Abstract

We consider players that have very limited knowledge about their own valuations. Specifically, the only information that a Knightian player $i$ has about the profile of true valuations, $\theta^*$, consists of a set of distributions, from one of which $\theta_i^*$ has been drawn.

We prove a “robustness” theorem for Knightian players in single-parameter domains: every mechanism that is weakly dominant-strategy truthful for classical players continues to be well-behaved for Knightian players that choose undominated strategies.
1 Introduction

In [CMZ14] we motivate the problem of mechanism design for Knightian players, and prove that (1) dominant-strategy mechanisms for single-good and multi-unit auctions cannot provide good social-welfare efficiency, but (2) the second-price and Vickrey mechanisms deliver good social-welfare performance, for these two settings, in undominated strategies.

In this report, we prove a “robustness” theorem for single-parameter domains. Namely, consider a mechanism $M$ for a single-parameter domain and suppose that $M$, when players have perfect information about their own valuations, is weakly dominant-strategy truthful. Now consider the same mechanism $M$, but with Knightian players that, not having any dominant strategy to play, choose to play undominated strategies. We prove that the set of undominated strategies is well-behaved, in the sense that these strategies do not deviate from the players’ approximate information about his own valuation.

2 Model

In a classical single-parameter domain, there is a set $A$, the set of all possible allocations; for each player $i$ there exists a publicly known subset $S_i \subseteq A$; and the set of possible valuations for player $i$, $\Theta_i$, consists of all functions mapping $A$ to the reals, subject to the following constraints: for each $\theta_i \in \Theta_i$,

1. $\theta_i(x) = 0 \quad \forall x \notin S_i$ and
2. $\theta_i(x) = \theta_i(y) \quad \forall x, y \in S_i$.

We denote the true valuation of player $i$ by $\theta_i^*$.

(The term “single-parameter” derives from the fact that each $\theta_i \in \Theta_i$ coincides with a single number: $i$’s value for, say, the lexicographically first element of $S_i$. The term “classical” emphasizes that each player knows exactly his own true valuation.)

The set of possible outcomes is $\Omega \defeq A \times \mathbb{R}_{\geq 0}^n$. If $(A, P) \in \Omega$, we refer $P_i$ as the price charged to player $i$. We assume quasi-linear utilities. That is, the utility function $U_i$ of a player $i$ maps a valuation $\theta_i$ and an outcome $\omega = (A, P)$ to $U_i(\theta_i, \omega) \defeq \theta_i(A) - P_i$. 

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If ω is a distribution over outcomes, we also denote by \( U_i(\theta_i, \omega) \) the expected utility of player \( i \).

Single-parameter domains are general enough to include several settings of interest: in particular, provision of a public good\(^1\) [Cla71], bilateral trades [MS83], and buying a path in a network [NR01].

### 2.1 Knightian Valuation Uncertainty

In our model, a player \( i \)'s sole information about \( \theta^* \) consists of \( K_i \), a set of distributions over \( \Theta_i \), from one of which \( \theta_i^* \) has been drawn. (The true valuations are uncorrelated.) That is, \( K_i \) is \( i \)'s sole (and private) information about his own true valuation \( \theta_i^* \). Furthermore, for every opponent \( j \), \( i \) has no information (or beliefs) about \( \theta_j^* \) or \( K_j \).

Given that all he cares about is his expected (quasi-linear) utility, a player \( i \) may 'collapse' each distribution \( D_i \in K_i \) to its expectation \( \mathbb{E}_{\theta_i \sim D_i}[\theta_i] \). Therefore, for single-parameter domains, a \textit{mathematically equivalent} formulation of the Knightian valuation model is the following:

**Definition 2.1** (Knightian valuation model). For each player \( i \), \( i \)'s sole information about \( \theta^* \) is a set \( K_i \), the candidate (valuation) set of \( i \), such that \( \theta_i^* \in K_i \subset \Theta_i \).

We refer to an element of \( K_i \) as a candidate valuation.

