Erratum

Erratum to: “Ruled surfaces with time like rulings”

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Abstract

In this article, a new type of ruled surfaces in a Lorentz 3-space $\mathbb{R}_1^3$ is obtained by a
strictly connected time-like oriented line moving with Frenet’s frame along a space-like
curve. These surfaces are classified into time-like and space-like surfaces. The well-
known theorems due to Bonnet and Chasles in the 3-dimensional Euclidean space are
proved for a time-like ruled surface.

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1. Introduction

Definition 1. Let $\mathbb{R}_1^3$ be a 3-dimensional Lorentzian space with the pseudo-metric $ds^2 = dx^2 - dy^2 + dz^2$. If $\langle \vec{X}, \vec{Y} \rangle = 0$ for all $\vec{X}$ and $\vec{Y}$, the vectors $\vec{X}$ and $\vec{Y}$ are
called perpendicular in the sense of Lorentz, where $\langle , \rangle$ is the induced inner
product in $\mathbb{R}_1^3$ [1].

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Definition 2. The norm of $\vec{X} \in \mathbb{R}^3_1$ is denoted by $\|\vec{X}\|$ and defined as

$$\|\vec{X}\| = \sqrt{|\langle \vec{X}, \vec{X} \rangle|}.$$  

The vector $\vec{X} \in \mathbb{R}^3_1$ is called a space-like, time-like and null (light-like) vector if $
abla \vec{X} > 0$ or $\vec{X} = 0$, $\langle \vec{X}, \vec{X} \rangle < 0$, and $\langle \vec{X}, \vec{X} \rangle = 0$ for $\vec{X} \neq 0$, respectively [5].

Definition 3. A regular curve $\alpha(s) : I \rightarrow \mathbb{R}^3_1$, $I \subset \mathbb{R}$ in $\mathbb{R}^3_1$ is said to be a space-like, time-like and null curve if the velocity vector $\vec{\alpha}'(s) = d\alpha/ds$ is a space-like, time-like or null vector respectively [4].

Definition 4. A surface in a 3-dimensional Lorentz space is called a time-like surface if the induced metric on the surface is a Lorentz metric, i.e., the normal on the surface is a space-like vector [7].

As revealed from the foregoing definitions, one can prove the following:

Lemma 1. In the Lorentz space $\mathbb{R}^3_1$, the following properties are satisfied:

(i) Two time-like vectors are never orthogonal.
(ii) Two null vectors are orthogonal if and only if they are linearly dependent.
(iii) A time-like vector is never orthogonal to a null (light-like) vector [3].

Let $\vec{z} = \vec{z}(s)$ be a unit speed space-like curve in $\mathbb{R}^3_1$; by $\kappa(s)$, $\tau(s)$ we denote the natural curvature and torsion of $\vec{z}(s)$ respectively. Consider the Frenet frame $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ attached to the space-like curve $\vec{z} = \vec{z}(s)$ such that $\vec{e}_2 = \vec{e}_2(s)$ is the principal normal vector field of type time-like and $\vec{e}_3 = \vec{e}_3(s)$ is the binormal vector field of type space-like and $\vec{e}_1 = \vec{e}_1(s)$ is the unit tangent vector field with the normalization [4].

$$\langle \vec{e}_1, \vec{e}_1 \rangle = -\langle \vec{e}_2, \vec{e}_2 \rangle = \langle \vec{e}_3, \vec{e}_3 \rangle = 1,$$
$$\langle \vec{e}_i, \vec{e}_j \rangle = 0 \quad \text{for} \quad i \neq j, \quad \text{det}(\vec{e}_1, \vec{e}_2, \vec{e}_3) = 1.$$  

(1.1)

The infinitesimal displacements of the frame are given as

$$\vec{e}_1'(s) = \kappa(s)\vec{e}_2, \quad \vec{e}_2'(s) = \kappa(s)\vec{e}_1 + \tau(s)\vec{e}_3, \quad \vec{e}_3'(s) = \tau(s)\vec{e}_2.$$  

(1.2)

One can easily see the following:

Lemma 2. Assume that $\vec{z}(s)$ is a unit speed space-like curve in $\mathbb{R}^3_1$ and $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ be its Frenet's frame field. Then

$$\vec{e}_1 \wedge \vec{e}_2 = \vec{e}_3, \quad \vec{e}_1 \wedge \vec{e}_3 = \vec{e}_2, \quad \vec{e}_2 \wedge \vec{e}_3 = \vec{e}_1.$$  

(1.3)
2. Ruled surfaces with time-like generators in $\mathbb{R}^3$

A time-like straight line $\vec{L}$ in $\mathbb{R}^3$ such that it is strictly connected to Frenet’s frame of the space-like curve $\vec{a} = \vec{a}(s)$ is represented, uniquely with respect to this frame, in the form

$$\vec{L}(s) = \sum_{i=1}^{3} \ell_i(s) \vec{e}_i(s), \quad \langle \vec{L}(s), \vec{L}(s) \rangle > 0,$$

where the components $\ell_i = \ell_i(s)$ ($i = 1, 2, 3$) are scalar functions of the arc length parameter of the base curve $\vec{a} = \vec{a}(s)$. Without loss of generality the family of ruled surfaces under investigation are characterized by the regular parametrization.

$$M : \Phi(s, v) = \vec{a}(s) + v\vec{L}(s),$$

$$\ell_1^2 - \ell_2^2 + \ell_3^2 = -1, \quad \vec{L}(s) \neq 0. \quad \text{(I)}$$

**Definition 5.** If there exists a common perpendicular to two constructive rulings in the skew surface, then the foot of the common perpendicular on the main ruling is called a central point. The locus of the central point is called the striction curve [7].

Using (1.2) and (2.1), it is easy to see that the parametrization of the striction curve on a ruled surface of type (I) is given by

$$\vec{b}(s) = \vec{a}(s) - \phi(s)\vec{L}(s), \quad \text{where } \phi(s) = \frac{\ell'_1 + \ell_2\kappa}{\langle \vec{L}(s), \vec{L}(s) \rangle}. \quad \text{(2.2)}$$

The unit normal vector $\vec{n}$ on the ruled surface of type (I) is given by

$$\vec{n} = \frac{\vec{x}'(s) \wedge \vec{L}(s) + v\vec{x}''(s) \wedge \vec{L}(s)}{||\vec{x}'(s) \wedge \vec{L}(s) + v\vec{x}''(s) \wedge \vec{L}(s)||}. \quad \text{(2.3)}$$

Thus, from (2.3) and (1.2) the unit normal vector to the surface $M$ at the point $(s, o)$ is

$$\vec{n}(s, o) = \frac{\ell_2 \vec{e}_3 + \ell_3 \vec{e}_2}{\sqrt{\ell_2^2 - \ell_3^2}}. \quad \text{(2.4)}$$

From (2.4), it follows that the base curve of the surface (I) to be a geodesic curve ($\vec{n}(s, o) = \vec{e}_2$) if $\ell_2 = 0$, $\ell_3 \neq 0$.

From (2.2) and (2.4) it follows that the Bonnet’s theorem for the ruled surface (I) can be formulated as
Theorem 1. If a space-like curve on a ruled surface (I) in $\mathbb{R}^3_1$ has two of the following properties, it has the third also

(i) It is a geodesic ($\ell_2 = 0$ or $\kappa = 0$).
(ii) It cut the rulings at a constant angle ($\ell_1 = \text{const.}$).
(iii) It is a striction curve ($\phi(s) = 0$).

The induced metric on the ruled surfaces (I) is given by [2]

\[ g = -(1 + \ell_1^2) - (2v\phi(s) + v^2)\langle \vec{L}', \vec{L}' \rangle. \]

Therefore, one can give a classification to the ruled surfaces (I) into time-like or space-like ruled surfaces as follows:

The class of ruled surfaces (I) is of type time-like ($g < 0$) if one of the following conditions:

(i) $v$ and $\phi(s)$ have the same sign,
(ii) $\phi(s) = 0$,
(iii) $v$ and $\phi(s)$ have opposite sign and

\[ (1 + \ell_1^2) + (v^2 - 2v\phi(s))\langle \vec{L}'(s), \vec{L}'(s) \rangle > 0 \]

is satisfied.

