Game theory
Lecture 5. Noncooperative strategic-form n-player games (continuation)

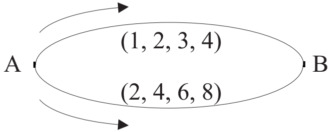
Potential games (L.S. Shapley and D. Monderer [1996])

A normal-form n-player game $\Gamma =< N, \{X_i\}_{i \in N}, \{H_i\}_{i \in N} >$.

Suppose that there exists a certain function $P : \prod_{i=1}^{n} X_i \to R$ such that for any $i \in N$ we have the inequality

$$H_i(x_{-i}, x_i') - H_i(x_{-i}, x_i) = P(x_{-i}, x_i') - P(x_{-i}, x_i)$$

for arbitrary $x_{-i} \in \prod_{j \neq i} X_j$ and any strategies $x_i, x_i' \in X_i$, where P is a potential function



Traffic jamming. Suppose that companies *I* and *II*, each possessing two trucks, have to deliver some cargo from point A to point B.

These points communicate through two roads. Numbers on the figure indicate the journey time on each road depending on the number of moving trucks.

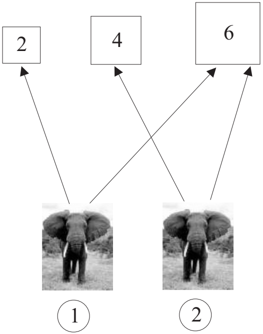
Payoff matrix:

$$\begin{array}{c} \\ (2,0) \\ (1,1) \\ (0,2) \end{array} \begin{array}{c} \begin{array}{ccc} (2,0) & (1,1) & (0,2) \end{array} \\ \begin{pmatrix} (-8,-8) & (-6,-5) & (-4,-8) \\ (-5,-6) & (-6,-6) & (-7,-12) \\ (-8,-4) & (-12,-7) & (-16,-16) \end{pmatrix} \end{array}.$$

Potential games

The game possesses the potential

$$P = \begin{array}{c} \\ (2,0) \\ (1,1) \\ (0,2) \end{array} \begin{array}{c} \begin{array}{ccc} (2,0) & (1,1) & (0,2) \end{array} \\ \left(\begin{array}{ccc} 13 & 16 & 13 \\ 16 & 16 & 10 \\ 13 & 10 & 0 \end{array} \right) \end{array}$$



Animal foraging. Two animals choose one or two areas among three areas for their foraging. These areas provide 2, 4 and 6 units of food, respectively. If both animals visit a same area, they equally share available food. The payoff of each player is the total units of food gained at each area minus the costs to visit this area (we set them equal to 1).

Payoff matrix:

$$\begin{array}{c} \\ (1) \\ (2) \\ (3) \\ (1,2) \\ (1,3) \\ (2,3) \end{array} \begin{array}{c} \begin{array}{cccccc} (1) & (2) & (3) & (1,2) & (1,3) & (2,3) \end{array} \\ \begin{pmatrix} (0,0) & (1,3) & (1,5) & (0,3) & (0,5) & (1,8) \\ (3,1) & (1,1) & (3,5) & (1,2) & (3,6) & (1,6) \\ (5,1) & (5,3) & (2,2) & (5,4) & (2,3) & (2,5) \\ (3,0) & (2,1) & (4,5) & (1,1) & (3,5) & (2,6) \\ (5,0) & (6,3) & (3,2) & (5,3) & (2,2) & (3,5) \\ (8,1) & (6,1) & (5,2) & (6,2) & (5,3) & (3,3) \end{pmatrix} \end{array}.$$

Potential:

$$P = \begin{array}{c} \\ (1) \\ (2) \\ (3) \\ (1,2) \\ (1,3) \\ (2,3) \end{array} \begin{array}{c} \begin{array}{cccccc} (1) & (2) & (3) & (1,2) & (1,3) & (2,3) \end{array} \\ \begin{pmatrix} 1 & 4 & 6 & 4 & 6 & 9 \\ 4 & 4 & 8 & 5 & 9 & 9 \\ 6 & 8 & 7 & 9 & 8 & 10 \\ 4 & 5 & 9 & 5 & 9 & 10 \\ 6 & 9 & 8 & 9 & 8 & 11 \\ 9 & 9 & 10 & 10 & 11 & 11 \end{pmatrix} \end{array}.$$

Potential games

Theorem. *Let an n-player game* $\Gamma =< N, \{X_i\}_{i \in N}, \{H_i\}_{i \in N} >$ *have a potential* P. *Then a Nash equilibrium in the game* Γ *represents a Nash equilibrium in the game* $\Gamma' =< N, \{X_i\}_{i \in N}, P >$, *and vice versa. Furthermore, the game* Γ *admits at least one pure strategy equilibrium.*

Proof. The first assertion follows from the definition of a potential.

