Emergence of Clusters in Growing Networks with Aging

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Abstract. We study numerically a model of nonequilibrium networks where nodes and links are added at each time step with aging of nodes and connectivity- and age-dependent attachment of links. By varying the effects of age in the attachment probability we find, with numerical simulations and scaling arguments, that a giant cluster emerges at a first-order critical point and that the problem is in the universality class of one dimensional percolation. This transition is followed by a change in the giant cluster's topology from tree-like to quasi-linear, as inferred from measurements of the average shortest-path length, which scales logarithmically with system size in one phase and linearly in the other.

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1. INTRODUCTION

The understanding of natural, technological and social phenomena through the network perspective has motivated a large body of research in the past 10 years [1, 2, 3], as many of their properties can be inferred from the simple idea that relations among elements in such systems can be interpreted, at a certain level of abstraction, as nodes and links of a complex network. Examples range from physically interacting proteins in the cell [4, 5] and the set of routers comprising the Internet [6] to social networks [7, 8] and, according to empirical measurements of a few network properties, like degree distribution and correlations, clustering and average shortest path, it is observed that many of these networks are strikingly similar (in the statistical sense) [2]. Thus, generic features of these networks can be predicted by the analysis of simple models. Since the classic work of Erdös and Rényi [9, 10], a whole set of interesting results in percolation theory have been brought up revealing the importance of network structure in determining critical properties [11, 12, 13, 14, 15, 16] . In particular, it has been found that in nonequilibrium (growing) networks with exponential [11] or power-law degree distribution [12] an infinite order critical point separates a phase with many finite clusters from another where a single macroscopic connected cluster emerges. It has been shown both numerically [11] and analytically [12] that all the derivatives of the average size of the largest cluster, taken as the order parameter, are zero at the critical point. Here we study a similar problem, where networks are grown by the addition of nodes and links that attach preferentially, but with an age-dependent probability. With computer simulations and scaling arguments we show that one has a first-order transition in the size of the giant connected cluster as one makes it less likely to attach links to older nodes. Following the transition the topology of the largest cluster changes from tree-like to one-dimensional, as one can infer from measurements of the average shortest-path length inside the largest cluster in each phase.

2. THE MODEL

Let a network grow from an initial cluster of m_0 fully-connected nodes (we use $m_0 = 2$) by the addition of a node and a link at each time step. This new link will randomly join a pair of nodes (i, j) with probability $\Pi(k_i, k_j, a_i, a_j, t) = \Pi(k_i, a_i, t)\Pi(k_j, a_j, t)$, where $\Pi(k, a, t) = C(A_0 + k)e^{-\alpha(t-a)}$ is the probability that a node with k links and added at time $a \leq t$ to the network receives a link ‡ and $C(t) = \sum_i \Pi(k_i, a_i, t)$. Self-links and multiple links between the same pair of nodes are forbidden. The parameter α plays the role of the inverse of a timescale β, which suppresses the attractiveness of older nodes for new connections. The case $\alpha = 0$ ($\beta \to \infty$) has been extensively studied in [12], who reported an infinite-order phase transition as the number b of links added per time step is varied around a critical value b_c. Here we analyze the effect of varying the

‡ an initial attractiveness [17] $A_0 > 0$ is necessary for the newly added node, which has $k = 0$, to participate in the dynamics. We set $A_0 = 1$ in the following.

memory parameter α and find another nontrivial phase transition at a critical value α_c separating a phase with multiple clusters from another where a single, $\mathcal{O}(N)$, giant component emerges.

For each value of α we grow 10^3 networks with up to $N = 15000$ nodes. Since we do not force the newly added node to immediately attach to a preexisting cluster, there is always a possibility to create networks with isolated clusters. For a given value of α, we identify these clusters in each network with breadth-first-search and calculate the average size of the largest cluster $\langle S(\alpha)\rangle$ in each network, normalized by the total number of nodes N (Fig. 1).

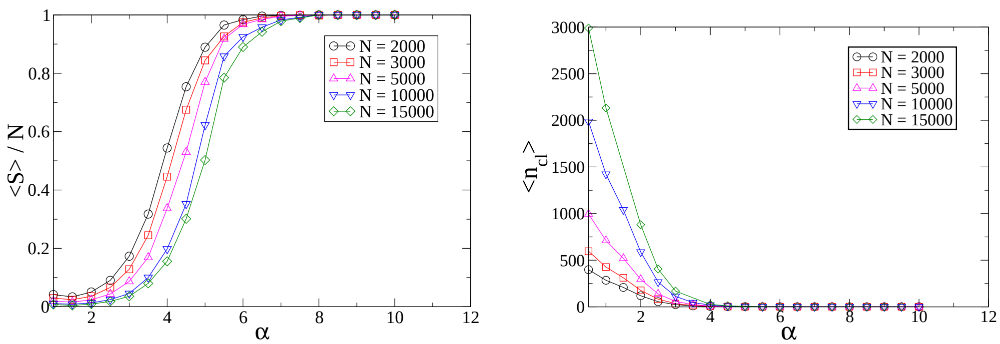

Figure 1. The relative size of the largest cluster (left) and the average number of clusters (right) as a function of α. We observe a phase transition at a given value α_c where one giant cluster emerges. Results are averaged over 10^3 samples and clusters are identified with breadth-first-search.

