Robust stabilization of polytopic discrete-time systems with time-varying state delay:
A convex approach

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Abstract

Convex conditions, expressed as linear matrix inequalities (LMIs), for stability analysis and robust design of uncertain discrete-time systems with time-varying delay are presented in this paper. Delay-dependent and delay-independent convex conditions are given. This paper is particularly devoted to the synthesis case where convex conditions are proposed to consider maximum allowed delay interval. It is also presented some relaxed LMIs that yield less conservative conditions at the expense of increasing the computational burden. Extensions to cope with decentralized control and output feedback control are discussed. Numerical examples, including real world motivated models, are presented to illustrate the effectiveness of the proposed approach.

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1. Introduction

Time delays are unavoidable in digital controlled systems and are intrinsically presented in many dynamical systems, such as robotics, informations, communications and networks. Metal cutting, chemical, thermal and rolling mill are examples of relevant processes that are affected by delays \cite{4,17,29}. Also, discrete-time delays (or concentrated delays) have been investigated in models of population dynamics, where they can yield...
destabilizing effects and a phenomenon of Hopf bifurcation [23]. It is well known that even small time delays can reduce the performance of systems and, in some cases, can lead them to instability. Due to their relevance, systems with delay have been widely studied in the last two decades specially on continuous-time domain [17].

In general, the available methods for stability analysis or controller design for systems with delay can be classified into delay-dependent or delay-independent in case they explicitly take into account or not, respectively, the value of the delay. Usually, delay-dependent conditions yield less conservative results, but for systems which stability does not depend on the time-delay value, the analysis performed through delay-dependent conditions can be very conservative. Also, it is interesting to note that delay-independent conditions cannot be obtained as a limit case of delay-dependent ones just by imposing the maximum delay value \( d_{\text{max}} \to +\infty \), leading to a gap between these two types of delay stability conditions [29, p. 146; 20]. Also note that in real world applications, delays are often found as time-varying and arise in systems in two main different ways: in the input [8] and in the state [28]. For systems with state delay, most of the used techniques for robust stability analysis and robust control design are based on Lyapunov–Krasovskii approach. This approach has been used to obtain convex optimization problems in terms of linear matrix inequalities (LMIs) which can be solved efficiently by specialized numerical algorithms [2]. Less attention has been paid to discrete-time systems with state delay mainly due to the possibility of study of an augmented delay-free system when the delay is known and time-invariant [16,20].

In cases of time-varying delay no obvious technique is available in the literature to deal with fundamental issues such as stability analysis or controller design, specially for convex design methods. In case of continuous-time systems, an extension of asymptotic stability analysis coupled with strictly dissipative characteristics, where norm-bounded uncertainties are considered with time-varying delay is presented in [41]. An extension to couple with absolute stability analysis of continuous-time systems with Lur'e type perturbations is found in [18] — where the delay is considered time-invariant — and dissipative analysis and synthesis is proposed in [27] for nonlinear systems with time-varying delays by means of nonconvex conditions. In special, see [1] for robust filter design depending on an upper bound on the time-derivative of the delay. However, none of these results have their discrete-time counterpart found in the literature. This lack of results is even stronger for the class of uncertain discrete-time system with time-varying state delay. This case represents a relevant challenge in control theory, since the complexity of the problem is augmented by both the uncertainties and the presence of time-varying delay. Such class of systems is investigated in this paper.

In the literature, different aspects of discrete-time systems with delayed states have been studied. In [15] LMI conditions for robust stability analysis of discrete-time delayed systems with saturation are proposed. In [37] bi-dimensional (2D) discrete-time systems with delayed states are investigated, and delay-independent conditions for norm-bounded uncertainties and constant delay are given by means of nonconvex formulations. In [26] convex conditions have been proposed for discrete-time singular systems with time-invariant delay. Discrete-time switched systems with delayed states have been studied in [13,14]. The former establishes the equivalence between the approach used here (Lyapunov–Krasovskii functions) and the one used, in general, for the stability of switched systems with time-varying delay. The latter gives nonconvex conditions for switched systems where each operation mode is subject to a norm-bounded uncertainty and with
constant delay. The problem of robust filtering for discrete-time uncertain systems with delayed states is considered in some papers. Delayed state systems with norm-bounded uncertainties are studied in [39,3,38] and with polytopic uncertainties in [7]. In the last case the delay is time-invariant. Recently, the problem of output feedback has attracted attention and [11,12,24] can be cited as examples of on going research. In special, [12] presents results for precisely known systems with time-varying delay including both static output feedback (SOF) and dynamic output feedback. The conditions, however, are presented as an interactive method that relax some matrix inequalities. In [11,24] time-varying delay is assumed and a nonlinear algorithm is proposed to obtain a stabilizing controller. In [24] the results of [11] are extended, including polytopic uncertainties and constant Lyapunov–Krasovskii matrices. An interesting application can be found in the context of network control system: although most of the studies in the literature on this subject deal with continuous-time models, nowadays there are some approaches using discrete-time models with delayed states [42]. See also [36] for a robust adaptive sliding mode control scheme applied to discrete-time systems with time-varying delay in the state and subject to norm-bounded uncertainties. In the context of discrete-time-varying systems with time-varying delay in the state, see [19,34] for convex approaches to the dynamic output feedback controller design problem.

Note that most of the problems mentioned up to here had their solutions obtained from more fundamental problems: the stability analysis or the stabilizing controller design. In this sense, since most of the cited papers use constant Lyapunov–Krasovskii matrices, it can be said that the research of these more fundamental problems is an important issue. In the literature, it is possible to observe that besides most of the conditions are structured over quadratic stability and the majority consider norm-bounded uncertainties. See, for example, [9,10,24]. Among these works, [10] presents the less conservative conditions, but with formulations only for stability analysis of norm-bounded systems. It is worth to mention that norm-bounded uncertainty is not addressed in the present paper. Therefore, it is clear that there is room for investigation on robust stability analysis and robust stabilization of polytopic uncertain discrete-time systems with time-varying delayed state. Recently, convex delay-independent conditions for robust stabilization of polytopic systems has been proposed in [20], where delay is time-varying and in [22], where delay is constant and a $H_{\infty}$ level of performance is assured to the closed-loop system.

In this paper, both delay-dependent and delay-independent convex conditions for both robust stability analysis and robust synthesis of state feedback gains for uncertain and time-invariant discrete-time systems with time-varying delay and polytopic uncertainties are proposed. The conditions employ parameter-dependent Lyapunov–Krasovskii functions and extra matrix variables. This issue is two fold: it achieves less conservative results and decouples the system matrices from the function matrices allowing better results in both analysis and design. Also, some relaxed LMIs are given to illustrate how the proposed conditions can be extended to cope with recent results using other parameter-dependent formulations. The main advantage of the present proposal is the existence of a convex formulation to the synthesis of robust feedback gains. It is shown how these conditions can be used to cope with some interesting problems such as decentralized control, SOF, controller fragility analysis and some special cases of input delay.

