A Bayesian Approach to Multipath Mitigation in GNSS Receivers

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Abstract—Multipath is known to be one of the most dominant sources of accuracy degradation in satellite-based navigation systems. Multipath may cause biased position estimates that could jeopardize high-precision applications. This paper considers the problem of tracking the time-variant synchronization parameters of both the line-of-sight signal (LOSS) and its multipath replicas. In particular, the proposed algorithm tracks time-delays, amplitudes, phases and proposes a procedure to extract Doppler shifts from complex amplitudes. However, the interest is focused on LOSS time-delay estimates, since those provide the means to compute user’s position. The undertaken Bayesian approach is implemented by a Particle Filter. The selection of the importance density function, from which particles are generated, is performed using a Gaussian approximation of the posterior function. This selection provides a particle generating function close to the optimal, which yields to an efficient usage of particles. The complex-linear part of the model, i.e., complex amplitudes, is tackled by a Rao-Blackwellization procedure that implements a Complex Kalman Filter for each generated particle, thus reducing the computational load. Computer simulation results are compared to other Bayesian filtering alternatives (namely, the Extended Kalman Filter, Unscented Kalman Filter and the Sequential Importance Resampling algorithms) and the Posterior Cramér-Rao Bound.

Index Terms—Monte Carlo methods, Bayes procedures, satellite navigation systems, synchronization, multipath channel.

I. INTRODUCTION

GLOBAL Navigation Satellite Systems (GNSS) is the general concept used to identify those systems that allow user positioning based on a constellation of satellites. Specific GNSS are the well-known American GPS or the forthcoming European Galileo. Both systems rely on the same principle: the user computes its position by means of measured distances between the receiver and the set of in-view satellites. These distances are calculated estimating the propagation time that transmitted signals take from each satellite to the receiver. At least 4 satellites are needed in order to compute user position and time [1]–[4]. Each satellite is uniquely identified by its own direct-sequence spread-spectrum (DS-SS) signal, which are transmitted synchronously by all satellites. GNSS receivers are only interested in estimating the delays of signals which are received directly from the satellites, hereafter referred to as line-of-sight signal (LOSS), since they are the ones that carry information of direct propagation time. Hence, reflections distort the received signal in a way that may cause a bias in delay and carrier-phase estimations [5], [6]. Multipath is probably the dominant source of error in high-precision applications since it can introduce a bias up to a hundred of meters when employing a 1-chip wide (standard) Delay Lock Loop (DLL) to track the delay, which is a common synchronization method used in DS-SS receivers [7], [8]. To the aim of reducing the multipath effect on time-delay estimates, several DLL-based methods have been proposed such as the Narrow Correlator [9], the Pulse Aperture Correlator (PAC) [10], the Strobe Correlator (which is a specific implementation of the Double Delta Correlator described in [11]), the Early1/Early2 (E1/E2) tracking technique [12] or the Multipath Elimination Technology (MET) [13]. A robust statistical approach to the multipath problem is the Multipath Estimating Delay Lock Loop (MEDLL) developed by Van Nee [14], where the Maximum Likelihood principle is applied considering the number of reflections known. Another recent approach is the Vision Correlator introduced by NovAtel [15], which is an implementation of a multipath mitigation method known as the Multipath Mitigation Technique (MMT), developed by Weill [16]. The use of Particle Filters (PFs) for multipath mitigation in GNSS receivers has been addressed in the literature. For example, in [17] a Rao-Blackwellized PF was proposed to integrate GPS measurements and Inertial Navigation Systems [18]. In [19] a PF algorithm that operates at the observable level (i.e., pseudoranges) was presented to mitigate multipath effect in the positioning solution, which also considered Rao-Blackwellization of a multipath indicator process. A particle filtering algorithm in the vein of the present paper was reported in [20], where the PF operates at the signal level in order to jointly track the LOSS and its replicas. Other PF usages can be found in the literature in the field of multisensor data fusion and switching observation models [21], [22]. Aside from these approaches, joint processing of satellite signals was seen to provide multipath mitigation capabilities in the Vector DLL (VDLL) architecture [1] and the Direct Position Estimation (DPE) approach, which was presented in [23]. Out of the scope of this paper are antenna array based approaches, where the spatial diversity provided by antenna arrays is exploited [24]–[26].

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In this paper, we present a particle filtering algorithm that tracks the parameters of satellite signals in the presence of multipath. The algorithm tracks both the LOSS and its multipath replicas of a given satellite signal, reducing dramatically the bias caused by multipath components. The algorithm is applicable to any GNSS-like signal and was initially proposed by the authors in a simpler version [20] where the proposal function for particle generation was quite heuristic, though simple to implement. However, this was seen to be inefficient in terms of the required number of particles and thus not interesting for on-line tracking purposes. In the present paper, two important variations are introduced with respect to previous work. Firstly, it considers a variance reduction technique, known as Rao-Blackwellization, that marginalizes linear parameters using a Kalman Filter (KF). Secondly, aiming at improving particle efficiency, an approximation of the optimal importance density function that is close to the optimal. This yields to an efficient usage of particles to characterize the posterior probability density function (pdf) that makes feasible the implementation of the algorithm for tracking purposes. Simulation results are shown, comparing the performance of the presented PF with the Extended Kalman Filter (EKF), the Unscented Kalman Filter (UKF), the Posterior Cramér-Rao Bound (PCRB) and the work in [20].

PFs are a set of Sequential Monte-Carlo (SMC) based algorithms used to compute the Bayesian recursion in general state-space models. Although we can find SMC basic ideas back in 1950s [27], it will not be until the end of 90’s when they emerge as an appealing tool to deal with filtering problems. The main reason for this abandon was the low computational burden achievable at that time, so that when technology was mature enough to implement SMC based methods, the scientific community returned to investigate the topic. In recent years, SMC methods have played an important role in many research areas such as signal detection and demodulation, target tracking, Bayesian inference, audio processing, financial modeling, computer vision, robotics, control or biology [28]–[32]. In this paper, we investigate their application to the design of GNSS receivers and the mitigation of the bias produced by multipath.

The paper is organized as follows. Section II outlines the signal model considered for GNSS receivers. Section III provides an introduction to PF and the description of the proposed algorithm, which includes: Rao-Blackwellization of linear states, selection of the importance density function, resampling strategy and a Doppler-shift tracking algorithm that accounts for the slow time-variation of that parameter. Section IV presents the derivation of the PCRB, which is the theoretical lower bound on the estimator variance. Simulation results are discussed in Section V, comparing the proposed PF algorithm to other state-of-the-art Bayesian filtering algorithms, and Section VI concludes the paper.