The ruled surfaces (I) are space-like ruled surfaces if and only if $v$ and $\phi(s)$ have opposite sign and

\[ (1 + \ell_1^2) + (v^2 - 2v\phi(s))\langle \vec{L}'(s), \vec{L}'(s) \rangle < 0. \]

Theorem 2. The class of time-like ruled surfaces is divided into three subclasses according to the conditions (i), (ii) and (iii) respectively.

2.1. Time-like ruled surface in $\mathbb{R}^3_1$

Here, we study a subclass of time-like ruled surfaces which is characterized by the condition (ii). For this type, the striction curve is the base curve of a ruled surface of type (I), i.e.,

\[
\begin{align*}
M' : \Phi(s, v) = \vec{x}(s) + v\vec{L}(s), & \quad \|\vec{L}'(s)\| \neq 0, \\
\ell_1^2 - \ell_2^2 + \ell_3^2 = -1, & \quad \ell_1' + \ell_2\kappa = 0.
\end{align*}
\]

(II)
The distribution parameter \( \lambda(s) \) of the time-like ruled surface \( M^t \) is defined as
\[
\lambda(s) = \frac{[\vec{x}'(s), \vec{L}(s), \vec{L}'(s)]}{\langle \vec{L}'(s), \vec{L}(s) \rangle}
\]
and from (1.2) and (2.1) we have
\[
\dot{\lambda}(s) = \frac{\ell_2\ell_3' - \ell_3\ell_2' + (1 + \ell_1^2)\tau - \ell_1\ell_3\kappa}{\langle \vec{L}'(s), \vec{L}(s) \rangle}.
\]
From (2.3) and (2.1) it is easy to see that, the unit normal vector to the time-like ruled surface \( M^t \) at \( (s, v) \) and \( (s, 0) \) are
\[
\vec{n}(s, v) = \frac{\lambda \vec{I}(s) + v\vec{L}(s) \wedge \vec{L}(s)}{\sqrt{\lambda^2 + v^2}}, \quad \vec{n}_o = \frac{\vec{L}(s)}{||\vec{L}'(s)||}.
\]
From Definition 4 and Lemma 1, one can see that \( \vec{n} \) and \( \vec{n}_o \) are unit space-like vectors. The angle \( \theta \) of rotation from the normal \( \vec{n}_o \) to the normal \( \vec{n} \) is given from
\[
\sin \theta = ||\vec{n}_o \wedge \vec{n}|| = \frac{v}{\sqrt{\lambda^2 + v^2}}.
\]
Hence Chasles theorem is valid for a time-like ruled surface \( M^t \) in \( \mathbb{R}^3_1 \).

As an immediate result we have the following:

**Corollary 1.** The tangent plane turns evidently through \( 180^\circ \) along a ruling in a non-developable \((\lambda \neq 0)\) time-like ruled surface \( M^t \).

The Gaussian curvature \( K \) of the surface in \( \mathbb{R}^3_1 \) is given by \( K = \frac{\varepsilon h_{\beta\beta}}{g_{\alpha\beta}} \), where \( \varepsilon = \langle \vec{n}, \vec{n} \rangle \) and \( g_{\alpha\beta}, h_{\alpha\beta} \) are the first and second fundamental quantities, respectively [6]. After simple calculation, one can see that the Gaussian curvature of the time-like ruled surface \( M^t \) is
\[
K = \frac{\lambda^2}{(\lambda^2 + v^2)^2}.
\]

Thus we have the following:

**Theorem 3.** The Gaussian curvature \( K \) of a time-like ruled surface \( M^t \) in \( \mathbb{R}^3_1 \) is non negative and \( K \) is equal to zero only along the ruling which meet the striction curve at a singular point \((\lambda = 0, v \neq 0)\).

Eq. (2.7) allows us to give a geometric interpretation of the (regular) central points of a time-like ruled surface. Indeed, the points of a ruling except perhaps the central point, are regular points of the surface if \( \lambda \neq 0 \). The function \( K(s, v) \) is a continuous function on the ruling \((s = \text{const.})\) and the central point is characterized by the fact that \( K(s, v) \) has a maximum there.
2.2. Time-like ruled surfaces with constant parameter of distribution