This paper is organized as follows: In Section 2 the problems studied in this paper are stated and some results employed here are presented. The main results are given in Section 3. Some physically motivated models are used as examples in Section 4. By last are given the conclusions.
Notation: $\mathbb{R}(\mathbb{R}^+)$ is the set of real (positive) numbers, $\mathbb{N}_+$ stands for the set of the natural numbers excluded the 0 and $\mathbb{I}[a, b]$ stands for the set of all integer $n$ such that $a \leq n \leq b$. $x_k \in \mathbb{R}^n$ is the state vector at sample time $k$. $\textbf{1}$ and $\textbf{0}$ are the identity vector of appropriate dimensions, respectively. $M = \text{block-diag}\{M_1, M_2\}$ stands for the block-diagonal matrix $M$ made up of the matrices $M_1$ and $M_2$ at the main diagonal. $M > 0 (M < 0)$ means that $M$ is positive (negative) definite. $M'$ stands for the transpose of $M$. $\ast$ is used to indicate diagonally symmetric blocks in the LMIs. $\otimes$ stands for Kronecker product. $\Phi_d$ denotes the space of discrete vector functions mapping the interval $\mathbb{I}[-d, 0]$ into $\mathbb{R}^p$ with a finite $d \in \mathbb{N}_+$. $\phi^d_i(k) \in \Phi_d$ denotes a sequence of $d+1$ vectors $x_k$ with $k \in \mathbb{I}[t-d, t]$. The $j$-th term of this sequence is $x_{t+j-1-d} = \phi^d_i(k) \in \mathbb{R}^p$. It is defined $\|\phi^d_i(k)\|_D = \max_{j \in \mathbb{I}[1, (d+1)]} \|\phi^d_i(k)\|$ where $\| \cdot \|$ stands for the Euclidian vector norm. $\Phi^\kappa_d$ is the set defined by $\Phi^\kappa_d = \{ \phi^d_i(k) : \|\phi^d_i(k)\|_D < \kappa \}$, with $\kappa \in \mathbb{R}_+$. $\hat{\phi}_d$ stands for the null sequence $\hat{\phi}_d = \{\textbf{0}, \ldots, \textbf{0}\}$.

2. Preliminaries

Consider the uncertain discrete-time system with delayed state given by

$$x_{k+1} = \tilde{A}(z)x_k + \tilde{A}_d(z)x_{k-d_k} + \tilde{B}(z)u_k$$

with $x_k \in \tilde{\phi}^d_0(k)$ for $k \in \mathbb{I}[-d, 0]$ and $k$ refers to the $k$-th sample. $\tilde{\phi}^d_0(k)$ is the initial condition, necessary to assure existence and uniqueness for the solutions of Eq. (1), with $d = \text{max}(d_k)$. $x_k = x(k) \in \mathbb{R}^n$ is the state vector, $x_{k-d_k}$ is the state vector at $d_k$ past samples and $d_k = d(k)>0$ is the time-varying delay. $u_k = u(k) \in \mathbb{R}^q$ stands for the control signal. The system matrices are denoted by $[\tilde{A}(z)\mid \tilde{A}_d(z)\mid \tilde{B}(z)] = [\tilde{A}\mid \tilde{A}_d\mid \tilde{B}](z) = \tilde{y}(z) \in \mathbb{R}^{n \times (2n+p)}$ with the uncertain parameter $z$ satisfying

$$\Omega \equiv \left\{ z : z \in \mathbb{R}^N, \sum_{i=1}^{N} z_i = 1, z_i \geq 0 \right\}$$

(2)

The matrices $\tilde{y}(z)$ belong to the polytope

$$\tilde{\mathcal{P}} \equiv \left\{ \tilde{y}(z) : \tilde{y}(z) = \sum_{i=1}^{N} \tilde{y}_i z_i, z \in \Omega \right\}$$

(3)

where the vertices $\tilde{y}_i = [\tilde{A}_i\mid \tilde{A}_d_i\mid \tilde{B}_i] \equiv [\tilde{A}\mid \tilde{A}_d\mid \tilde{B}]_i$ are known. Time delay is supposed to be time-varying subject to

$$d_k \in \mathbb{I}[d, d]$$

(4)

The following state feedback control law is considered in this paper:

$$u_k = Kx_k + K_d x_{k-d_k}$$

(5)

where $[K\mid K_d] \in \mathbb{R}^{p \times 2n}$ are constant state feedback gains assuring the robust stability of the closed-loop (1)–(4). That is, the robust stability of Eqs. (1) and (2) with Eqs. (4) and (5) is assured $\forall z \in \Omega$. This yields an uncertain closed-loop system, with time-varying delay, given by

$$x_{k+1} = A(z)x_k + A_d(z)x_{k-d_k}$$

(6)
with $x_k \in \phi_0^d(k)$ for $k \in \mathcal{I}[-\overline{d}, 0]$, $[A(x)|A_d(x)] = [A|A_d](z) = \mathcal{V}(z)$ belonging to the polytope

$$
\mathcal{P} \equiv \left\{ \mathcal{V}(z) : \mathcal{V}(z) = \sum_{i=1}^{N} Y_i z_i, z \in \Omega \right\}
$$

where the vertices $Y_i = [A_i|A_{di}] = [A|A_{di}]$, are given by

$$
Y_i = [\tilde{A}_i + \tilde{B}_i K \tilde{A}_{di} + \tilde{B}_i K_d]
$$

Note that, if the delay $d_k$ is available at each sample time, then it is possible to use both gains, $K$ and $K_d$, to improve the closed-loop performance of Eq. (6). On the other hand, if the delay $d_k$ is not available, the approach presented in this paper is still valid, being enough to make $K_d = 0$ in Eq. (5).

If $x_k \in \phi_i^d(k) = \phi_d^\sigma$ for $k \in \mathcal{I}[t-\overline{d}, t]$, then an equilibrium condition is achieved for the closed-loop system (6), since $x_{k+1} = x_k = 0$, $\forall k > t$ and $\forall z \in \Omega$.

**Definition 1.** For a given $z \in \Omega$, the trivial solution of Eq. (6) is said uniformly asymptotically stable if for any $\kappa \in \mathbb{R}_+$ such that for all initial conditions $x_k = \phi_i^d(k) \in \Phi_{d_i}^\sigma$, $k \in \mathcal{I}[-\overline{d}, 0]$, it is verified

$$
\lim_{t \to \infty} \phi_i^d(k) = 0, \quad \forall j \in \mathcal{I}[t-\overline{d}, t]
$$

This allows the following definition:

**Definition 2.** System (6) subject to Eqs. (2), (4) and (7) is said robustly stable if its respective trivial solution is uniformly asymptotically stable $\forall z \in \Omega$.

The main objective of this work is to present convex conditions that can solve the following fundamental problems:

**Problem 1.** Determine if system (6) subject to Eqs. (2), (4) and (7) is robustly stable.

**Problem 2.** Determine a pair of static feedback gains, $K$ and $K_d$, such that Eqs. (1)–(4) controlled by Eq. (5) is robustly stable.

