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FAIR DIVISION OF A FIXED COST DEFINES THE SHAPLEY VALUE

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La règle de partage de coût dérivée de la valeur de Shapley est l'unique règle de partage qui alloue uniformément un coût fixe.

The cost sharing rule derived from the Shapley value is the unique sharing rule which uniformly divides a fixed cost.

There is a large literature on cost sharing based on solutions defined for cooperative games with transferable utility (see Moulin [1] or Young [3]). The allocation of a fixed cost is however not explicitly addressed. Fairness suggests that a fixed cost should be uniformly allocated among players. That's what does the sharing rule derived from the Shapley value [2]. We show that it is actually the only sharing rule which does so, offering as a byproduct an axiomatization which does not require additivity nor the null player axiom.

Given a set $N = \{1...n\}$ of players, a *cost sharing game* is defined by a cost function C which associates to each coalition $S \subset N$ a cost $C(S)$, with $C(\emptyset) = 0$. The set $G(N)$ of cost functions defined on N is a linear space of dimension $p = 2^n - 1$.

A *cost sharing rule* is a mapping φ which associates to a cost function $C \in G(N)$ an allocation $y = \varphi(C) \in I\!R^n$ such that $y(N) = C(N)$ and $C(S) = 0$ for all $S \subset N$ implies $\varphi(C) = 0$. The sharing rule derived from the Shapley value allocates to each player a weighted sum of his/her marginal costs :

$$\varphi_i(C) = \sum_{S \subset N} \alpha(s)[C(S) - C(S \backslash i)]$$

where the weights $\alpha(s)$ are given by

$$\alpha(s) = \frac{(s-1)!(n-s)!}{n!}.$$

A fixed cost f which affects a coalition T is defined by :

$$\begin{aligned} C_f(S) &= C(S) + f \quad \text{for all } S \subset N \text{ such that } S \cap T \neq \emptyset \\ &= C(S) \quad \text{for all } S \subset N \backslash T \end{aligned}$$

The amount to be allocated is equal to $C(N) + f$. Fairness requires that only the players who are concerned contribute and that they contribute equally :

Fairness For any cost function $C \in G(N)$, fixed cost f and subset T,

$$\begin{aligned} R_i(C_f) &= R_i(C) + \tfrac{1}{t} f \quad \text{for all } i \in T \\ &= R_i(C) \quad \text{for all } i \in N \backslash T \end{aligned}$$

Proposition *The Shapley sharing rule is the unique sharing rule satisfying fairness.*

The proof follows Shapley's original proof using a basis of the linear space $G(N)$ given by the elementary game e_T defined by :

$$\begin{aligned} e_T(S) = 1 \quad &\text{if} \quad S \cap T \neq \emptyset \\ e_T(S) = 0 \quad &\text{if} \quad S \subset N \backslash T \end{aligned}$$

These are simple games describing decision problems where a coalition is winning if and only if it contains at least one member of T. Shapley uses simple games where players in T are veto players : a coalition is winning if and only if it contains all members of T. This give rise to a $p \times p$ matrix whose determinant is equal to 1, instead of -1 for our basis.

Bibliography

[1] H. Moulin, *Axioms of Cooperative Decision Making,* Cambridge University Press, Cambridge, 1988.

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[3] P. Y. Young (ed.), *Fair Allocation,* Proceedings of Symposia in Applied Mathematics 33, AMS, Providence, Rhode Island, 1985.