In Knightian valuation model, a mechanism’s performance will of course depend on the inaccuracy of the players’ candidate sets, which we measure as follows.

**Definition 2.2.** Let \( K_i^\perp \overset{\text{def}}{=} \inf K_i \) and \( K_i^\top \overset{\text{def}}{=} \sup K_i \).

The candidate set \( K_i \) of a player \( i \) is (at most) \( \delta \)-approximate if \( K_i^\top - K_i^\perp \leq \delta \).

A single-parameter domain is (at most) \( \delta \)-approximate if each \( K_i \) is \( \delta \)-approximate.

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\(^1\)Indeed, in the provision of a public good, \( A \) has just two elements, \( a \) (i.e., the good is provided), which different players may value differently, and \( b \) (i.e., the good is not provided), which all players value 0.

\(^2\)Whatever the auction mechanism used, this equivalence holds for any auction where each \( \Theta_i \) is a \textit{convex} set. In particular, this includes unrestricted combinatorial auctions of \( m \) distinct goods.
2.2 Social Welfare, Mechanisms, and Knightian Dominance

Social welfare. The social welfare of an allocation \( A \in \mathcal{A} \), \( SW(A) \), is defined to be \( \sum_i \theta_i^*(A) \); and the maximum social welfare, \( MSW \), is defined to be \( \max_{A \in \mathcal{A}} SW(A) \). (That is, \( SW \) and \( MSW \) continue to be defined relative to the players’ true valuations \( \theta_i^* \), whether or not the players know them exactly.)

More generally, the social welfare of an allocation \( A \) relative to a valuation profile \( \theta \), \( SW(\theta, A) \), is \( \sum_i \theta_i(A) \); and the maximum social welfare relative to \( \theta \), \( MSW(\theta) \), is \( \max_{A \in \mathcal{A}} SW(\theta, A) \). Thus, \( SW(A) = SW(\theta^*, A) \) and \( MSW = MSW(\theta^*) \).

General mechanisms and strategies. A mechanism \( M \) specifies, for each player \( i \), a set \( S_i \). We interchangeably refer to each member of \( S_i \) as a pure strategy/action/report of \( i \), and similarly, a member of \( \Delta(S_i) \) a mixed strategy/action/report of \( i \).

After each player \( i \), simultaneously with his opponents, reports a strategy \( s_i \) in \( S_i \), \( M \) maps the reported strategy profile \( s \) to an outcome \( M(s) \in \Omega \).

If \( M \) is probabilistic, then \( M(s) \in \Delta(\Omega) \). Thus, as per our notation, \( U_i(\theta_i, M(s)) \overset{\text{def}}{=} \mathbb{E}_{\omega \sim M(s)}[U_i(\theta_i, \omega)] \) for each player \( i \).

Note that \( S_i = \Theta_i \) for the direct mechanisms in the classical setting, but may be arbitrary in general.

Knightian undominated strategies. Given a mechanism \( M \), a pure strategy \( s_i \) of a player \( i \) with a candidate set \( K_i \) is (weakly) undominated, in symbols \( s_i \in UD_i(K_i) \), if \( i \) does not have another (possibly mixed) strategy \( \sigma_i \) such that

1. \( \forall \theta_i \in K_i \forall s_{-i} \in S_{-i} \quad \mathbb{E}U_i(\theta_i, M(\sigma_i, s_{-i})) \geq U_i(\theta_i, M(s_i, s_{-i})) \), and
2. \( \exists \theta_i \in K_i \exists s_{-i} \in S_{-i} \quad \mathbb{E}U_i(\theta_i, M(\sigma_i, s_{-i})) > U_i(\theta_i, M(s_i, s_{-i})) \).

If \( K \) is a product or a profile of candidate sets, that is, if \( K = (K_1, \ldots, K_n) \) or \( K = K_1 \times \cdots \times K_n \), then \( UD(K) \overset{\text{def}}{=} UD_1(K_1) \times \cdots \times UD_n(K_n) \).