II. SYSTEM MODEL

Let us consider a detailed signal model that accounts for both the LOSS and the multipath signals. The received complex baseband DS-SS signal of a given satellite, affected by \( M - 1 \) multipath replicas, is modeled as

\[
x(t) = \sum_{m=0}^{M-1} \alpha_m(t) q(t - \tau_m(t)) e^{j\phi_m(t)} + n(t),
\]

where \( \alpha_m(t), \tau_m(t) \) and \( \phi_m(t) \) stand for the amplitude, delay and phase of the \( m \)-th received signal, respectively. These parameters are time-varying processes, which is explicitly expressed with their time dependence. \( n(t) \) is zero-mean additive Gaussian noise (AGN) with variance \( \sigma_n^2 \). In the model, the subscript \( m = 0 \) stands for the LOSS parameters. The contribution of the rest of satellites can be neglected considering that GNSS systems use pseudorandom noise (PRN) codes with a high processing gain (e.g., \( \sim 43 \text{dB} \)) in case of GPS L1 C/A signal. Thus, the influence of other satellites can be considered as Gaussian noise and included in the thermal noise term since those signals are below the noise floor.

Notice that Doppler-shifts are also taken into consideration in the model, being included in the phase parameter:

\[
\phi_m(t) = 2\pi \nu_m(t) T_c + \phi_m(0),
\]

where \( \nu_m(t) \) and \( \phi_m(0) \) are the time-varying Doppler-shift and carrier phase of the \( m \)-th signal, respectively.

The DS-SS signal of the tracked satellite is denoted by \( q(t) \), being composed of the sequence of data symbols \( \{d(t)\} \) and the PRN sequence \( \{c(t)\} \). For the sake of clarity we neglect indices referring each satellite since we consider the tracking of each one separately, in a parallel implementation approach. The rate of the spreading sequence is related to the chip period, \( T_c \). Data symbols are transmitted at a lower bit rate, denoted by \( T_b \). Thus, being \( g(t) \) the chip-shaping pulse, we define:

\[
q(t) = \sum_{l_1=\infty}^{\infty} d(l_1) p(t - l_1 T_b) \quad \text{and}
\]

\[
p(t) = \sum_{l_2=0}^{N_c} \sum_{l_3=1}^{L_c} c(l_3) g(t - l_2 L_c T_c - l_3 T_c),
\]

where \( p(t) \) is the spreading waveform and \( L_c = T_b/T_c \) is the chip length of the PRN sequence. \( N_c \) are the number of code epochs per data bit. Notice that, since the PRN sequence and the chip-shaping pulse are known at the receiver, \( g(t) \) can be considered also known, up to \( 180^\circ \) phase variations due to data-bit changes. However, \( d(t) \) is not likely to vary within the observation time, which is typically much shorter than the bit period. This model is quite general and encompasses both GPS [33] and Galileo [34] signal structures.

Defining:

\[
\alpha(t) = [\alpha_0(t), \ldots, \alpha_{M-1}(t)]^T \in \mathbb{R}^{M \times 1}
\]

\[
\phi(t) = [\phi_0(t), \ldots, \phi_{M-1}(t)]^T \in \mathbb{R}^{M \times 1}
\]

\[
\tau(t) = [\tau_0(t), \ldots, \tau_{M-1}(t)]^T \in \mathbb{R}^{M \times 1}
\]

\[
q(l; \tau(t)) = [q(t - \tau_0(t)), \ldots, q(t - \tau_{M-1}(t))]^T
\]

\[
\Phi(t) = \text{diag}(e^{j\phi(t)}) \in \mathbb{C}^{M \times M},
\]

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\]
we can obtain the compact form of (1) as

$$x(t) = q^T(t; \tau(t)) \Phi(t) \alpha(t) + n(t).$$

(5)

Considering the Software Defined Radio (SDR) philosophy [26], [35]–[39], a GNSS receiver records \( K \) snapshots with a sampling period of \( T_s \) (see Figure 1). Thus, if we use \( k \in \mathbb{N} \) to denote the \( k \)-th record of \( K \) samples, the model (in 5) is extended to

$$x_k = Q_k^T(\tau_k) a_k + n_k,$$

where \( \tau_k \triangleq \tau(k KT_s), \Phi_k \triangleq \Phi(k KT_s), \alpha_k \triangleq \alpha(k KT_s), \) \( a_k \triangleq \Phi_k \alpha_k \) and

$$Q_k(\tau_k) = [q((k - 1) KT_s + T_s; \tau_k), \ldots, q(k KT_s; \tau_k)]$$

(7)

is known as the basis-function matrix and contains \( K \) samples from the delayed narrowband envelopes of each \( M \) signals, i.e., \( Q_k(\tau_k) \in \mathbb{C}^{M \times K} \). The vectors containing the composite signal and the zero-mean AGN are expressed as \( x_k, n_k \in \mathbb{C}^{K \times 1} \), respectively. The covariance matrix of the noise is left arbitrary \( \Sigma_{n,k} \).

Notice that the unknown parameters \( (\alpha_k, \tau_k, \phi_k(\phi_k(k KT_s))) \) are time-varying processes, as explicitly expressed by subscript \( k \). However, we assume that they are piecewise constant during the observation interval of \( K \) samples. In addition, we take into account that amplitudes and phases can be obtained as the modulus and phase of complex amplitudes \( a_k \), respectively. Then, the considered real state vector is \( z_k \triangleq \{\Re[a_k], \Im[a_k], \tau_k\}^T \in \mathbb{R}^{3M \times 1} \). Accordingly, measurement equation (6) is parameterized by the elements of \( z_k \)

$$x_k = Q_k^T(\tau_k)T a_{s,k} + n_k,$$

where

$$a_{s,k} = 
\begin{bmatrix}
\Re\{a_k\} \\
\Im\{a_k\}
\end{bmatrix} \in \mathbb{R}^{2M \times 1}
$$

$$T = \begin{bmatrix} I_M & jM \end{bmatrix} \in \mathbb{C}^{M \times 2M},$$

(9)

(10)

with \( I_M \) being the \( M \)-dimensional identity matrix and \( a_{s,k} \) the vector that stacks real and imaginary parts of complex amplitudes.

Following [40], states time evolution within intervals is modeled by a Markovian prior which is a first-order autoregressive model

$$z_k \sim p(z_k|z_{k-1}) = \mathcal{N}(F_k z_{k-1}, \Sigma_{z,k}),$$

(11)

where we defined \( F_k = \text{diag}\{F_{a,k}, F_{s,k}\}, \Sigma_{z,k} = \text{diag}\{\Sigma_{a,k}, \Sigma_{s,k}\} \) and \( \Sigma_{z,k} = \text{diag}\{\Sigma_{a,k}, \Sigma_{s,k}\} \), using \( F_{a,k}, F_{s,k} \) and \( \Sigma_{a,k} \) the respective transitional matrices of each state, defined as

$$F_{a,k} = \begin{pmatrix}
\mu_{a_0} & \cdots & 0 \\
0 & \ddots & \vdots \\
0 & \cdots & \mu_{a_{M-1}}
\end{pmatrix} \in \mathbb{R}^{M \times M},$$

$$F_{s,k} = \begin{pmatrix}
\mu_{s_0} & \cdots & 0 \\
0 & \ddots & \vdots \\
0 & \cdots & \mu_{s_{M-1}}
\end{pmatrix} \in \mathbb{R}^{M \times M},$$

$$F_{r,k} = \begin{pmatrix}
\mu_{r_0} & \cdots & 0 \\
0 & \ddots & \vdots \\
0 & \cdots & \mu_{r_{M-1}}
\end{pmatrix} \in \mathbb{R}^{M \times M},$$

(12)

with \( \mu_\cdot \) controls the dynamics of the corresponding parameter. \( \Sigma_{a,k}, \Sigma_{s,k} \) and \( \Sigma_{r,k} \) denote the covariance matrices of the evolving states. The above state evolution model has been used in related publications due to its simplicity, see [40]–[43] for example. The values of the transitional matrices depend on the considered scenario and the dynamics of the receiver, and must be tuned accordingly.