For small values of α the network is fragmented in many small clusters, whereas after a (size-dependent) value $\alpha_c(N)$ a giant component most likely exists. In order to determine the value of $\alpha_c(N)$ we analyze the fluctuations of the order parameter $\chi(\alpha, N) = \langle S^2\rangle - \langle S\rangle^2$ as a function of α and identify the position of the maximum of χ with $\alpha_c(N)$ (Fig. 2 on the left). Finite-size scaling analysis of usual (first or second order) phase transitions suggests a power-law scaling for the critical point shift

$$\alpha_c(N) = \alpha_c(\infty) + A\ N^{-1/\nu}, \tag{1}$$

where A is a constant and $\nu = 1/d$ at a first-order (discontinuous) phase-transition [19]. Nevertheless, we find that for this model the position of the maxima of the susceptibility-like parameter $\chi(\alpha_c(N), N)$ scale as

$$\ln \alpha_c(N) = \ln \alpha_c(\infty) - K\ N^{-1}, \tag{2}$$

where $\ln \alpha_c(\infty) = 1.717(1)$ and $K = 516.1(7)$. In the thermodynamic limit ($N \to \infty$) we obtain $\alpha_c = 5.568(2)$ (We refer to $\alpha_c(\infty)$ as α_c in the following).

As another unusual feature of this phase transition, we find that the relative size of the largest cluster $\langle S\rangle/N$ is a function of the ratio $\alpha/\alpha_c(N)$, as can be seen in Fig. 3. This might result from the fact that, close to α_c, the dimensionless scaling variable for the average size of the largest cluster $\langle S\rangle$ is $x = \log(\alpha)$, as appears from the scaling

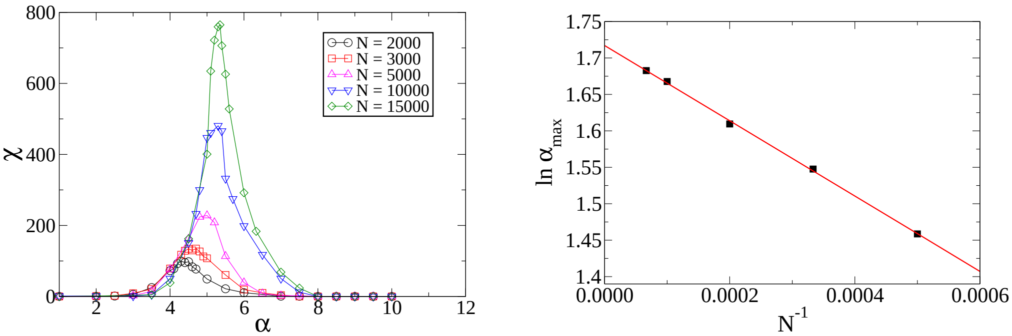

Figure 2. Fluctuation of the order parameter for different network sizes as a function of α (left) and the position of their maxima $\alpha_c(N)$ versus the inverse of network size N^{-1} (right). The straight line corresponds to the best fit ln $y = 1.7171 - 516.17\,x$.

of the rounding of the phase transition [Eq. (2)], so a function of distance in variables x translates into a function of the ratio in variables α. It is noteworthy to mention that the same occurs in $1d$ percolation [18], where the characteristic length $\xi(p)$ scales as $\log(p/p_c)^{-1}$ which can be expanded as $\xi(p) \approx |p - p_c|^{-1}$ when p is close enough to $p_c = 1$.

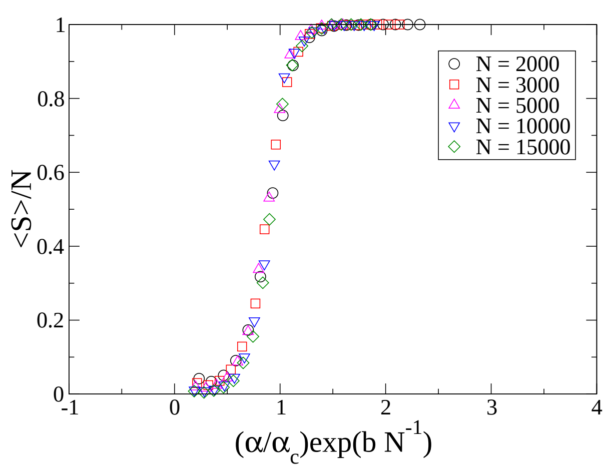

Figure 3. Scaling plot of the order parameter $< S > /N$ as a function of $\frac{\alpha}{\alpha_c(N)}$, where $\alpha_c(N) = \alpha_c e^{-bN^{-1}}$. The values of the parameters are $\alpha_c = 5.5682$ and $b = 516.17$.