Note that the above notion of an undominated strategy is a natural extension of its classical counterpart, but other extensions are possible.

Weakly dominant-strategy truthfulness in classical settings. Finally, let us recall what it means for a mechanism \( M \) to be weakly dominant-strategy truthful.
(weakly DST) when every player $i$ knows $\theta_i^*$ exactly. Namely, for each player $i$:

(0) $S_i = \Theta_i$

(1) $\forall v_i \in \Theta_i \forall v'_i \in \Theta_i \forall v_{-i} \in \Theta_{-i} \quad U_i(v_i, M(v_i, v_{-i})) \geq U_i(v_i, M(v'_i, v_{-i}))$

(2) $\forall v_i \in \Theta_i \forall v'_i \in \Theta_i \setminus \{v_i\} \exists v_{-i} \in \Theta_{-i} \quad U_i(v_i, M(v_i, v_{-i})) > U_i(v_i, M(v'_i, v_{-i}))$.

(For comparison, the notion of a DST mechanism omits the last condition above.)

3 Result

We prove the Knightian robustness of many mechanisms at once as follows.

**Theorem 1.** Let $M$ be a weakly dominant-strategy truthful mechanism for classical single-parameter domains. Then, in this domain with Knightian valuation uncertainty, for every player $i$, $\text{UD}(K_i) \subseteq [K_i^\perp, K_i^\top]$.

**Discussion.** The above theorem implies that the behavior of (weakly dominant-strategy truthful) mechanisms in a $\delta$-approximate single-parameter domains gracefully degrades with $\delta$. In particular, it implies that, when applied to the provision of a public good in the presence of $n$ Knightian players, the VCG mechanism guarantees, in undominated strategies, a social welfare $\geq \text{MSW} - 2n\delta$. As another example, when applied to buying paths in a network, the VCG mechanism guarantees a social welfare $\geq \text{MSW} - 2m\delta$, where $m$ is the number of edges in the network. Finally, we note that the proof of Theorem 1 easily extends to imply an analogous result for the VCG mechanism for single-minded combinatorial auctions, which are not quite single-parameter domains.\(^3\)

More generally, Theorem 1 implies that, for all weakly dominant-strategy mechanisms $M$ (which include those of [Cla71, MS83, NR01])

\begin{quote}
the outcome $M(v)$ is sufficiently good whenever $\max_i |v_i - \theta_i^*|$ is sufficiently small for all $i$ and $\theta_i^* \in K_i$.
\end{quote}

\(^3\)In such an auction, there are $m$ distinct goods, and each player $i$ values, positively and for the same amount $\theta_i^*$, only the supersets of a given subset $S_i$ of the goods. This auction is not single-parameter because $S_i$ is *private*, that is, known solely to $i$. Accordingly, $i$’s true valuation can be fully described only by the number $\theta_i^*$ and the subset $S_i$. The VCG mechanism for single-minded auctions ensures, in undominated strategies, a social welfare that is at least $\text{MSW} - 2\min\{n, m\}\delta$. 

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Proof. The theorem is obvious when $K_i = \{\theta_i^*\}$ is a singleton: since reporting
the truth is a weakly dominant strategy, it dominates all other strategies so that
UD($K_i$) = $\{\theta_i^*\}$ must also be a singleton. For the rest of the proof we assume that $K_i$
has at least two distinct valuations.

We begin by recalling the following fact about dominant-strategy truthful mechan-
isms in single-parameter domains where each player perfectly knows his own true
valuation [AT01]:

Let $M$ be a mechanism for a single-parameter domain, and let $f_i(v) \in [0, 1]$
be the probability that the allocation chosen by $M$, under strategy profile $v$, is in player $i$’s set $S_i$. Then, $M$ is dominant-strategy truthful if and only if
(a) $f$ is monotonically non-decreasing, i.e., $f_i(v_i, v_{-i}) \leq f_i(v_i', v_{-i})$ whenever $v_i \leq v_i'$, and (b) player $i$’s expected price on input $v$, denoted by $p_i(v)$, equals
to $v_i \cdot f_i(v_i, v_{-i}) - \int_0^{v_i} f_i(z, v_{-i})dz$.