III. PARTICLE FILTERING ALGORITHM

The discrete state-space (DSS) approach is adopted to deal with the non-linear Bayesian filtering problem, this is to recursively compute estimates of states \( z_k \) given measurements \( x_k \in \mathbb{C}^{K \times 1} \) at time index \( k \). On the one hand, state equation in (11) models the evolution of target states as a discrete-time stochastic model. On the other hand, equation (8) models the relation between measurements and states, which completes the DSS model representation. The objective is to estimate recursively the posterior pdf of the states given all available measurements at time \( k \), \( x_{1:k} = \{x_1, \ldots, x_k\} \):

$$p(z_{0:k}|x_{1:k}) = \frac{p(x_k|z_k)p(z_k|z_{k-1})}{p(x_k|x_{1:k-1})} p(z_{0:k-1}|x_{1:k-1}) .$$

(13)

In particular, the filtering problem considers the marginal distribution \( p(z_k|x_{1:k}) \), which can be computed recursively. \( z_{0:k} \) is the state trajectory up to time \( k \). However, in general this recursion cannot be solved analytically. There are few cases where the posterior pdf can be characterized by a sufficient statistic, e.g. linear-Gaussian models where the Kalman Filter (KF) yields to the optimal solution [44]. Unfortunately, this is not the case in the problem under study since measurements depend non-linearly on states. When the model is non-linear, the KF must be modified in order to cope with this situation. A common approach to deal with non-linear state-spaces is to resort to the Extended Kalman Filter (EKF) [45] by linearizing the model at some point of interest. Another alternative, based on the KF, is the Unscented Kalman Filter (UKF) [46]–[48] where the model is not linearized and, instead, a Gaussian posterior is considered which is characterized by a set of deterministically chosen sample points. In the simulations presented in Section V, the EKF solution proposed in [40], [49] for a similar problem and a UKF approach are used for comparison purposes.
Particle Filters (PFs) are a set of Sequential Monte-Carlo (SMC) based algorithms used to compute the Bayesian recursion in general state-space models. SMC methods are simulation-based techniques that obtain a characterization of the posterior distribution in a sequential manner [29]–[31]. PFs rely on the Sequential Importance Sampling (SIS) concept to characterize this density. Basically, it involves the approximation of the posterior by a set of \( N_s \) random samples taken from an importance density function, \( \pi(\cdot) \), with associated importance weights \( w^i_k \).

For a set of generated particles at time \( k, \{z_{\theta,k}^i, w^i_k\}_{i=1}^{N_s} \), the characterization of the marginal posterior distribution is given by

\[
p(z_{\theta,k}^i | x_{1:k}) = \sum_{i=1}^{N_s} w^i_k \delta(z_{\theta,k}^i - z_{\theta,k}^i), \tag{14}
\]

with \( \delta(\cdot) \) being the Dirac’s delta function. This approximation converges almost surely to the true posterior as \( N_s \to \infty \) under weak assumptions, according to the Strong Law of Large Numbers [50], [51]. These assumptions hold if the support of the chosen importance density includes the support of the true posterior.

### A. Rao-Blackwellized Particle Filter

Observing the DSS model in (8) and (11), we see that the state-space can be partitioned into two sub-spaces corresponding to its linear and non-linear parts. Whereas the latter requires the use of a PF (or any other filter capable to cope with non-linearities), the former can be dealt analytically using a KF conditional on non-linear states. This marginalization procedure is referred to as Rao-Blackwellization and constitutes a variance reduction technique that aims at improving PF efficiency [52]–[55]. Thus, complex amplitudes, i.e., \( a_{s,k} \), can be considered nuisance parameters optimally handled by a KF. Our focus is then on the characterization of the posterior pdf of non-linear states \( p(\theta_k | x_{1:k}) \).

According to the SIS concept, the \( i \)-th particle in a PF is generated and weighted respectively as

\[
\tau^i_k \sim \pi(\tau_k | \theta_{0:k-1}, x_{1:k}), \tag{15}
\]
\[
\tilde{w}_k^i \propto w_{k-1}^i \frac{p(x_k | \theta_{0:k-1}, x_{1:k}) p(\tau_k | \theta_{0:k-1})}{\pi(\tau_k | \theta_{0:k-1}, x_{1:k})}, \tag{16}
\]

where \( \tilde{w}_k^i \) is the unnormalized importance weight. The selection of a proper importance density function is an important issue, which has to consider both particle efficiency and a feasible implementation. This point is delved into in Section III-B.

The algorithm requires the evaluation of the likelihood function \( p(x_k | \theta_{0:k}, x_{1:k-1}) \) in order to compute the \( i \)-th weight in (16). The distribution can be expressed as

\[
p(x_k | \theta_{0:k}, x_{1:k-1}) = \int_{a_{s,k}} p(x_k | \theta^i_k, a_{s,k}) p(a_{s,k} | \theta_{0:k-1}, x_{1:k-1}) da_{s,k}, \tag{17}
\]

where, from equation (8), we now that

\[
p(x_k | \theta^i_k, a_{s,k}) = \mathcal{N}(Q^i_k(\tau^i_k)T a_{s,k}, \Sigma_{n,k}) \tag{18}
\]

and

\[
p(a_{s,k} | \theta_{0:k-1}, x_{1:k-1}) = \mathcal{N}\left(\hat{a}^i_{s,k|k-1}, \Sigma_{n,k} + Q^i_k(\tau^i_k)P^i_{k|k-1}Q^i_k(\tau^i_k)^T\right) \tag{19}
\]

can be sequentially computed using a KF for each generated particle. Notice that, due to the Rao-Blackwellization procedure, the likelihood distribution is a function of the predictive posterior distribution of linear states [41]. Therefore, the integral in (17) can be solved as [56]

\[
p(x_k | \theta_{0:k}, x_{1:k-1}) = \mathcal{N}\left(Q^i_k(\tau^i_k)T \hat{a}^i_{s,k|k-1}, \Sigma_{n,k} + Q^i_k(\tau^i_k)P^i_{k|k-1}Q^i_k(\tau^i_k)^T\right) \tag{20}
\]

with the equations of the prediction step of the KF being

\[
\hat{a}^i_{s,k|k-1} = F a_{s,k-1} - K_k(\hat{x}_{k} - G_k^i a_{s,k|k-1}) \tag{21}
\]
\[
P_k^i = P_k^i - K_k^i G_k^i P_k^i G_k^i + \Sigma_{n,k} \tag{22}
\]

After generating each particle, the corresponding KF is conditionally updated following:

\[
\hat{a}^i_{s,k|k} = \hat{a}^i_{s,k|k-1} + K_k^i (\hat{x}_k - G_k^i \hat{a}^i_{s,k|k-1}) \tag{23}
\]
\[
P_k^i = P_k^i - K_k^i G_k^i P_k^i G_k^i + \Sigma_{n,k} \tag{24}
\]

where for the sake of clarity we define \( G_k^i \triangleq Q^i_k(\tau^i_k)T \).

