu_p \), where \( \mu_c \) is countably additive and \( \mu_p \) is purely finitely additive. Moreover, if \( \mu \geq 0 \), then \( \mu_c \) and \( \mu_p \) are non-negative, too.

For higher regularity of multipliers, the following assumption is needed: Define, for \( \delta > 0 \), the \( \delta \)-active sets

\[
M^\delta_i := \{ x \in \Omega : g_i(x, \tilde{y}(x), \tilde{u}(x)) \geq -\delta \}.
\]

Assume that there exist \( \delta > 0 \) and \( \tilde{u} \in L^\infty(\Omega) \) such that

\[
\frac{\partial g_i}{\partial u}(x, \tilde{y}(x), \tilde{u}(x)) \tilde{u}(x) \geq 1 \quad \text{a.e. on } M^\delta_i
\]

holds for all \( i \in \{1, \ldots, k\} \).

This requirement is equivalent to a "uniformly positive linear independency condition", cf. Dmitruk (3). For some types of constraints, this assumption is automatically satisfied. In other cases, the optimal solution must fulfill a certain separation condition.

**Theorem 2:** If \( \tilde{u} \in U, \tilde{y} \in Y \) and \( \mu_i \in L^\infty(\Omega)^*, \mu_i \geq 0, \ i = 1, \ldots, k \), satisfy the first-order optimality conditions of Theorem 1 and (5) is satisfied, then the purely finitely additive parts of all \( \mu_i \) are vanishing so that all \( \mu_i, \ i = 1, \ldots, k \), can be represented by densities in \( L^1(\Omega) \).

The proof follows the one given by Dmitruk (3) for the case of ordinary differential equations.

If the functions \( \varphi \) and \( g_i, i = 1, \ldots, k \), are assumed to be Lipschitz with respect to \((x, y, u)\), then locally optimal controls enjoy Lipschitz continuity, too. In associated parabolic control problems, Hölder continuity of the optimal controls can be derived.

**REFERENCES**