Sufficient convex formulations for solving Problems 1 and 2 are obtained by means of parameter-dependent Lyapunov–Krasovskii functions as a counterpart of delay-dependent conditions for continuous-time systems with delayed states [17]. In this paper the following Lyapunov–Krasovskii function candidate is considered

$$
V(z, k) = \sum_{i=1}^{5} V_i(z, k) > 0
$$
with
\[ V_1(z, k) = x_k'P(z)x_k, \quad V_2(z, k) = \sum_{j=k-d_k}^{k-1} x_j'Q(z)x_j \] (10)

\[ V_3(z, k) = \sum_{\ell=2-d}^{1} \sum_{j=k+\ell-1}^{k-1} x_j'Q(z)x_j \] (11)

\[ V_4(z, k) = \sum_{\ell=-d}^{-1} \sum_{m=k+\ell}^{k-1} y_m'Z(z)y_m, \quad V_5(z, k) = \sum_{j=k-d_k}^{k-1} y_j'Z(z)y_j \] (12)

\[ y_j = x_{j+1} - x_j \] (13)

In the Lyapunov–Krasovskii function above, if only the functions \( V_v(z, k), v = 1, \ldots, 3 \) are used, as done for instance in [20], then delay-independent conditions are obtained. On the other hand, by adding functions \( V_4(z, k) \) and \( V_5(z, k) \) — see Eqs. (12) and (13) — delay-dependent convex formulation for both robust analysis stability and robust synthesis design are achieved. The parameter dependency of Eq. (9) is a relevant issue to deal with uncertain systems as it may reduce the conservatism of analysis and synthesis conditions. To be a Lyapunov–Krasovskii function, the candidate (9) must be positive definite and satisfy
\[ \Delta V(z, k) = V(z, k + 1) - V(z, k) < 0 \] (14)

\[ \forall [x_k'x_{k-d_k}'] \neq 0 \text{ and } \forall z \in \Omega. \]

The following result is employed in this paper to decouple the matrices of the parameter-dependent Lyapunov–Krasovskii candidate from the matrices of the system.

**Lemma 1 (Finsler’s Lemma).** Let \( \omega \in \mathbb{R}^n, Q(z) = Q(z)' \in \mathbb{R}^{n \times n} \) and \( B(z) \in \mathbb{R}^{m \times n} \) such that \( \text{rank} (B(z)) < n \). The following statements are equivalent:

(i) \( \omega'Q(z)\omega < 0, \forall \omega : B(z)\omega = 0, \omega \neq 0; \)
(ii) \( \exists \lambda(z) \in \mathbb{R}^{n \times m} : Q(z) + \lambda(z)B(z) + B(z)\lambda(z)' < 0. \)

**Proof.** The proof follows similar steps of the proof presented in [6] replacing the precisely known matrices by parameter-dependent matrices. ∎

For conciseness, in what follows, the dependency on \( z \) of terms \( V_v(z, k) \) is omitted and they are denoted by \( V_v(k), v = 1, \ldots, 5 \).

### 3. Main results

In this section, the conditions for the robust stability analysis of Eq. (6) and for the design of robust state feedback gains to robust stabilization of Eq. (1).
3.1. Robust stability analysis

\textbf{Lemma 2.} If there exist symmetric matrices $0 < P(x) \in \mathbb{R}^{n \times n}$, $0 < Q(x) \in \mathbb{R}^{n \times n}$, $0 < Z(x) \in \mathbb{R}^{n \times n}$, a matrix $X(x) \in \mathbb{R}^{m \times n}$, $d_k \in \mathbb{I}[d, \bar{d}]$ with $d$ and $\bar{d}$ belonging to $\mathbb{N}_*$, such that

$$\mathcal{P}(x) = \text{block-diag}\{M(x), -Q(x), -Z(x), -Z(x)\} + X(x)B(x) + B(x)X(x)' < 0$$

(15)

with

$$M(x) = \begin{bmatrix} P(x) & 0 \\ 0 & \beta Q(x) - P(x) \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \otimes (\bar{d} + 1)Z(x)$$

(16)

$$\beta = \bar{d} - d + 1$$

(17)

and

$$B(x) = [I - A(x) - A_d(x)] \begin{bmatrix} 0 & 0 \end{bmatrix}$$

(18)

is verified $\forall z \text{ admissible}$, then system (6) subject to Eq. (4) is robustly stable. Besides, Eqs. (9)–(13) are a Lyapunov–Krasovskii function assuring the robust stability of the considered system.

\textbf{Proof.} The positivity of the function (9) is clearly assured by $P(x) = P(x)' > 0$, $Q(x) = Q(x)' > 0$, $Z(x) = Z(x)' > 0$. Eq. (14) is calculated taking into account the following terms:

$$\Delta V_1(k) = x_{k+1}^tP(x)x_{k+1} - x_k^tP(x)x_k$$

(19)

$$\Delta V_2(k) = x_k^tQ(x)x_k - x_k^tQ(x)x_{k-d_k} + \sum_{i=k+1-d_{k+1}}^{k-1} x_i^tQ(x)x_i - \sum_{i=k+1-d_k}^{k-1} x_i^tQ(x)x_i$$

\leq x_k^tQ(x)x_k - x_k^tQ(x)x_{k-d_k} + \sum_{i=k+1-d_k}^{k-d} x_i^tQ(x)x_i$$

(20)

$$\Delta V_3(k) = (\bar{d} - d)x_k^tQ(x)x_k - \sum_{i=k+1-\bar{d}}^{k-d} x_i^tQ(x)x_i$$

(21)

$$\Delta V_4(k)(y_k) = \bar{d}y_k^tZ(x)y_k - \sum_{j=-\bar{d}}^{-1} y_{k+j}^tZ(x)y_{k+j}$$

\leq \bar{d}y_k^tZ(x)y_k - y_{k-d}^tZ(x)y_{k-d} - \sum_{i=k+1-\bar{d}}^{k-d} y_i^tZ(x)y_i$$

(22)

$$\Delta V_5(k) = y_k^tZ(x)y_k - y_{k-d}^tZ(x)y_{k-d} + \sum_{i=k+1-d_{k+1}}^{k-1} y_i^tZ(x)y_i - \sum_{i=k+1-d_k}^{k-1} y_i^tZ(x)y_i$$

\leq y_k^tZ(x)y_k - y_{k-d}^tZ(x)y_{k-d} + \sum_{i=k+1-\bar{d}}^{k-d} y_i^tZ(x)y_i$$

(23)
Therefore, Eq. (14) can be bounded using Eqs. (19)–(23) which yields
\[
\Delta V(k) \leq \Delta V_{\min}(k) + \Delta V_{\max}(k) \\
\leq x_{k+1}^T P(x) x_{k+1} + x_k^T \left[ \beta Q(z) - P(z) \right] x_k - x_{k-d_k}^T Q(z) x_{k-d_k} \\
+ y_k^T (\overline{d} + 1) Z(z) y_k - y_{k-d_k}^T Z(z) y_{k-d_k} < 0
\] (24)
with \( \beta \) given by Eq. (17). Replacing \( y_k \) by the right side of Eq. (13) in Eq. (24) and taking into account Eq. (6) it is possible to apply Lemma 1 with
\[
Q(z) = \text{block-diag}(M(z), -Q(z), -Z(z), -Z(\bar{z}))
\]
with \( \mathcal{X}(z) \in \mathbb{R}^{5n \times n}, M(z) \) and \( B(z) \) given by Eqs. (16) and (18), respectively, yielding \( \Psi(z) = Q(z) + \mathcal{X}(z) B(z) + B(z) \mathcal{X}(z)' < 0 \) given by Eq. (15). Thus, this is equivalent to verify Eq. (24) subject to Eq. (6), which assures the robust stability of the considered system. □

An important issue in Lemma 2 is that there is no product between the matrices of the system and the matrices of the Lyapunov–Krasovskii proposed function, Eq. (9). This is exploited in this paper to reduce conservatism in both analysis and synthesis methods. The dependency on \( \overline{d} \) and \( d \) is discussed in this paper mainly for synthesis procedure.