### B. Selection of Importance Density: Laplace’s method

One of the key points of a PF algorithm is the choice of a good importance density function, \( \pi(\cdot) \). This is to propose an importance density function close to the optimal, which is the posterior distribution, in the sense that it minimizes the variance of importance weights. However, it is only possible to
draw samples from this distribution in limited cases and other alternatives must be explored [31, 52]. The simplest approach is to consider the prior distribution as the importance function, in which case the weights are proportional to the likelihood function in (20). Nevertheless this option was shown to be inefficient as it requires a large number of samples to effectively characterize the posterior [20], [43], [57]. Many importance function alternatives can be found in the literature, for instance local model linearization [52], Gaussian approximations [58] or use of the unscented transform [48].

In this paper, in order to improve particle efficiency, we aim at using an importance density function which is close to the optimal $p(\tau_k|\tau_{0:k-1}, x_{1:k})$. Since it is not possible to sample from such distribution, we consider the approximation¹:

$$\pi(\tau_k|\tau_{0:k-1}, x_{1:k}) \approx p(\tau_k|\tau_{0:k-1}, x_k) \propto p(x_k|\tau_k)p(\tau_k|\tau_{0:k-1}),$$

(23)

which is proportional to the likelihood and the transition prior. The idea behind is that if both $p(x_k|\tau_k)$ and $p(\tau_k|\tau_{0:k-1})$ are unimodal and normal distributions, then $\pi(\tau_k|\tau_{0:k-1}, x_k)$ is also normal and closed form expressions for its mean and covariance matrix can be readily obtained. Wherein the prior attains these conditions, the likelihood has to be approximated. Then, a unimodal Gaussian characterization of $p(x_k|\tau_k)$ is obtained by a Laplacian approximation [59] (refer to Appendix I for details). This method for selecting the importance density function was proposed in [60] for an acoustic multitarget tracking problem. Laplace’s method yields analytical Gaussian approximations of densities from a Taylor series expansion at the mode of the density, being the inverse Hessian of the log-likelihood used as a covariance approximation [61].

Thus, we aim at obtaining the parameters that characterize the likelihood as a Gaussian distribution: $p(x_k|\tau_k) \approx \mathcal{N}(\tau_k, \Sigma_k)$. These two parameters are the mode $\tau_k$ of the likelihood and the inverse Hessian evaluated at the mode $\Sigma_k$, serves as a covariance matrix estimator. Then, the Gaussian approximation reduces to find the mode of the log-likelihood by maximizing it with respect to $\tau_k$.

Manipulating equation (6), under the Gaussian assumption, it can be shown (see [62]) that the maximization of the log-likelihood is equivalent to minimizing the following non-linear Least Squares cost function with respect to $\tau_k$:

$$\Lambda_k (\tau_k, a_k) = || x_k - Q_k^T (\tau_k) a_k ||^2 .$$

(24)

Deriving with respect to $a_k$ and equating to zero, one is able to find an analytical expression for $a_k$ which is a nuisance parameter in this optimization problem. Then, after substituting the solution for $a_k$, the mode $\hat{\tau}_k$ is found by minimizing

$$\Lambda_k (\hat{\tau}_k) = x_k^H (I - \Pi_k) x_k = || \Pi_k x_k ||^2,$$

(25)

with $\Pi_k (\hat{\tau}_k) \triangleq H_k (H_k^H H_k)^{-1} H_k^H$ being the projection matrix onto the subspace spanned by the columns of $H_k$.

¹Note that equation (20) provides an expression for $p(\tau_k|\tau_{0:k-1}, x_{1:k})$ which is conditional on nonlinear states that cannot be used in the generation of particles since $\tau_k$ is not already known.

$$\mathbf{Q}_k^T (\tau_k) \text{ and } \mathbf{P}_k^T (\tau_k) \text{ its orthogonal complement. A regularization term is introduced in order to constrain the search space to be in the } \Sigma_k \text{-neighborhood of the propagated prior estimate } \mu_r = \mathbf{F}_{r,k} \hat{\tau}_{k-1} \text{ to avoid divergence, mimicking [60]. Thus, the optimization problem }

\tau_k = \arg \min_{\tau_k} \{ \Lambda_k (\tau_k) + r (\mu_r, \Sigma_r) \} = \arg \min_{\tau_k} \{ \Lambda_k^* \}$

(26)

where

$$r (\mu_r, \Sigma_r) = (\tau_k - \mu_r)^T \Sigma^{-1}_r (\tau_k - \mu_r),$$

(27)

can be solved via the Newton-Raphson recursive algorithm⁴:

$$\tau_{k+1} = \tau_k - \lambda^{\ell} \left[ H_{\tau_k}^{-1} (A^T_{2\ell})_0 \right] \nabla_r (A^T_{\ell})_0,$$

(28)

where index $\ell$ denotes iteration and $\lambda^{\ell}$ is the step-size, implemented with backtracking. The expressions for the Gradient and the Hessian of $\Lambda_k^*$ are given in Appendix II.

Given the previous likelihood approximation, we now incorporate the information of each propagated particle to form the Gaussian importance density as follows:

$$\pi(\tau_k|\tau_{0:k-1}, x_k) = \mathcal{N} (\mu_k^\ell, \Sigma_k^\ell),$$

(29)

where

$$\mu_k^\ell = \Sigma_k^\ell \left( \Sigma_k^{-1} \hat{\tau}_k + \Sigma_r^{-1} \mathbf{F}_{r,k} \hat{\tau}_{k-1} \right)$$

$$\Sigma_k^\ell = \left( \Sigma_k^{-1} + \Sigma_r^{-1} \mathbf{F}_{r,k} \right)^{-1},$$

(30)

as shown in Appendix I. After approximating the optimal importance function, particles can be generated and weighted according to (15) and (16). Recall that conditional linear states are taken into account when computing the likelihood in (20). Then, we can easily obtain the MMSE estimate of delays and complex amplitudes at instant $k$

$$\hat{\tau}_k = \sum_{i=1}^{N_s} w^i_k \tau^i_k,$$

$$\hat{a}_{s,k|k} = \sum_{i=1}^{N_s} w^i_k \hat{a}^i_{s,k|k} .$$

(31)

C. Resampling

Particle filtering algorithms suffer from the so-called degeneracy phenomenon, which states that variance of importance weights can only increase over time [29], [31], [63]. In other words, after a certain number of recursive steps, all but one particle will have negligible normalized weights. A measure of the degeneracy is the effective sample size $N_{eff}$, which is estimated as $\hat{N}_{eff} = \left( \sum_{i=1}^{N} (w^i)^2 \right)^{-1}$ where small values of $\hat{N}_{eff}$ indicates severe degeneracy.