The analogy with percolation on a ring extends further when we look at the cluster size distribution (Fig. 4). In the limit $\alpha \to 0$, depending on the number b of bonds added per time step, there is a critical phase without a giant connected component or a normal phase with a giant cluster (as predicted by Mendes et al. in [12]). Since $b = 1$ in our case, there is no giant cluster at $\alpha \to 0$, and the network consists of clusters of

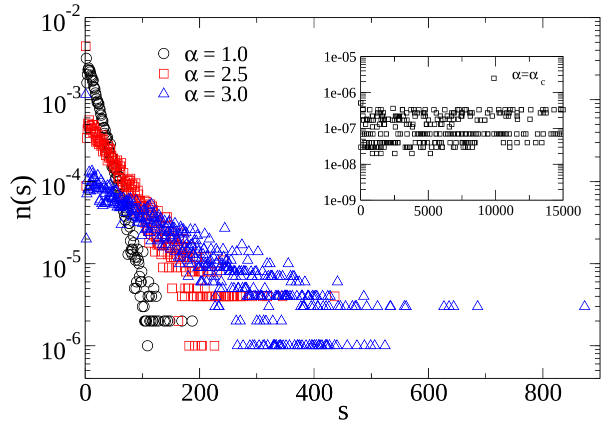

Figure 4. The relative cluster size distribution $n(s)$ for some values of the parameter α and $N = 15000$ nodes. The exponential decay of $n(s)$ for every value of α suggests that $\tau = 0$, as in 1−dimensional percolation.

many different sizes s, distributed in a pure exponential form

$$n(s) \sim e^{-s/\xi(\alpha)} \tag{3}$$

which suggests that $\tau = 0$ §. Moreover, as seen in the inset of Fig. 4, the whole $\alpha > \alpha_c$ phase is critical, in the sense that $\xi(\alpha \geq \alpha_c) = \infty$. One might guess this result by noting that when $\alpha \gg \alpha_c$, the most typical configuration is a line of nodes, that is, each added node gets connected to its immediate neighbor in the past. Given $P(a, b) = C(k_a + 1)(k_b + 1)e^{-\alpha(t_a+t_b)}$, the probability of joining nodes a and b with ages t_a and t_b and degrees k_a and k_b, respectively, and neglecting, as a first approximation, terms with contributions smaller than $e^{-4\alpha}$, the probability that a newly added node gets connected to this line is $P_{con} \approx 4/27(2e^{-\alpha} + 3e^{-2\alpha} + 3e^{-3\alpha} + 3e^{-4\alpha})$ and a new cluster emerges with probability $P_n \approx (24/27)e^{-4\alpha}$. These probabilities get comparable when $4 < \alpha < 5$ but, since there is always a nonzero chance of not connecting the new node to the previous ones one finds that for large values of α the system should have scaling properties of a critical one-dimensional percolation network.

We also find power-law scaling for the divergence of the susceptibility at the critical point

$$\chi(\alpha_c) \sim N^{\gamma}, \tag{4}$$

with $\gamma = 1.0(2)$, as depicted in Fig. 5. Sufficiently close to α_c, one can approximate the equation for the phase transition shift, Eq. (2), by $\ln(\alpha_c(\infty)/\alpha_c(N)) \approx (\alpha_c(N) - \alpha_c(\infty))/\alpha_c(\infty) \sim N^{-1}$, and we find that $\nu = 1$. This is what one expects at first-order phase transitions in one-dimensional systems, based on scaling and renormalization group arguments [19].

§ The general scaling function for the distribution of cluster sizes writes $n(s) \sim s^{-\tau}e^{-s/\xi(\alpha)}$ with a characteristic cluster size diverging near α_c as $\xi(\alpha \to \alpha_c) \sim |p - p_c|^{-\nu}$. In 1$d$ one has $\nu = 1$ and $\tau = 0$.