From (2.5), one can see that a time-like ruled surface $M_t$ with a constant distribution parameter satisfies the following differential equation:

$$
\ell_2 \ell_3' - \ell_3 \ell_2' + (1 + \ell_3^2)\tau - \ell_1 \ell_3 R = C \langle \vec{L}'(s), \vec{L}(s) \rangle, \quad \phi(s) = 0,
$$

where $C$ is constant. Using (II), one can see that

$$
\ell_1 = - \int \ell_2 \kappa \, ds + c_1,
$$

$$
\ell_3 = \int \ell_2 \left( \frac{C\ell(s)\vec{L}(s)}{1 + \ell_1^2} - \tau \right) \, ds + c_2, \quad \ell_2 \neq \ell_3, \quad \ell_1^2 - \ell_2^2 + \ell_3^2 = -1.
$$

The second equation of the triple (2.8) is an integral equation for the unknown $\ell_3 = \ell_3(S)$. Therefore, if we given $\ell_2 = \ell_2(s)$ we get $\ell_1 = \ell_1(s)$ and $\ell_3 = \ell_3(s)$. Thus, we have the existence theorem:

**Theorem 4.** The range of existence of a one parametric time-like ruled surfaces $\{M_t\}$ with constant parameter of distribution comprises within one arbitrary function of one variable.

Developable time-like ruled surface are special class of the ruled surfaces described by Theorem 4 ($\lambda = c = 0$).

If a point P be displaced orthogonally along $\vec{L}$ to a neighbouring point $P_o$, we have an orthogonal trajectory. The condition that the point P be displaced orthogonally to the ruling is $\langle d\vec{c}'/ds, \vec{L} \rangle = 0$. One can see that

$$
v = \int \ell_1 \, ds. \tag{2.9}
$$

From (2.9) and Theorem 1, for the time-like ruled surface $M_t$, one can give an interpretation to the parameters $v, \ell_1$, as

**Theorem 5.** For a time-like ruled surface $M_t$ in $\mathbb{R}^3_1$. If the base curve is geodesic, then the distance $v$ along a ruling from the base curve $\vec{x} = \vec{x}(s)$ to an orthogonal trajectory is proportional to the arc length of the base curve.

2.3. The geodesic and normal curvature

Let $\vec{\gamma} = \vec{\gamma}(s)$ be a curve on the time-like ruled surface $M_t$, then it can be represented in the form

$$
\vec{\gamma}(s) = \vec{x}(s) + v(s)\vec{L}(s).
$$
Using (1.2), it is easy to see that the unit tangent vector along the curve \( \vec{\gamma} = \vec{\gamma}(s) \) is

\[
\vec{\gamma}'(s) = \frac{\xi_1 \vec{e}_1 + \xi_2 \vec{e}_2 + \xi_3 \vec{e}_3}{\sqrt{R}},
\]

where

\[
\begin{align*}
\xi_1 &= 1 + v' \ell_1, & \xi_2 &= v \varphi_1 + v' \ell_2, & \xi_3 &= v \varphi_2 + v' \ell_3, \\
\varphi_1 &= \ell_1 \kappa + \ell_2' + \ell_3 \tau, & \varphi_2 &= \ell_2 \tau + \ell_3', \\
R &= |\xi_1^2 - \xi_2^2 + \xi_3^2|.
\end{align*}
\]

Therefore, one can see that

\[
\vec{\gamma}''(s) = R^{-3/2} ([(\xi_1' + \xi_2 \kappa) \vec{e}_1 + (\xi_2' + \xi_1 \kappa + \xi_3 \tau) \vec{e}_2 + (\xi_3' + \xi_2 \tau) \vec{e}_3] R \\
- \frac{R'}{2} (\xi_1 \vec{e}_1 + \xi_2 \vec{e}_2 + \xi_3 \vec{e}_3)).
\]

Using (2.6), the unit normal vector field on the time-like ruled surface \( M' \) along the curve \( (v = v(s)) \) is

\[
\vec{n}(s, v(s)) = \frac{v(\ell_3 \varphi_1 - \ell_2 \varphi_2) \vec{e}_1 + (\ell_3 - v \ell_1 \varphi_1) \vec{e}_2 + (\ell_2 - v \ell_1 \varphi_1) \vec{e}_3}{\sqrt{v^2(\ell_3 \varphi_1 - \ell_2 \varphi_2)^2 + (\ell_3 - v \ell_1 \varphi_1)^2 + (\ell_2 - v \ell_1 \varphi_1)^2}}.
\]