In order to combat this effect, resampling step is introduced. Basically, it consists in discarding samples with low importance weights and replicate samples with high importance weights. In the simulations, we apply a resampling with

⁴In the sequel, we denote the Gradient and the Hessian of a function $f$ with respect to an arbitrary vector $a$ as $\nabla_a f$ and $\mathcal{H}_a f$, respectively.
replacement of \(N_s\) particles according to their importance weights. Particles are resampled when \(N_{e_{eff}}\) falls below a threshold, which in our case has been set to \(N_{e_{eff}} < \frac{2}{3}N_s\). Resampling constitutes the bottle neck in any parallel implementation of PFs [64], since all particles must be combined in this process. Some schemes that alleviate these constraints can be found in [65].

### D. Tracking Doppler-shifts

According to equation (2), the linear part of the phase is due to the Doppler deviation, while the other term corresponds to the carrier phase. Hence, the Doppler-shift of the \(m\)-th signal can be tracked from the estimated complex amplitude, with \(m \in \{0, \ldots, M-1\}\). Since the variation of this parameter is typically slow, compared to the sampling frequency and the estimation rate (1 estimate per \(K\) samples), a simple linear regression can be used to extract \(v_m\) from a set of phase estimates. Thus, one has to consider an observation window for \(\hat{a}_k\) such that the Doppler deviation has not changed significantly. We denote the length of this window by \(L\) estimates or, equivalently, \(LKT_s\) seconds. Hence, Doppler estimates at instant \(k\) are obtained as the solution to a linear Least Squares problem

\[
\hat{\beta}_k = (\Gamma_{k-L:k}^T \Gamma_{k-L:k})^{-1} \Gamma_{k-L:k} \hat{\phi}_{m_{k-L:k}} \tag{32}
\]

where

\[
\hat{\beta}_k = \begin{bmatrix} \hat{v}_{m_k}, \hat{\phi}_{m,0} \\ 2 \pi (k-L) KT_s & 1 \\ \vdots & \vdots \\ 2 \pi k KT_s & 1 \end{bmatrix}
\]

\[
\hat{\phi}_{m_{k-L:k}} = \begin{bmatrix} \hat{\phi}_{m_{k-L}}, \ldots, \hat{\phi}_{m_k} \end{bmatrix}^T
\]

with the latter being the set of phase estimates obtained as the angle of the complex amplitude estimate given by (31), i.e.,

\[
\hat{\phi}_{m_k} = \angle \hat{a}_{k|k} \tag{34}
\]

The tracking of the Doppler-shift for the LOSS can be done whether considering overlapped windows or not. The choice might depend on the variability of that parameter, and thus on the receiver dynamics. If tracking is performed at each \(k\), the computational cost increases so that one has to design the algorithm depending on the scenario requirements. In that sense, the design of \(L\) depends on a number of factors and must ensure that the Doppler deviation remains piecewise constant during the window. On the one hand, it depends on the variation rate of the Doppler for a given scenario. On the other hand, the designed value of \(L\) depends on parameters of the receiver such as the sampling period \(T_s\) and the number of samples used to estimate delays and complex amplitudes \((K)\). A longer window length will provide smoother Doppler estimates, but the designed \(L\) has to ensure that the parameter remains piecewise constant within the interval.

### Algorithm 1 Rao-Blackwellized Laplace’s Particle Filter tracking algorithm

**Require:** \(x_{1:k}, \phi_{0:k}, p(\phi_0)\)

**Ensure:** \(\bar{\alpha}_{0:k}, \phi_{0:k}, \nu_{0:k}, \tau_{0:k}\)

**Initialization:**

1: \(\hat{z}_0 \sim p(\phi_0)\)
2: \(\sigma_{\tau_0}^2 = \Sigma_r (m, m), m = 0, \ldots, M-1\)
3: for \(i = 1\) to \(N_s\) do
   4:   Generate \(\tau_{0i}^0 \sim N(\hat{\tau}_0, \sigma_{\tau_0}^2)\)
5:   Generate \(\tau_{0i}^1 \sim N(\tau_{0i}^0) + U(0, \sigma_{\tau_0})\)
6:   \(w_i = 1/N_s\)
7:   \(\Gamma_{0:0} = \text{diag}\{\Sigma_{\alpha,0}, \Sigma_{\phi,0}\}\)
8: end for

**Tracking:**

9: for \(k = 1\) to \(\infty\) do
10: Run Newton-Raphson algorithm in (28) \(\Rightarrow \tau_{k}, \Sigma_k\)
11: Calculate \(\Sigma_k\) according to (30)
12: for \(i = 1\) to \(N_s\) do
13:   \(i\)-th Kalman prediction according to (21)
14:   Calculate \(\mu_i^1\) according to (30)
15:   Generate \(\tau_{1i}^k \sim N(\mu_i^1, \Sigma_{i})\)
16:   Calculate \(\omega_i^k = \frac{w_k^{-1}}{N} \Sigma_i \left( \begin{array}{c} \tau_{1i}^k - \mu_i^1 \\ \Sigma_{i} \end{array} \right)^{-1} \)
17:   \(i\)-th Kalman update according to (22)
18: end for
19: Normalize weights, \(w_i^k\).
20: Compute MMSE estimates.
21: Track Doppler-shift when required, according to (32).
22: if \(N_{e_{eff}} < \frac{2}{3}N_s\) then
23: Resample Particles, \(\{\tau_k, w_k\}_{i=1}^{N_s}\)
24: end if
25: end for

### IV. Posterior Cramer-Rao Bound

Let \(x_k \in \mathbb{C}^{n_x}\) be a vector of measured data, \(z_k \in \mathbb{R}^{n_z}\) an unknown random parameter and \(\hat{z}_k(x_{1:k})\) an estimator of \(z_k\) considering available data at time instant \(k\), \(x_{1:k} = \{x_1, \ldots, x_k\}\). For the filtering problem, the minimum theoretical achievable error variance is given by the Posterior Cramér-Rao Bound (PCRB) [31], [66], [67]. The PCRB states that the covariance matrix of the estimation error \((C_k)\) is bounded by the inverse of the Bayesian Information Matrix, \(J_k \in \mathbb{R}^{n_x \times n_z}\), i.e.,

\[
C_k = \text{Var}(\hat{z}_k(x_{1:k}) - z_k(x_{1:k}) - z_k^T) \geq J_k^{-1}
\]

where the expectation is with respect to both measurements and states. The inequality in (35) means that the difference \(C_k - J_k^{-1}\) is a positive semidefinite matrix and, if the equality holds, the estimator is said to be statistically efficient. Defining \(z_{0:k} = \{z_0, \ldots, z_k\}\) as the entire trajectory of state-vectors, which can be decomposed as \(z_{0:k} = [z_{0:k-1}, z_k^T]\), it arises that the elements of \(J(z_{0:k})\) are given by the lower-right corner of the Trajectory Information Matrix, \(J(z_{0:k}) \in\)
\(\mathbb{R}^{(k+1)n_z \times (k+1)n_z}\), defined as

\[
J(z_{0:k}) = \mathbb{E}_{x,z}\left\{-\Delta z_{0:k} \ln p(x_{1:k}, z_{0:k})\right\}.
\] (36)