Having recalled the above fact, we now prove that, for any Knightian player $i$ with
candidate set $K_i = [K_i^+, K_i^-]$,

$v_i \in \text{UD}_i(K_i) \implies v_i \in [K_i^+, K_i^-]$.

Let $v_i^+ \overset{\text{def}}{=} K_i^+$ and $v_i^- \overset{\text{def}}{=} K_i^-$, and consider any strategy $v_i \in \text{UD}_i(K_i)$. If $v_i \in K_i = [v_i^+, v_i^-]$ then we are done. Otherwise, suppose that $v_i < v_i^+$. (The other case, $v_i > v_i^-$, can be shown analogously.)

We first claim that, for player $i$, reporting $v_i^+$ is no worse than reporting $v_i$. Indeed, fixing any (pure) strategy sup-profile $v_{-i}$ for the other players and any possible true
valuation $\theta_i \in K_i$, and letting $v^+ = (v_i^+, v_{-i})$ and $v = (v_i, v_{-i})$, we compute that

\[
\mathbb{E}[U_i(\theta_i, M(v^+))] - \mathbb{E}[U_i(\theta_i, M(v))] = (f_i(v^+) - f_i(v)) \cdot \theta_i - (p_i(v^+) - p_i(v))
\]

\[
= (f_i(v^+) - f_i(v)) \cdot \theta_i - \left( v_i^+ \cdot f_i(v^+) - \int_0^{v_i^+} f_i(z, v_{-i})dz - v_i \cdot f_i(v) + \int_0^{v_i} f_i(z, v_{-i})dz \right)
\]

\[
= (f_i(v^+) - f_i(v)) \cdot (\theta_i - v_i^+) + \int_{v_i}^{v_i^+} (f_i(z, v_{-i}) - f_i(v))dz.
\]

Now note that $\theta_i \in K_i$ implies that $\theta_i - v_i^+ = \theta_i - K_i^+ \geq 0$. Moreover, by the
monotonicity of $f$, whenever $z \geq v_i$, it holds that $f_i(z, v_{-i}) \geq f_i(v)$. Therefore we
deduce that the above difference is greater than or equal to zero. We conclude that reporting \( v_i^+ \) is no worse than reporting \( v_i \).

Next there are two subcases. If \( \mathbb{E}[U_i(\theta_i, M(v^+))] - \mathbb{E}[U_i(\theta_i, M(v))] \) equals to zero for all \( \theta_i \in K_i \) and for all \( v_{-i} \), then, using the fact that \( K_i \) has at least two distinct valuations, we conclude that for \( i \), the allocation probability and (expected) price in outcomes \( M(v_i, v_{-i}) \) and \( M(v_i^+, v_{-i}) \) are the same, independent of \( v_{-i} \). This contradicts the fact that \( M \) is weakly dominant-strategy truthful in the classical setting, since \( U_i(v_i, M(v_i, v_{-i})) \) must be strictly greater than \( U_i(v_i, M(v_i^+, v_{-i})) \) at least for some \( v_{-i} \).

Otherwise, if there exist some \( \theta_i^* \) and some \( v_{-i}^* \) that make the difference \( \mathbb{E}[U_i(\theta_i, M(v^+))] - \mathbb{E}[U_i(\theta_i, M(v))] \) non-zero, it must follow that the difference is strictly positive. For such \( \theta_i^* \) and \( v_{-i}^* \), reporting \( v_i^+ \) is therefore strictly better than reporting \( v_i \), so by definition \( v_i^+ \) weakly dominates \( v_i \) for player \( i \), leading to a contradiction to \( v_i \in UD_i(K_i) \).

This concludes the proof of Theorem 1.

References