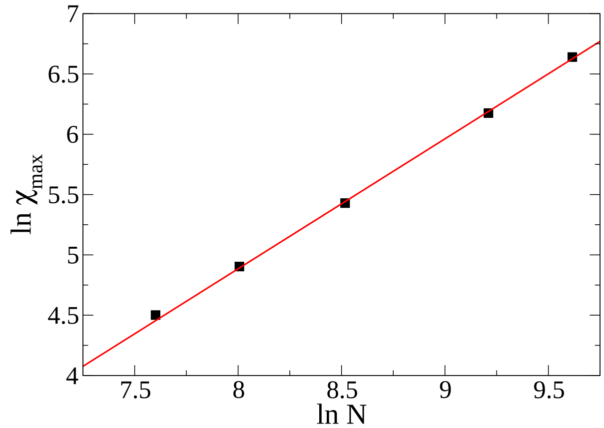

Figure 5. Maximum of the susceptibility $\chi(\alpha_c(N), N)$ versus N. The best fit indicates a power-law divergence $\chi(\alpha_c) \sim N^{\gamma}$ with $\gamma = 1.0(2)$.

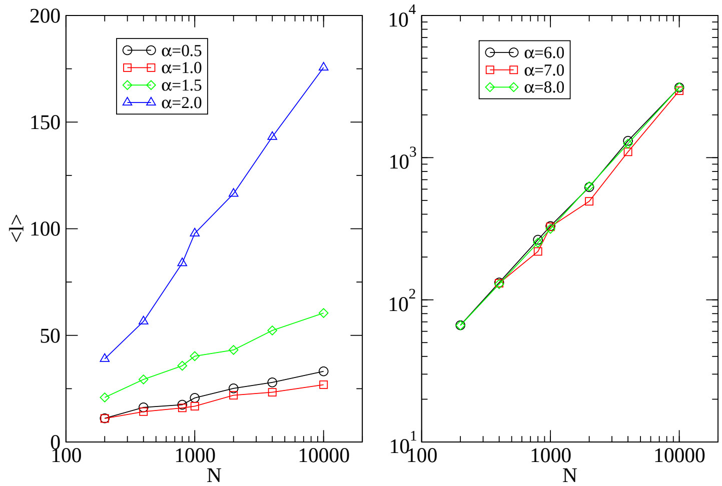

Figure 6. Scaling of the average shortest-path between every pair of nodes in the largest connected component with system size N, for values of α below (left panel) and above the critical point α_c (right panel). The emergence of a giant connected component is followed by a change from tree-like to quasi-one dimensional topology, as indicates the crossover from logarithmic to linear scaling of $\langle l \rangle$ with N.

In the limit $\alpha \to 0$, when the age-dependence becomes vanishingly small, links between nodes are added in a random fashion (although with different rates on nodes with different degrees) and the finite clusters are locally trees in the large size limit [12, 15]. In this limit one should expect a logarithmic dependence of the average shortest-path length $\langle l(N,\alpha)\rangle$ with system size N

$$\bar{l} = \frac{1}{N}\sum_{j=1}^{N}\langle l_{ij}\rangle \sim \mathcal{O}\ln(N), \tag{5}$$

where l_{ij} is the minimum number of links that must be traversed to join nodes i and j, and $\langle\ \rangle$ means average over realizations. On the other side, on d-dimensional networks $\bar{l}$ is proportional to the linear dimension L. Thus, as one increases the effects of aging in the preferential attachment of links, one expects a transition from "small" to "large-world" networks [20], or a change from logarithmic to linear scaling of the average shortest-path with system size. In Fig. 6 we show the average shortest-path $\langle l(N,\alpha)\rangle$ for every pair of nodes in the largest cluster of each network generated for different system sizes N. For $\alpha < \alpha_c$ we find logarithmic scaling of $\langle l\rangle$ with N, while $\langle l\rangle \sim N$ for $\alpha > \alpha_c$, supporting our view of a first-order transition in the universality class of $1d$ percolation for this problem.

3. Conclusions

We have studied the percolating properties of growing networks with age and degree preferential attachment of links: nodes introduced earlier in time are exponentially less likely to acquire new links than "younger" ones and links attach preferentially to nodes with high degree. One node and one link are introduced at each time step and the effect of aging on a node with age a and k links is varied by changing the exponent α in the age- and connectivity- dependent attachment probability $P(k,a) \propto (1+k)e^{-\alpha a}$. This model has a discontinuous transition in the relative size of the largest connected component: below a critical value α_c there is an extensive number of topologically tree-like clusters of connected nodes and at a first-order critical point a giant cluster of linearly connected nodes emerges. We analyzed the properties of this phase transition with numerical simulations, finding that fluctuations of the order parameter scale linearly with system size and that, close to the critical point, the inverse of a characteristic length scale vanishes linearly with the distance from the critical point, suggesting that the transition is in the universality class of $1d$ percolation.

Acknowledgements

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