A result due to [68] shows that \(J_k\) can be obtained using the following recursion

\[
J_{k+1} = D_{k}^{22} - D_{k}^{21} (J_k + D_{k}^{11})^{-1} D_{k}^{12},
\] (37)

where

\[
D_{k}^{11} = \mathbb{E}_{z_{k}, z_{k+1}}\left\{-\Delta z_{k} \ln p(z_{k+1}|z_k)\right\},
\]

\[
D_{k}^{12} = \mathbb{E}_{z_{k}, z_{k+1}}\left\{-\Delta z_{k} \ln p(z_{k+1}|z_k)\right\},
\]

\[
D_{k}^{21} = \mathbb{E}_{z_{k}, z_{k+1}}\left\{-\Delta z_{k} \ln p(z_{k+1}|z_k)\right\} = [D_{k}^{22}]^T,
\]

\[
D_{k}^{22} + \mathbb{E}_{z_{k+1}, z_{k+1}}\left\{-\Delta z_{k} \ln p(z_{k+1}|z_k)\right\}.
\] (38)

and the initialization is done considering the prior density of the states, \(p(z_0)\),

\[
J_0 = \mathbb{E}_{z_0}\left\{-\Delta z_0 \ln p(z_0)\right\}.
\] (39)

The recursion in (37) is extremely useful in many cases where the computation of the PCRB is mathematically untractable. In addition, matrices involved in the recursive formula are \(z_t \times z_t\), in contrast to the problem in equation (36) which has a dimension that increases with \(k\).

Some simplifications apply to equations (38) and (39) when assuming the state-space model considered in this paper, i.e., linear states evolution and non-linear measurements with AGN. As presented in (11), states are drawn from a Gaussian pdf and \(p(z_0) = \mathcal{N}(z_0, \Sigma_{0,z})\). Thus, after mathematical manipulation of equation (39), we obtain that \(J_0 = \Sigma_{0,z}^{-1}\). Following the result in [31], [68], matrices in (38) simplify to:

\[
D_{k}^{11} = F_k^T \Sigma_{z,k}^{-1} F_k,
\]

\[
D_{k}^{12} = -F_k^T \Sigma_{z,k}^{-1},
\]

\[
D_{k}^{22} = \Sigma_{z,k}^{-1} + \mathbb{E}_{z_{k+1}}\left\{H_{k+1}^T \Sigma_{k+1,z}^{-1} H_{k+1}\right\},
\] (40)

where we define the Jacobian of \(h_k(z_k)\) evaluated at the true value of \(z_k\), as \(H_k\).

Notice that, after simplifications due to the model at hand, matrices \(D_{k}^{11}\), \(D_{k}^{12}\) and \(D_{k}^{21}\) are deterministic and can be easily obtained. However, due to the non-linearity in the measurement model, the expectation operator in the computation of \(D_{k}^{22}\) cannot be dropped out. In order to compute this expectation, a Monte Carlo approximation can be applied by creating a number of state-vector realizations, calculating the corresponding PCRBs and averaging them to obtain the theoretical PCRB of the system under study [68].

\[5\]where \(\Delta f = \nabla_a \nabla_a (f)^T\) stands for the second-order partial derivatives operator of a function \(f\). Notice that \(\mathcal{H}_a(f) \triangleq \Delta a f\), according to the nomenclature used in the paper.

V. SIMULATION RESULTS

In order to assess the performance of the proposed tracking algorithm, and without loss of generality, we simulate a GPS L1 C/A signal (though the algorithm can be readily used with other GNSS signal structures) assuming an scenario composed of a LOSS and a multipath replica \((M = 2)\). In GNSS, the otherwise rather simple model of one single reflection is quite representative of the multipath effect in many situations from a statistical point of view, even in the urban canyon environment (see [69], [70] for an experimental investigation, including a high resolution measurement campaign, of the land mobile satellite navigation multipath channel).

In the following experiments, a carrier-to-noise density ratio of 45 dB-Hz for the LOSS, a signal-to-multipath ratio of 6 dB and the LOSS and multipath to be in-phase, the worst possible case [5], are considered. The received signal is filtered with a 2 MHz bandwidth filter, down-converted to baseband and then digitized at a sampling frequency of \(f_s = 5.714\) MHz. Thus, 1 ms of recorded data corresponds to \(K = 5714\) samples. The chip period is \(T_c = 1/(1.023\) MHz) in GPS C/A code. The covariance matrices involved in the simulations are \(\Sigma_{0,z}^2 = \Sigma_z^2 = [\delta(\tau_c,2T_c)]\), the receiver dynamics correspond to a vehicle with a constant velocity of 22 m/s. The multipath relative delay is \(\tau_1 = 0.35T_c\) with respect to the time-varying delay of the LOSS, \(\tau_0(t)\). The scenario is simulated with a realistic GNSS signal generator programmed in MATLAB, thus all states evolve according to vehicle and satellite dynamics. The evolution model considers that \(F_k = I_{3M}\), i.e., the algorithm considers a small variation between consecutive instants.

Figure 2 shows the RMSE performance [71] (averaged over 100 Monte-Carlo simulations) of the proposed Laplace’s PF when considering different numbers of particles. The results are compared to the PCRB, which are obtained by Monte Carlo simulation as explained in Section IV. The left-hand axis shows the RMSE of time-delay estimates normalized to \(T_c\) and the right axis corresponds to the resulting pseudorange estimation error in meters, which is propagated in the computation of user’s position [1]. It arises from simulation results that Laplace’s PF approaches the PCRB as \(N\) increases. In addition, Figure 3 presents the results obtained with the EKF and the UKF for the same scenario. It is seen from both figures that all algorithms yield similar performances for that scenario. However, it is well known that KF-based approaches are highly dependant on a proper choice of covariance matrices, whereas particle filtering is more robust to such modeling errors.