It is worth of mentioning that conditions given in Lemma 2 are not convex in \( x \) when this parameter belongs to a continuous domain. To overcome this issue and to provide a solution to Problem 1, the same polytopic structure of the system is assumed for the Lyapunov–Krasovskii matrices \( P(z), Q(z) \) and \( Z(z) \):
\[
P(z) = \sum_{i=1}^{N} z_i P_i, \quad Q(z) = \sum_{i=1}^{N} z_i Q_i, \quad Z(z) = \sum_{i=1}^{N} z_i Z_i, \quad z \in \Omega
\] (25)
leading to the following result.

**Theorem 1.** If there exist symmetric matrices \( 0 < P_i \in \mathbb{R}^{n \times n}, 0 < Q_i \in \mathbb{R}^{n \times n}, 0 < Z_i \in \mathbb{R}^{n \times n}, i = 1, \ldots, N \), a matrix \( \mathcal{X} \in \mathbb{R}^{5n \times n}, d_k \in \mathcal{D}[\overline{d}, \overline{d}] \) with \( \overline{d} \) and \( d \) belonging to \( \mathbb{N}_+ \), such that
\[
A_i = \text{block-diag}(M_i, -Q_i, -Z_i, -Z_i) + \mathcal{X} \mathcal{B}_i + \mathcal{B}_i' \mathcal{X}' < 0, \quad i = 1, \ldots, N
\] (26)
with
\[
M_i = \begin{bmatrix} P_i & 0 \\ 0 & \beta Q_i - P_i \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \otimes (\overline{d} + 1) Z_i
\] (27)
\[
\mathcal{B}_i = [I - A_i - A_{di} \ 0 \ 0]
\] (28)
and \( \beta \) is given by Eq. (17) is verified, then system (6) subject to Eqs. (4) and (3) is robustly stable, characterizing a solution to Problem 1. Besides, Eqs. (9)–(13) with matrices \( P(z), Q(z) \) and \( Z(z) \) given in Eq. (25) and \( z \in \Omega \) are a Lyapunov–Krasovskii function assuring the robust stability of the considered system.

**Proof.** Condition (26) is sufficient for Eq. (15), given in Lemma 2. Note that, the positivity of the function (9)–(13) with Eq. (25) is clearly assured by \( P_i = P_i' > 0, \quad Q_i = Q_i' > 0, \quad Z_i = Z_i' > 0 \). Besides assuming Eq. (25), the extra matrix \( \mathcal{X}(z) \) given in Eq. (15) is constrained to be parameter independent to obtain a convex formulation. Thus, \( \mathcal{X}(z) = \mathcal{X} \in \mathbb{R}^{5n \times n} \) is a restriction on Finsler’s multiplier, leading to a sufficient condition for
\[ \Psi(z) < 0 \] given in Eq. (15). This allows to recover \( \Psi(z) = \sum_{i=1}^{N} A_i z_i < 0 \) for \( z \in \Omega \), which is the continuous domain of uncertainty assumed for Eq. (6). \( \square \)

It is worth of mentioning that the presented approach does not introduce any model transformation neither extra dynamics. The use of Finsler’s Lemma allows the introduction of slack matrix variable \( X \), which role is to lead to less conservative results by enlarging the search space. It can be found in the literature some similar results using the addition of a null term to Eq. (24) as done by [35]. However, the use of null term can be viewed as a special case of Finsler’s lemma, where some extra variables are set to zero.

In this robust stability analysis some relaxed LMIs can be introduced to avoid the constraint made over \( \lambda(z) \). If this extra matrix is parameter dependent, the approach employed in [21,31] can be used in a similar way to get less conservative conditions. However, the drawback of this approach is to increase the computational burden. Thus, considering

\[ \lambda(z) = \sum_{i=1}^{N} x_i X_i, \quad z \in \Omega \] (29)

the following condition can be stated.

**Corollary 1.** If there exist symmetric matrices \( 0 < P_i \in \mathbb{R}^{n \times n}, 0 < Q_i \in \mathbb{R}^{n \times n}, 0 < Z_i \in \mathbb{R}^{n \times n}, i = 1, \ldots, N \), matrices \( X_i \in \mathbb{R}^{n \times n}, d_k \in \mathbb{R}[d, d], d \) belonging to \( \mathbb{N}_+ \), such that

\[ L_i = \text{block-diag}\{M_i, -Q_i, -Z_i, -Z_i\} + \lambda_i B_i + B_i^r \lambda_r < 0, \quad i = 1, \ldots, N \] (30)

\[ L_{ij} = \text{block-diag}\{M_{ij}, -Q_i, -Z_i - Z_j, -Z_i - Z_j\} + \lambda_i B_j + B_j^r \lambda_r < 0 \]

\[ \lambda_{ij} X_i B_j + B_j^r \lambda_r < 0, \quad i = 1, \ldots, N-1; j = i + 1, \ldots, N \] (31)

with \( M_i \) given in Eq. (27),

\[ M_{ij} = \begin{bmatrix} P_i & P_j \\ 0 & \beta(Q_i + Q_j) - P_i - P_j \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \otimes (\bar{d} + 1)(Z_i + Z_j) \] (32)

\[ B_m = [I - A_m - A_{d,m} 0 0], \quad m \in \{i, j\} \] (33)

and \( \beta \) given by Eq. (17) is verified, then system (6) subject to Eqs. (4) and (3) is robustly stable, characterizing a solution to Problem 1. Besides, Eqs. (9)–(13) with matrices \( P(z), Q(z) \) and \( Z(z) \) given in Eq. (25) and \( z \in \Omega \), are a Lyapunov–Krasovskii function assuring the robust stability of the considered system.