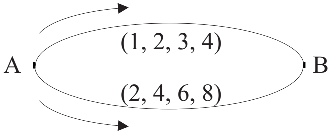
$$H_i(x^*_{-i}, x_i) \leq H_i(x^*), \forall x_i, \quad P(x^*_{-i}, x_i) \leq P(x^*), \forall x_i$$

Potential games

Now, we argue that the game Γ' always has a pure strategy equilibrium. Let x^* be the pure strategy profile maximizing the potential $P(x)$ on the set $\prod_{i=1}^{n} X_i$. For any $x \in \prod_{i=1}^{n} X_i$, the inequality $P(x) \leq P(x^*)$ holds true at this point, particularly,

$$P(x^*_{-i}, x_i) \leq P(x^*), \forall x_i.$$

Therefore, x^* represents a Nash equilibrium in the game Γ' and, hence, in the game Γ.

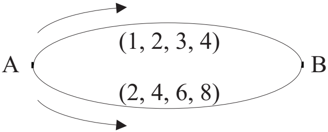


A game without potential. A game may have no potential, even if a pure strategy equilibrium does exist.

Suppose that the costs of players are defined by the maximal journey time of their trucks on both roads.

Payoff matrix:

$$\begin{array}{c} \\ (2,0) \\ (1,1) \\ (0,2) \end{array} \begin{array}{c} \begin{array}{ccc} (2,0) & (1,1) & (0,2) \end{array} \\ \begin{pmatrix} (-4,-4) & (-3,-3) & (-2,-4) \\ (-3,-3) & (-4,-4) & (-6,-6) \\ (-4,-2) & (-6,-6) & (-8,-8) \end{pmatrix} \end{array}.$$



The described game has no potential. We demonstrate this fact rigorously. Assume that a potential P exists; then:

$$P(1,1) - P(3,1) = H_1(1,1) - H_1(3,1) = -4 - (-4) = 0,$$
$$P(1,1) - P(1,2) = H_2(1,1) - H_2(1,2) = -4 - (-3) = -1.$$

And so,

$$P(3,1) - P(1,2) = -1.$$

On the other hand,

$$P(1,2) - P(3,2) = H_1(1,2) - H_1(3,2) = -3 - (-6) = 3,$$
$$P(3,1) - P(3,2) = H_2(3,1) - H_2(3,2) = -2 - (-6) = 4,$$

whence it follows that

$$P(3,1) - P(1,2) = 1.$$

This two facts contradicts, the game possesses no potential.

Congestion games

Definition. *A symmetrical congestion game is an n-player game* $\Gamma =< N, M, \{S_i\}_{i \in N}, \{c_i\}_{i \in N} >$, *where* $N = \{1, ..., n\}$ *stands for the set of players, and* $M = \{1, ..., m\}$ *means the set of some objects for strategy formation. A strategy of player* i *is the choice of a certain subset from* M. *The set of all feasible strategies makes the strategy set of player* i, *denoted by* S_i, $i = 1, ..., n$. *Each object* $j \in M$ *is associated with a function* $c_j(k), 1 \leq k \leq n$, *which represents the payoff (or costs) of each player from* k *players that have selected strategies containing* j. *This function depends only on the total number* k *of such players.*

Gongestion games

Players have chosen strategies $s = (s_1, \ldots, s_n)$. The payoff function of player i is determined by the total payoff on each object:

$$H_i(s_1, \ldots, s_n) = \sum_{j \in s_i} c_j(k_j(s_1, \ldots, s_n)).$$

Here $k_j(s_1, \ldots, s_n)$ gives the number of players whose strategies incorporate object j, $i = 1, \ldots, n$.

Theorem. *A symmetrical congestion game is potential, ergo admits a pure strategy equilibrium.*

$$P(s_1, \ldots, s_n) = \sum_{j \in \cup_{i \in N} s_i} \left(\sum_{k=1}^{k_j(s_1, \ldots, s_n)} c_j(k) \right)$$

Auctions

Players simultaneously bid for the item (suggest prices $(x_1, ..., x_n)$, respectively). The item passes to a player announcing the highest price.