Let us assume now a different scenario in which the multipath replica is present for a given period of time, then suddenly switches off (modeling occultation) and finally it re-appears again. The total observation interval is 1 second and the interval in which no multipath is present in the signal is \(t \in (300, 500)\) milliseconds. Figures 4 and 5 present the results obtained after 100 Monte Carlo runs in that scenario for Laplace’s PF and KF-based solutions, respectively. The most evident conclusion from these results is that Laplace’s PF is by far more robust than EKF or UKF to mismodeling effects, i.e., to assume that there is a multipath replica when actually
there is not. In fact, the performance of the particle filtering algorithm is hardly altered, whereas KF-based approaches suffer a severe degradation due to the bias produced by the inaccurate model, which could even cause the algorithm to diverge. This is related to the initialization of the covariance matrices in the KF: large entries in the covariance matrices aid the algorithm to converge when states evolution suffer abrupt changes, though the error due to the absence of multipath is high. On the contrary, small covariance matrices, reduce the bias effect in mismodeling situations, but the algorithm is very likely to diverge. This is shown in Figure 5.

In real world scenarios, multipath signals will appear/disappear arbitrarily depending on a number of phenomena such as occultation or vehicle motion and inclination. Simulations shown in 4 and 5 are meaningful since they highlight the robustness of the presented PF to this effect. Whereas EKF and UKF would require a multipath detection and/or number of replicas estimation for proper operation, the presented algorithm could use a fixed number of replicas to operate (as said \( M = 2 \) is quite representative in many scenarios). However, we agree in that having knowledge of the exact number of replicas would improve the efficiency of the algorithm at the expenses of increasing the computational cost. Since the model assumes that the LOSS is always present, the algorithm fails when attempting to track it when the signal is actually blocked or not present. This drawback is shared by the other tested algorithms and could be overcome taking into account the effect in the signal model, also with a switching Markov model, and modifying the algorithm accordingly.

Figure 6 compares the RMSE performance of the Sampling Importance Resampling (SIR) algorithm proposed by the authors in [20] and the Laplace’s PF proposed in this work in terms of RMSE versus the number of particles. Actually, the former has been used as a batch processing algorithm more likely to fall in the Population Monte-Carlo (PMC) category [72]. That resulted in an inefficient usage of particles, since a large number of samples are required to increase the performance significantly (which has already been confirmed in [43]). Figure 6 highlights the efficiency of Laplace’s PF in terms of \( N_s \), when compared to SIR PF. The efficiency increase comes from the choice of an importance density function which is closer to the optimal. In contrast, using the transitional prior yields to a solution that does not take into account observations to generate particles. Thus, the posterior characterization is poorer and it requires a larger number of particles to attain similar performances as Laplace’s PF does.

VI. CONCLUSIONS

This paper proposed a Particle Filter (PF) algorithm that mitigates the multipath effect on the estimation of GNSS signal parameters, one of the dominant sources of error in high-precision applications. Previous solutions based on batch processing and the Sequential Importance Resampling (SIR) concept required a large number of particles in order to obtain performances close to the Posterior Cramér-Rao Bound. This work addressed the efficiency of such approach in two ways. On the one hand, the selection of the importance density function was performed by first using a Laplacian approximation of the likelihood distribution. This resulted in an importance density choice that is close to the optimal, the posterior distribution, which yielded to an efficient PF algorithm in terms of number of particles. As a consequence, Laplace’s PF effectively tracked time-varying parameters which was not feasible at all with SIR PF. On the other hand, the presented algorithm considered a variance reduction technique, known as Rao-Blackwellization, that estimated the linear/Gaussian part of the state-space using a Kalman Filter, i.e., complex amplitudes. Computer simulation results showed that, in the presence of coherent multipath, the performance of the proposed PF improves the results of a SIR PF and that it got closer to the PCRB as the number of samples increased.

Furthermore, it has been observed that a solution based on the KF is very sensible to abrupt changes in the scenario, i.e., appearance/disappearance of multipath replicas. In contrast, Laplace’s PF was robust to this situation being its performance
not altered. A further improvement of the presented algorithm can be envisaged by including a switching Markov model, as proposed in [19]. Indeed, the latter could model better the appearance and disappearance of multipath signals. Improvements can be achieved at the expenses of a higher complexity.

To sum up, Laplace’s PF provides an efficient algorithm to track evolving GNSS signal parameters in the presence of on/off multipath replicas that requires weaker model assumptions than KF-based solutions.

**APPENDIX I**

**SELECTION OF IMPORTANCE DENSITY USING LAPLACE’S APPROXIMATION**

The idea behind the choice of \( \pi(\tau_k | \tau_{k-1}, x_k) \propto p(x_k | \tau_k) p(\tau_k | \tau_{k-1}) \) in equation (23) is that if both likelihood and prior distributions are unimodal and Gaussian, the resulting importance density is also unimodal and Gaussian. Furthermore, analytical expressions for its mean and covariance can be derived from those of \( p(x_k | \tau_k) \) and \( p(\tau_k | \tau_{k-1}) \). In our application the prior pdf is already Gaussian and unimodal, but the likelihood has to be approximated. The latter is done using Laplace’s method, which yields analytical Gaussian approximations of densities from a Taylor series expansion at the mode of the density (\( \tau_\alpha \)). Considering second order terms, the log-likelihood can be approximated as

\[
\mathcal{L}(\tau_k) \triangleq \ln p(x_k | \tau_k) \\
\approx \mathcal{L}(\tau_\alpha) + \frac{1}{2} \nabla x \mathcal{L}|_{\tau_\alpha} (\tau_k - \tau_\alpha)^T \Sigma_k^{-1} (\tau_k - \tau_\alpha),
\]

where we have recognized that \( \nabla\tau \mathcal{L}|_{\tau_\alpha} = 0 \). Identifying terms in (41), the following Gaussian approximation of the likelihood function is obtained

\[
p(x_k | \tau_k) \approx \mathcal{N} \left( \tau_k, \Sigma_k \right),
\]

where the mode \( \tau_\alpha \) and the inverse Hessian evaluated at the mode \( \Sigma_k = \mathcal{H}^{-1}_k(\mathcal{L})|_{\tau_\alpha} \) are its mean and covariance.

Hence, the importance density is Gaussian and can be obtained as a product of Gaussians:

\[
\pi(\tau_k | \tau_{k-1}, x_k) \propto p(x_k | \tau_k) p(\tau_k | \tau_{k-1})
\]

\[
= \mathcal{N} \left( \tau_k, \Sigma_k \right) \mathcal{N} \left( \tau_k - F_{\tau, k} \tau_{k-1}, \Sigma_{\tau, k} \right),
\]

where \( \pi(\tau_k | \tau_{k-1}, x_k) \) can also be expressed as \( \mathcal{N}(\mu_\mu, \Sigma_\mu) \), being the problem to analytically obtain \( \mu_\mu \) and \( \Sigma_\mu \).