**Proof.** The proof follows from Eq. (15) by noting that it can be rewritten as \( \Psi(z) = \sum_{i=1}^{N} a_i^2 L_i + \sum_{i=1}^{N} \sum_{j=i+1}^{N} a_i a_j L_{ij} < 0 \) and imposing Eqs. (30), (31) and (29). For details, see [21] or [31]. \( \square \)

Also notice that a quadratic stability based condition can be recovered from Theorem 1 by defining \( P_i = P, \ Q_i = Q \) and \( Z_i = Z, i = 1, \ldots, N \) in Eq. (26). The resulting condition is a set of LMIs with slack matrix variables and constant parameter-independent Lyapunov–Krasovskii matrices. In this case, the system matrices can be time-varying, i.e., \( a_i \) can be replaced by \( a(k)_i \), with \( \sum_{i=1}^{N} a(k)_i = 1, a(k)_i \geq 0 \).
Another remark is that delay-independent conditions can be obtained by considering a simpler Lyapunov–Krasovskii function given by \( V(x, k) = \sum_{i=1}^{3} V_i(k) \), with \( V_i(k) \) as in Eqs. (10) and (11). In this case, the obtained conditions depend on the difference \((\overline{d} - d)\) but not on the delay itself. This condition is presented in the next corollary.

**Corollary 2.** If there exist symmetric matrices \( 0 < P_i \in \mathbb{R}^{n \times n}, 0 < Q_i \in \mathbb{R}^{n \times n}, i = 1, \ldots, N, \) \( \dot{X} \in \mathbb{R}^{3n \times n}, d_k \in \mathbb{T}[d, \overline{d}] \) with \( d \) and \( \overline{d} \) belonging to \( \mathbb{N}_u \), such that

\[
\dot{X} = \text{block-diag}\{P_i, \beta Q_i - P_i, -Q_i\} + \dot{X}\hat{B}_i + \hat{B}_i'\dot{X}' < 0, \quad i = 1, \ldots, N
\]

with

\[
\hat{B}_i = [I - A_i - A_d]
\]

and \( \beta \) given by Eq. (17) is verified, then system (6) subject to Eqs. (7), (4) and \( |d_{k+1} - d_k| \leq (\overline{d} - d) \) is robustly stable, independently of the delay value. Besides, \( V(x, k) = \sum_{i=1}^{3} V_i(x(k)) > 0 \) with \( V_r, v = 1, \ldots, 3 \), given by Eqs. (9)–(11) and matrices \( P(x) \) and \( Q(x) \) given in Eq. (25) with \( x \in \Omega \), is a Lyapunov–Krasovskii function assuring the robust stability of the considered system.

It is important to observe that the conditions of Corollary 2 cannot be obtained from Eq. (26) just by taking its limit as \( \overline{d} \to \infty \). Note that, Corollary 2 is delay-independent and, thus, it does not depend on \( d \) itself, but it does depend on the difference \( \overline{d} - d \). Also note that the same approach used in Corollary 1 can be applied in this case yielding the following corollary.

**Corollary 3.** If there exist symmetric matrices \( 0 < P_i \in \mathbb{R}^{n \times n}, 0 < Q_i \in \mathbb{R}^{n \times n}, i = 1, \ldots, N, \) integer constants \( \overline{d} \in \mathbb{N}_u, d_k \in \mathbb{T}[d, \overline{d}] \) with \( d \) and \( \overline{d} \) belonging to \( \mathbb{N}_u \), such that

\[
\dot{X} = \text{block-diag}\{P_i, \beta Q_i - P_i, -Q_i\} + \dot{X}\hat{B}_i + \hat{B}_i'\dot{X}' < 0, \quad i = 1, \ldots, N
\]

with

\[
\hat{B}_i = [I - A_i - A_d], \quad m \in \{i, j\}
\]

and \( \beta \) given by Eq. (17) is verified, then system (6) subject to Eqs. (7), (4) and \( |d_{k+1} - d_k| \leq (\overline{d} - d) \) is robustly stable, independently of the delay value. Besides, \( V(x, k) = \sum_{i=1}^{3} V_i(x(k)) > 0 \) with \( V_r, v = 1, \ldots, 3 \), given by Eqs. (9)–(11) and matrices \( P(x) \) and \( Q(x) \) given in Eq. (25) with \( x \in \Omega \), is a Lyapunov–Krasovskii function assuring the robust stability of the considered system.

**Proof.** The proof follows the same ideas described in the proof of Corollary 1. □

Obviously, if \( d = \overline{d} \), then the delay is taken as constant and uncertain. Finally, it is also worth of noting that the conditions proposed in Theorem 1 and Corollary 1 can be used to verify the robust stability of the dual system obtained by replacing \( A(x) \) and \( A_d(x) \) by
$A(x)'$ and $A_d(x)'$, respectively. This leads to similar changes on the conditions of Corollaries 2 and 3.

Another relevant issue is that the technique proposed to obtain Corollaries 1 and 3 can be improved by supposing that the matrices of the Lyapunov–Krasovskii function (9) depend polynomially on the parameter $x$. In this case, the results proposed by [30] can be similarly applied at the expense of increasing computational effort. However this extension is not dealt here, once the main objective in this paper is to obtain a convex condition for the synthesis of robust feedback gain.

It is interesting to note that the presence of the term $V_3(x,k)$ in the Lyapunov–Krasovskii function, see Eq. (11), results on conditions that are dependent on the difference $\bar{d} - d$. This difference is related to the maximum variation rate allowed to the delay, which is such that $|d_{k+1} - d_k|$ be equal to or less than $(\bar{d} - d)$. Therefore, although delay-independent, the conditions presented in Corollaries 2 and 3 depend on the maximum variation of the delay between two samples.

### 3.2. Robust feedback gain design

The robust analysis conditions given in Section 3.1 are exploited to obtain convex conditions for robust feedback gain design yielding a solution to Problem 2.

**Theorem 2.** If there exist matrices $W \in \mathbb{R}^{p \times n}$, $W_d \in \mathbb{R}^{p \times n}$, and matrices $F \in \mathbb{R}^{n \times n}$, $0 < P_i = P_i' \in \mathbb{R}^{n \times n}$, $0 < Q_i = Q_i' \in \mathbb{R}^{n \times n}$, $0 < Z_i = Z_i' \in \mathbb{R}^{n \times n}$, $i = 1, \ldots, N$, $d_k \in \mathbb{Z}[d, \bar{d}]$ with $\bar{d}$ and $d$ belonging to $\mathbb{N}_a$, such that

$$
\begin{bmatrix}
P_i + (\bar{d} + 1)Z_i + F + F' & -((\bar{d} + 1)Z_i + F'\tilde{A}_1 + W_1\tilde{B}_1) \\
* & \beta Q_i - P_i + (\bar{d} + 1)Z_i \\
* & * & -Q_i
\end{bmatrix} < 0,
$$

with $\beta$ given by Eq. (17) is verified, then the uncertain system (1)–(4) subject to the control law (5) with

$$K = WF^{-1} \quad \text{and} \quad K_d = W_dF^{-1}$$

is robustly stable, thus providing a solution to Problem 2. Besides, Eqs. (9)–(13) with matrices $P(x)$, $Q(x)$ and $Z(x)$ given in Eq. (25) and $x \in \Omega$, are a Lyapunov–Krasovskii function assuring the robust stability of the closed-loop system.