The geodesic curvature of the curve \( \vec{\gamma} = \vec{\gamma}(s) \) is given by

\[
k_g = \frac{1}{R||\vec{n}(s, v(s))||} \begin{vmatrix}
\xi_1' + \xi_2 \kappa & \xi_2' + \xi_1 \kappa + \xi_3 \tau & \xi_3' + \xi_2 \tau \\
\xi_1 & \xi_2 & \xi_3 \\
v(\ell_3 \varphi_1 - \ell_2 \varphi_2) & \ell_3 - v \ell_1 \varphi_1 & \ell_2 - v \ell_1 \varphi_1
\end{vmatrix}.
\]  

(2.10)

Then, the differential equation of the geodesic curves on the time-like ruled surface \( M' \) is given by

\[
f(s)v'' + h(s)(v')^2 + f(s)v' + r(s)v + \ell_2 \kappa = 0,
\]

where

\[
\begin{align*}
f(s) &= 1 + \ell_1^2 + v^2 (\varphi_2^2 - \varphi_1^2), \\
h(s) &= v[\varphi_1^2 - \varphi_2^2 + \ell_2' \varphi_1 - \ell_3 \varphi_2 - (\ell_1 \kappa + \ell_3 \tau) \varphi_1], \\
g(s) &= v \ell_1 [\varphi_2^2 + \varphi_1^2 - 2 \varphi_1 (\ell_1 \kappa + \ell_3 \tau) + \ell_3' \varphi_2 - \ell_3 \varphi_2' + 2 \ell_2 \tau \varphi_2 - \ell_2' \varphi_1] \\
&\quad + v^2 (\varphi_1 \varphi_1' - \varphi_2 \varphi_2') + v \ell_1 \ell_2 \kappa, \\
r(s) &= \ell_2 \varphi_1' - \ell_3 \varphi_2' - \ell_1 \kappa \varphi_1 + \ell_2 \tau \varphi_2 - \ell_3 \tau \varphi_1 \\
&\quad + v [\ell_1 (\varphi_2 \varphi_2' - \varphi_1 \varphi_1') + \ell_2 \kappa (\varphi_2^2 - \varphi_1^2)] \\
&\quad + v^2 [(\varphi_1 \varphi_2 - \varphi_1 \varphi_2')(\ell_2 \varphi_2 - \ell_3 \varphi_1) + (\varphi_1^3 - \varphi_1 \varphi_2^2)(\ell_1 \kappa + \ell_3 \tau)].
\end{align*}
\]
From (2.10), the geodesic curvature of the base curve \((v = 0)\) is

\[
(k_g)_b = \frac{\ell_2 \kappa}{\sqrt{\ell_2^2 - \ell_3^2}}.
\]

(2.10a)

Then, the normal curvature of a curve \(\vec{\gamma} = \vec{\gamma}(s)\) is

\[
k_n = \frac{1}{R^{1/2} ||\vec{n}(s, \nu(s))||} \left( \nu \left[ \frac{\xi'_1 + \xi_2 \kappa - \frac{R'}{2R} \xi_1}{2} \right] (\ell_3 \varphi_1 - \ell_2 \varphi_2) 
- \left[ \frac{\xi'_2 + \xi_1 \kappa + \xi_3 \tau - \frac{R'}{2R} \xi_2}{2} \right] (\ell_3 - \nu \ell_1 \varphi_2) 
+ \left[ \frac{\xi'_3 + \xi_2 \tau - \frac{R'}{2R} \xi_3}{2} \right] (\ell_2 - \nu \ell_1 \varphi_1) \right).
\]

(2.11)

Then, the normal curvature of the base curve \((v = 0)\) is

\[
(k_n)_b = \frac{-\ell_3 \kappa}{\sqrt{\ell_2^2 - \ell_3^2}}.
\]

(2.11a)

Therefore, from (2.10a), (2.11a) and (I) we have

**Corollary 2.** The geodesic curvature \(k_g\) and the normal curvature \(k_n\) of a spacelike base curve on the time-like ruled surface \(M^t\) in \(\mathbb{R}^3_1\) space satisfy

\[
(k_g)^2 - (k_n)^2 = \kappa^2.
\]

**References**