Taking logarithms and neglecting additive constants, we can express (43) as

\[
\ln \mathcal{N}(\mu_\mu, \Sigma_\mu) \propto \frac{1}{2} (\tau_k - \mu_\mu)^T \Sigma_{\mu, k}^{-1} (\tau_k - \mu_\mu)
\]

\[
+ \frac{1}{2} (\tau_k - F_{\tau, k} \tau_{k-1})^T \Sigma_{\tau, k}^{-1} (\tau_k - F_{\tau, k} \tau_{k-1}),
\]
which can be reorganized, neglecting the multiplicative constant, as
\[
\ln \mathcal{N}(\mu, \Sigma) \propto \tau_k^T \left( \Sigma_k^{-1} + \Sigma_r^{-1} \right) \tau_k - \tau_k^T \left( \Sigma_k^{-1} \tau_k + \Sigma_r^{-1} F_{r,k} \tau_{k-1} \right) - \tau_k^T \left( \Sigma_k^{-1} \tau_k + \Sigma_r^{-1} F_{r,k} \tau_{k-1} \right)^T \tau_k
\]
\[
+ \left( \tau_k^T \Sigma_k^{-1} \tau_k + \tau_k^T F_{r,k} \Sigma_r^{-1} F_{r,k} \tau_{k-1} \right),
\]
yielding to
\[
\tau_k^T \Sigma_k^{-1} \tau_k = \tau_k^T \left( \Sigma_k^{-1} + \Sigma_r^{-1} \right) \tau_k
\]
\[
\tau_k^T \Sigma_k^{-1} \mu_x = \tau_k^T \left( \Sigma_k^{-1} \tau_k + \Sigma_r^{-1} F_{r,k} \tau_{k-1} \right)
\]
\[
\mu_x^T \Sigma_k^{-1} \tau_k = \left( \Sigma_k^{-1} \tau_k + \Sigma_r^{-1} F_{r,k} \tau_{k-1} \right)^T \tau_k
\]
\[
\mu_x^T \Sigma_k^{-1} \mu_x = \left( \tau_k^T \Sigma_k^{-1} \tau_k + \tau_k^T F_{r,k} \Sigma_r^{-1} F_{r,k} \tau_{k-1} \right).
\]
From the latter, it is straightforward (but straightforward) to obtain the mean and covariance of the Gaussian importance density:
\[
\mu_x = \Sigma_x \left( \Sigma_k^{-1} \tau_k + \Sigma_r^{-1} F_{r,k} \tau_{k-1} \right)
\]
\[
\Sigma_x = \left( \Sigma_k^{-1} + \Sigma_r^{-1} \right)^{-1},
\]
under the assumption that \( \tau_k \approx F_{r,k} \tau_{k-1} \), s.t. \( \mu_x^T \Sigma_k^{-1} \mu_x \) in (46) holds. This is reasonable since the evolution of the parameter is supposed to be small compared to the observation interval of \( K \) samples.

**APPENDIX II**

**GRADIENT AND HESSIAN OF COST FUNCTION \( \Lambda_k^r (\tau_k) \)**

The expressions for the Gradient and the Hessian of \( \Lambda_k^r (\tau_k) \) can be obtained as
\[
\nabla_r (\Lambda_k^r) = -x_k^H \nabla_r (\Pi_k (\tau_k)) x_k + \Sigma_r^{-1} (\tau_k - \mu_x)
\]
\[
\mathcal{H}_r (\Lambda_k^r) = -x_k^H \mathcal{H}_r (\Pi_k (\tau_k)) x_k + \Sigma_r^{-1},
\]
respectively.

From the definition of the Projection matrix \( \Pi_k \) it arises, after lengthy but straightforward mathematical manipulation, that its Gradient and Hessian are
\[
\nabla_r (\Pi_k) = \Pi_k^H \nabla_r (H_k) H_k^H + \left( \Pi_k^H \nabla_r (H_k) H_k^H \right)^H
\]
\[
\mathcal{H}_r (\Pi_k) = \nabla_r (\Pi_k^H) \nabla_r (H_k) H_k^H + \Pi_k^H \mathcal{H}_r (H_k) H_k^H
+ \Pi_k^H \nabla_r (H_k) \nabla_r (H_k) H_k^H + \left( \Pi_k^H \nabla_r (H_k) \nabla_r (H_k) H_k^H \right)^H,
\]
where \( \Pi_k^H \triangleq I - \Pi_k \) and thus \( \nabla_r (\Pi_k^H) = -\nabla_r (\Pi_k) \). A detailed derivation of the expressions in (49) can be found in [26, pages 96-98], where a similar problem was addressed.

For the sake of clarity, we introduce the Moore-Penrose pseudoinverse in the notation
\[
H_k^H \triangleq (H_k H_k^H)^{-1} H_k^H,
\]
whose Gradient can be obtained as
\[
\nabla_r (H_k^H) = (H_k^H H_k)^{-1} \nabla_r (H_k^H) H_k^H - H_k^H \nabla_r (H_k) H_k^H.
\]

Thus, after obtaining \( \nabla_r (H_k^H) \) and \( \mathcal{H}_r (H_k^H) \), the expressions in equation (49) can be evaluated, and one is able to compute \( \nabla_r (\Lambda_k^r) \) and \( \mathcal{H}_r (\Lambda_k^r) \) in (48).

**APPENDIX III**

**GRADIENT AND HESSIAN OF MATRIX \( H_k \)**

First, we note that \( \nabla_r (H_k) \) can be decomposed as the derivative with respect to each element of \( \tau \)
\[
\nabla_r (H_k) = \left[ \nabla_r (H_k), \ldots, \nabla_r (H_k) \right]^T
\]
and recall that \( H_k \triangleq Q_k^T (\tau_k) \), where \( Q_k \) has been defined as the basis function matrix in equation (8).

Then, the Gradient of \( H_k \) with respect to the \( i \)-th element of \( \tau \) is
\[
\nabla_r (q_i(k'; \tau_i,k)) = -\dot{q}_i(k'; \tau_i,k),
\]
where \( \dot{q}_i(k'; \tau_i,k) \) stands for the derivative of time of waveform \( q_i(t; \tau_i,k) \) evaluated at instant \( k' \).

The Hessian of \( H_k \) is defined as
\[
\mathcal{H}_r (H_k) = \left( \begin{array}{c} \Delta_r^0 (H_k) \ \cdots \ \Delta_r^0 (H_k) \\ \vdots \ \ddots \ \vdots \\ \Delta_r^{M-1} (H_k) \ \cdots \ \Delta_r^{M-1} (H_k) \end{array} \right),
\]
where
\[
\Delta_r^i (H_k) = 0, \forall i \neq j
\]
and
\[
\Delta_r^i (H_k) = \left( \begin{array}{c} 0 \ \cdots \ 0 \\ \Delta_r^i (q_i((k-1)K + 1; \tau_i,k)) \ \cdots \ \Delta_r^i (q_i((k-1)K + 1; \tau_i,k)) \\ \vdots \ \ddots \ \vdots \\ 0 \ \cdots \ 0 \end{array} \right),
\]
\[
\]
with \( \dot{q}_i(k'; \tau_i,k) \) standing for the second derivative of time of waveform \( q_i(t; \tau_i,k) \) evaluated at instant \( k' \).

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