**Proof.** Firstly, note that $x_{k_1} = Ax_k + A_dx_{k_d}$ is stable if and only if $x_{k_1} = A'x_k + A'_d x_{k_d}$ is stable. Then, from Theorem 1 by replacing $A_i$ and $A_{di}$ by $(\tilde{A}_1 + \tilde{B}_1K)'$ and $(\tilde{A}_{di} + \tilde{B}_1K_d)'$, respectively, defining $\chi = [F \; 0_{n \times 4n}]$, $K_F = W$ and $K_dF = W_d$, and noting that the resulting condition can be simplified once the required positiveness of $Z_i$ is already included in the hypothesis of the theorem. Also note that the regularity of matrix $F$ is assured by block (1,1) of Eq. (39), once it yields $F + F' < -(P_i + (\bar{d} + 1)Z_i) < 0$, $i = 1, \ldots, N$. □

Similarly what has been done in the analysis case, for the synthesis condition it is also possible to present a delay-independent condition that allows a delay variation given by $|d_{k+1} - d_k| \leq \bar{d} - d$. This is presented in the next corollary.
Corollary 4. If there exist matrices $W \in \mathbb{R}^{p \times n}$ and $W_d \in \mathbb{R}^{p \times n}$, $F \in \mathbb{R}^{n \times n}$, $0 < P_i = P_i'$, $0 < Q_i = Q_i'$, $i = 1, \ldots, N$, $|d_{k+1} - d_{k}| \leq \overline{d} - \underline{d}$ with $\overline{d}$ and $\underline{d}$ belonging to $\mathbb{N}_+$, such that
\[
\begin{bmatrix}
P_i + F + F' & -(F' \tilde{A}_i + W' \tilde{B}_i) & -(F' \tilde{A}'_{di} + W' \tilde{B}'_i) \\
\ast & \beta Q_i - P_i & 0 \\
\ast & \ast & -Q_i
\end{bmatrix} < 0, \quad i = 1, \ldots, N
\] (41)
with $\beta$ given by Eq. (17) is verified, then system (1) subject to Eqs. (2)–(4) is robustly stabilizable, for any value of the delay $d_k$, by means of static robust feedback gains $K$ and $K_d$ given in Eq. (40). Besides, $V(x, k) = \sum_{v=1}^{3} V_v(x, k) > 0$ with $V_v$, $v = 1, \ldots, 3$, given by Eqs. (9)–(11) and matrices $P(x)$ and $Q(x)$ given in Eq. (25) with $z \in \Omega$, is a Lyapunov–Krasovskii function assuring the robust stability of the closed-loop.

Proof. The proof can be obtained from the conditions of Corollary 2, replacing $A_i$ and $A_{di}$ by $(\tilde{A}_i + \tilde{B}_i K)'$ and $(\tilde{A}'_{di} + \tilde{B}'_i K_d)'$, respectively, and defining $\mathcal{X} = [F 0_{n \times 2n}]$, $KF = W$ and $K_d F = W_d$. □

Quadratic stability conditions can be obtained from both Eqs. (39) and (41) simply imposing $P_i = P$, $Q_i = Q$ and $Z_i = Z$, $i = 1, \ldots, N$ ($Z = 0$ in Eq. (41)). In this case, system matrices can be time-varying encompassing, for example, actuators faults.

Note that conditions presented in Corollary 1 or 3 are not suitable for a convex synthesis procedure once the parameter dependency on $\mathcal{X}_i$ does not allow, by convex means, the design of constant gains $K$ and $K_d$.

The proposed conditions can be specialized to deal with some constrained control design such as decentralized control, SOF and input delay. This is discussed in the sequence.

3.2.1. Decentralized control

This issue refers to decentralized control, employed when interconnected systems must be controlled by means of local information only. In this case, decentralized control gains $K = K_D$ and $K_d = K_{dD}$ can be obtained by imposing block-diagonal structure to matrices $W$, $W_d$ and $F$ as follows:
\[
W = W_D = \text{block-diag}\{W^1, \ldots, W^q\}
\]
\[
W_d = W_{dD} = \text{block-diag}\{W^1_d, \ldots, W^q_d\}
\]
\[
F = F_D = \text{block-diag}\{F^1, \ldots, F^q\}
\]
where $q$ denote the number of defined subsystems. In this case, one gets the robust block-diagonal state feedback gains $K_D = W_D F_D^{-1}$ and $K_{dD} = W_{dD} F_D^{-1}$. It is worth to mention that the matrices of the Lyapunov–Krasovskii function, $P(x)$, $Q(x)$ and $Z(x)$, do not have any restrictions in their structures, which leads to less conservative designs.

3.2.2. Static output feedback

This is the case when only a linear combination of the states is available for feedback and the output signal is given by $y_k = \tilde{C} x_k$. If $\tilde{C}$ is full row rank, then it is always possible to find a regular matrix $T$ such that $\tilde{C} T^{-1} = [I_q 0]$. Using such matrix $T$ in a similarity
transformation applied to Eq. (1) it leads to
\[ \tilde{x}_{k+1} = \tilde{A}(z)\tilde{x}_k + \tilde{A}_d(z)\tilde{x}_{k-dk} + \tilde{B}(z)u_k \] (42)
where $\tilde{A}(z) = T\tilde{A}(z)T^{-1}$, $\tilde{A}_d(z) = T\tilde{A}_d(z)T^{-1}$ and $\tilde{B}(z) = T\tilde{B}(z)$, $\tilde{x}_k = TX_k$ and the output signal is given by $y_k = [I_q \ 0]\tilde{x}_k$. Thus, the objective here is to find robust static feedback gains $K \in \mathbb{R}^{p \times q}$ and $K_d \in \mathbb{R}^{p \times q}$ such that Eq. (42) is robustly stabilizable by the control law
\[ u_k = K_y + K_d y_{k-dk} \] (43)
These gains can be determined by using the conditions of Theorem 2 or Corollary 4 and imposing the following structures:
\[ F = \begin{bmatrix} F_{01}^{11} & 0 \\ F_{01}^{21} & F_{02}^{22} \end{bmatrix}, \quad W = \begin{bmatrix} W_K & 0 \\ W_{K_d} \end{bmatrix}, \quad W_d = \begin{bmatrix} W_{K_d} & 0 \end{bmatrix} \]
with $F_{01}^{11} \in \mathbb{R}^{q \times q}$, $F_{01}^{21} \in \mathbb{R}^{(n-q) \times q}$, $F_{02}^{22} \in \mathbb{R}^{(n-q) \times (n-q)}$, $W_K \in \mathbb{R}^{p \times q}$, $W_{K_d} \in \mathbb{R}^{p \times q}$ which yields
\[ K = [K|0] \quad \text{and} \quad K_d = [K_d|0] \]
Note that, similar to the decentralized case, no constraint is taken over the Lyapunov–Krasovskii function matrices leading to less conservative conditions.