First-price auction. A winner (a player suggesting the maximal price) gets the item and actually pays nothing. The rest players have to pay the price they have announced (for participation). If several players bid the maximal price, they equally share the payoff.

$$H_i(x_1, ..., x_n) = \begin{cases} -x_i, & \text{if } x_i < y_{-i}, \\ \frac{V}{m_i(x)} - x_i, & \text{if } x_i = y_{-i}, \\ V, & \text{if } x_i > y_{-i}, \end{cases} \qquad (*)$$

Theorem. *A first-price auction with the payoff function (*) admits the mixed strategy equilibrium*

$$F^*(x) = \left(\frac{x}{V + x}\right)^{1/(n-1)},$$

and the game value is zero.

Second-price auction. Here all players pay their announced prices for participation in an auction, while a winner pays merely the second highest price. Such auctions are called Vickrey auctions. If several players make the maximal bid, they share V equally.

$$H_i(x_1, ..., x_n) = \begin{cases} -x_i, & \text{if } x_i < y_{-i}, \\ \frac{V}{m_i} - x_i, & \text{if } x_i = y_{-i}, \\ V - y_{-i}, & \text{if } x_i > y_{-i}, \end{cases} \qquad (**)$$

Theorem. *Consider a second-price auction with the payoff function (**). An equilibrium consists in the mixed strategies*

$$F(x) = \left(1 - \exp(-\frac{x}{V})\right)^{\frac{1}{n-1}}.$$

Each player has one bullet and can shoot at any instant from the interval $[0,1]$. Let $A(t)$ be the probability of target hitting provided that shooting occurs at instant $t \in [0,1]$. We believe that the function $A(t)$ is differentiable, $A'(t) > 0, A(0) = 0$ and $A(1) = 1$.

Suppose that all players choose same mixed strategies with a distribution function $F(t)$ and density function $f(t), a \leq t \leq 1$, where $a \in [0,1]$ is a parameter. If player 1 shoots at instant x and other players apply mixed strategies $F(t)$, his expected payoff becomes:

$$H_1(x, \overbrace{F, \cdots, F}^{n-1}) = \begin{cases} A(x), & \text{if } 0 \leq x < a, \\ A(x)\left[1 - \int\limits_a^x A(t)f(t)dt\right]^{n-1}, & \text{if } a \leq x \leq 1. \end{cases}$$

Let v be the optimal payoff common for all players. Then the sufficient condition of an equilibrium takes the form

$$H_1(x, F, \cdots, F) \left\{ \begin{array}{c} \leq \\ = \end{array} \right\} v, \ \text{ for} \left\{ \begin{array}{c} 0 \leq x < a \\ a \leq x \leq 1 \end{array} \right\}$$

In the case of $a \leq x \leq 1$, apply the first-order necessary optimality conditions to (8.1) to obtain the differential equation

$$\frac{f'(x)}{f(x)} = -\frac{2n-1}{n-1} \left[\frac{A'(x)}{A(x)} - \frac{A''(x)}{A'(x)} \right].$$

Theorem. *Let* α_n *be a unique root of the equation*

$$\alpha^{\frac{n}{n-1}} + n\alpha - 1 = 0 \tag{8.11}$$

within the interval $[0, 1]$.
Then the game admits the mixed strategy Nash equilibrium

$$f^*(x) = \frac{1}{n-1} (\alpha_n)^{\frac{1}{n-1}} (A(x))^{-\frac{2n-1}{n-1}} A'(x), \quad \textit{for } A^{-1}(\alpha_n) = a_n \leq x \leq 1. \tag{8.12}$$

In the equilibrium, the optimal payoffs of players constitute α_n.

Example. Now, choose $A(x) = \dfrac{e^x - 1}{e - 1}$. Consequently,

$$a_n = A^{-1}(\alpha_n) = \log\{1 + (e-1)\alpha_n\}.$$

Hence, a_n decreases if n goes up. In the case of a duel ($n = 2$),

$$a_n = \log\left\{(\sqrt{2} - 1)(e + \sqrt{2})\right\} \approx 0.5375.$$

For a truel ($n = 3$), we obtain

$$a_n \approx 0.3964.$$

The optimal strategies are defined by the density function

$$f^*(x) = \tfrac{1}{n-1}(\alpha_n)^{\frac{1}{n-1}}(e-1)^{-1}(e^x - 1)^{-\frac{2n-1}{n-1}} e^x, \quad \text{for } a_n \le x \le 1.$$