3.2.3. Input delay
A relevant issue in Control Theory is the study of stability and stabilization of input delay systems, which is quite frequent in many real systems [39,3]. The approach presented in this paper can be used to handle input delays as summarized in the sequel. Consider the controlled system given by
\[ x_{k+1} = \tilde{A}(z)x_k + \tilde{B}(z)u_{k-dk} \] (44)
with $\tilde{A}(z)$ and $\tilde{B}(z)$ belonging to polytope (3), $\tilde{A}_{di} = 0$ and $z \in \Omega$. This type of system is detailed investigated in [40], where the problem is converted into an optimization problem in Krein space with a stochastic model associated. Here, the delayed input control signal is considered as
\[ u_{k-dk} = K_d x_{k-dk} \] (45)
The closed-loop system is given by
\[ x_{k+1} = \tilde{A}(z)x_k + \tilde{B}(z)K_d x_{k-dk} \] (46)
Thus, with known $K_d$ closed-loop system (46) is similar to Eq. (6) with $A(z)$ replaced by $\tilde{A}(z)$ and $A_{di}(z)$ by $\tilde{B}(z)K_d$. This leads to simple analysis stability conditions obtained from Theorem 1, replacing $A_i$ by $\tilde{A}_i$ and $A_{di}$ by $\tilde{B}(z)K_d$, $i = 1, \ldots, N$. Besides, similar replacements can be used with conditions presented in Corollaries 1–3. It is interesting to note that this issue is two fold once both controller fragility and input delay can be addressed by this formulation. In the former it is required that no uncertainty affects the input matrix, i.e., $\tilde{B}(z) = \tilde{B}$, $\forall z \in \Omega$, while the latter can be used to investigate, for example, the bounds of stability of a closed-loop system where there exists delay due to digital processing or information propagation.

In case of $K_d$ design, it is possible to take similar steps with conditions of Theorem 2 and Corollary 4. In this case, it is necessary to impose $\tilde{A}_i = \tilde{A}_i$, $A_{di} = 0$, $i = 1, \ldots, N$ and $W = 0$ leading to $K = 0$. Also, the Schur stability of $\tilde{A}(z)$ is required in this case. Although
possible, the design of a stabilizing $K_d$ for Eq. (46) usually leads to a gain with very small entries modulus. On the other hand, since Theorem 2 gives a convex way to determine $K$ and $K_d$, it is expected that an approach similar to that used in this paper can be used to derive convex conditions to deal with some performance specification, such as guaranteed $\mathcal{H}_\infty$ index. In this case, the design of $K$ and $K_d$ or only $K_d$ certainly becomes even more interesting. This also can be the case when dealing with saturating actuators as done in the continuous-time domain by [33].

Finally, observe that static delayed output feedback control can be additionally addressed here by considering what is pointed out in Section 3.2.2.

3.3. Numerical complexity

The numerical complexity of the proposed conditions depend on the number of scalar variables, $\mathcal{V}$, and on the number of rows, $\mathcal{G}$, in the LMIs. In this paper it has been used Matlab with the interface YALMIP [25] and the solver SeDuMi [32] whose number floating point operations performed to the same problems has an order given by $\mathcal{V}^2\mathcal{G}^{5/2} + \mathcal{G}^{7/2}$. It has been verified by the authors that SeDuMi usually presents an important time reduction in solving the proposed optimization problems comparatively to other commercially available softwares, for the tests performed in this paper. For the conditions present here, it is possible to determine the following relationships:

$$
\begin{align*}
\mathcal{V}_{T1} &= \frac{n^2(3N + 10) + 3Nn}{2}, & \mathcal{G}_{T1} &= 8Nn, & \mathcal{V}_{T2} &= \frac{n^2(3N + 2) + n(3N + 4p)}{2}, & \mathcal{G}_{T2} &= 5Nn, \\
\mathcal{V}_{C2} &= \frac{n^2(2N + 6) + 2Nn}{2}, & \mathcal{G}_{C2} &= 6Nn, & \mathcal{V}_{C4} &= n^2(N + 1) + n(N + 2p), & \mathcal{G}_{C4} &= 3Nn, \\
\mathcal{V}_{C1} &= \frac{(13n + 3)Nn}{2}, & \mathcal{G}_{C1} &= (N^2 + 6N)n/2, & \mathcal{V}_{C3} &= (6n + 1)Nn, & \mathcal{G}_{C3} &= (N^2 + 4N)n/2.
\end{align*}
$$

Thus, it is evident that the number of scalar variables and the number of LMI lines increase significantly with the conditions of Corollaries 1 and 3 w.r.t. Theorem 1 and Corollary 2, respectively.

4. Numerical examples

In all examples it has been used a Intel Core 2 Duo, T 7600, 2.33 GHz, with 2 Gb of RAM, using MatLab, the parser YALMIP [25] and SeDuMi toolbox [32].

**Example 1** (Stabilization design). A physically motivated problem is considered in this example. It consists of a fifth order state space model of an industrial electric heater investigated in [4]. This furnace is divided into five zones, each of them with a thermocouple and a electric heater as indicated in Fig. 1. The state variables are the temperatures in each zone ($x_1, \ldots, x_5$), measured by thermocouples, and the control inputs are the electrical power signals ($u_1, \ldots, u_5$) applied to each electric heater. The temperature

![Fig. 1. Schematic diagram of the industrial electric heater.](image-url)
of each zone of the process must be regulated around its respective nominal operational conditions (see [4] for details). The dynamics of this system is slow and can be subject to several load disturbances. Also, a time-varying delay can be expected, since the velocity of the displacement of the mass across the furnace may vary. A discrete-time with delayed state model for this system has been obtained as given by Eq. (1) with $d = 15$, where

$$
\tilde{A} = \tilde{A}_0 = \begin{bmatrix}
0.97421 & 0.15116 & 0.19667 & -0.05870 & 0.07144 \\
-0.01455 & 0.88914 & 0.26953 & 0.11866 & -0.22047 \\
0.06376 & 0.12056 & 1.00049 & -0.03491 & -0.02766 \\
-0.05084 & 0.09254 & 0.28774 & 0.82569 & 0.02570 \\
0.01723 & 0.01939 & 0.29285 & 0.03544 & 0.87111
\end{bmatrix}
$$

(47)

and

$$
\tilde{A}_d = \tilde{A}_{d0} = \begin{bmatrix}
-0.01000 & -0.08837 & -0.06989 & 0.18874 & 0.20505 \\
0.02363 & 0.03384 & 0.05282 & -0.09906 & -0.00191 \\
-0.04468 & -0.00798 & 0.05618 & 0.00157 & 0.03593 \\
-0.04082 & 0.01153 & -0.07116 & 0.16472 & 0.00083 \\
-0.02537 & 0.03878 & -0.04683 & 0.05665 & -0.03130
\end{bmatrix}
$$

(48)

where $\tilde{A}_0$, $\tilde{A}_{d0}$, and $\tilde{B}_0$ are called here the nominal matrices of this system. Note that, this nominal system has unstable modes. The design of a stabilizing state feedback gain for this system has been considered in [4] by using optimal control theory. In this case, an augmented delay-free system with order equal to 85 has to be used and the control gain has been determined for the time-invariant delay $d = 15$ by means of a Riccati equation.

Table 1

Feasible structures (✓) achieved by conditions of Theorem 2 (T2), Corollary 4 (C4), and quadratic stability approach (QS) obtained from Theorem 2 and Corollary 4, respectively QS-T2 and QS-C4. The symbol ‘−’ means that no feasible solution has been obtained.

<table>
<thead>
<tr>
<th>Structure</th>
<th>$K$ and $K_d$</th>
<th>Only $K$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>T2</td>
<td>C4</td>
</tr>
<tr>
<td>{1,1,3}</td>
<td></td>
<td></td>
</tr>
<tr>
<td>{1,3,1}</td>
<td></td>
<td></td>
</tr>
<tr>
<td>{3,1,1}</td>
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</tr>
<tr>
<td>{2,3}</td>
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</tr>
<tr>
<td>{3,2}</td>
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<td></td>
</tr>
<tr>
<td>{1,4}</td>
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<td>✓</td>
</tr>
<tr>
<td>{4,1}</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>{5}</td>
<td>✓</td>
<td>✓</td>
</tr>
</tbody>
</table>
In [22] an uncertain system given by
\[ \dot{A}(\rho) = (1 + \rho)\hat{A}_0, \quad \dot{A}_d(\eta) = (1 + \eta)\hat{A}_{d0} \quad \text{and} \quad \hat{B}(\sigma) = B_d(\sigma) = (1 + \sigma)\hat{B}_0 \] (50)
has been considered in the context of $\mathcal{H}_\infty$ control with uncertain and time-invariant delay and delay-independent conditions. In this case, the parameters $\rho$, $\eta$ and $\sigma$ are the uncertainties. This set of uncertainties defines a polytope with eight vertices, obtained by combination of the extremum values of uncertain parameters.

Here the stabilization of this uncertain system with \[ |\rho| \leq 0.3, \quad |\eta| \leq 0.3 \quad \text{and} \quad |\sigma| \leq 0.08 \] (51)
is investigated under time-varying delay assumption $10 \leq d_k \leq 20$. Both, delay-dependent and delay-independent conditions have been used. Different gain structures, as discussed in Section 3.2.1, have been investigated. The obtained results are summarized in Table 1. The size of each square block in the decentralized control structure is indicated in the first column.

Note that quadratic stability conditions (QS) identified in Table 1 refer to the conditions of Theorem 2 and of Corollary 4 with $P_i = P$, $Q_i = Q$ and $Z_i = Z$, $i = 1, \ldots, N$. In Table 1, the notation $\{s_1, s_2, s_3\}$ used in the first column means that the controller is structured into a block-diagonal matrix where the submatrices have dimensions $s_i \times s_i$, $i = 1, 2, 3$. These conditions can be obtained directly from the conditions of Theorem 2 or Corollary 4 and, in this case, they will have extra matrices decoupling system matrices from the Lyapunov–Krasovskii function. This is not verified if quadratic stability conditions can be obtained from the conditions of Theorem 2 or Corollary 4 and, in this column.

It is clear by this example that some decentralized control structures are possible only by using $K$ and $K_d$, or equivalently, $x_k$ and $x_k-d_k$ in the feedback loop. These structures, namely $\{3, 1, 1\}$, $\{2, 3\}$, $\{3, 2\}$, $\{4, 1\}$ and $\{1, 4\}$, can be of interest in real applications since their respective control implementations can be less expensive than using full controllers. In special, note that for structures $\{3, 1, 1\}$, $\{2, 3\}$, $\{3, 2\}$ the quadratic stability based approach cannot achieve a feasible solution, while the present proposal does.

In special, consider the control structure given by $\{3, 1, 1\}$ and suppose that the delay value ($10 \leq d_k \leq 20$) is available at each control instant. In this case the following gains are obtained with Theorem 2:

\[
K = \text{block-diag} \left\{ \begin{array}{ccc} -1.3166 & -0.9610 & -0.2231 \\ -1.1835 & -2.3621 & 0.1941 \\ 0.3637 & 0.7230 & -0.9838 \end{array} \right\}, -1.1373, -1.9104
\]

\[
K_d = \text{block-diag} \left\{ \begin{array}{ccc} -0.0847 & 0.1932 & -0.2354 \\ -0.1451 & 0.1269 & -0.2713 \\ 0.1243 & -0.0573 & 0.1432 \end{array} \right\}, -0.2156, -0.0546
\]
Suppose that system (1) with Eqs. (47)–(51) is subject to an initial condition given by \( \phi_0^{20}(k) \), where each vector belonging to \( \phi_0^{20}(k) \) is equal to \([10, -10, 5, -5, 8]^T\). These values correspond to typical deviations in the real furnace zones temperature studied in [4]. In Fig. 2 it is shown the states behavior of the closed-loop system. At each subplot is shown eight curves, one for each vertex of the polytope describing Eqs. (47)–(51). The time-varying delay employed in these simulations is given by \( d_k = 15 + \text{round}(5\sin(0.4\pi k)) \), as shown in Fig. 3. It is interesting to note that the majority of the time-responses reach the zero between samples 5 and 8. However, the delay, typically about 15, acts as

![Fig. 2. State response for each vertex of Eqs. (47)–(51).](image)

![Fig. 3. Time-varying delay used in the simulations.](image)
a perturbation for almost time-response about samples between 10 and 20. For the curves presented in Fig. 3 it is possible to say that they reach the equilibrium condition after 40 samples.

**Example 2 (Robust stability analysis).** Consider Eq. (6) with

\[
A = \begin{bmatrix} 0.6 & 0 \\ 0.35 & 0.7 \end{bmatrix}, \quad A_d = \begin{bmatrix} 0.1 & 0 \\ 0.2 & 0.1 \end{bmatrix}
\]  \hspace{1cm} (52)

This system has been investigated in [24]. Using Theorems 1, 2 and 5 of [24] with \(d = 1\) it is achieved, respectively, \(\overline{d} = 9, 12\) and 8. Using Corollary 2 of the present proposal the stability is assured for \(1 \leq d_k \leq 8\), the same bound achieved by [24, Theorem 5], which is the only one from [24] that can deal with uncertain system. Once Corollary 2 and [24, Theorem 5] are delay-independent, their conditions can assure the stability of the considered system for \(d_k \in [d, d + 7]\) with \(1 \leq d \leq \infty\). Besides, the delay-dependent conditions provide in Theorems 1 and 2 of [24] cannot be used to deal with uncertain systems.

Consider a slight modified system where matrices \(A\) and \(A_d\) are both affected by a multiplicative uncertainty given by \((1 + \rho)\). This yields a polytopic representation with two vertices given by \(V_i = [A_i A_{di}], \) with \(A_i = A(1 + (-1)^i \rho), A_{di} = A_d(1 + (-1)^i \rho), \) \(i = 1, 2\) and \(\rho \geq 0\). Fig. 4 has been obtained by varying \(\rho\) from 0 and searching with conditions from Corollary 2 and from [24, Theorem 5] the maximum value of \(\overline{d}\) that this system is verified to be stable, for \(d = 1\). Both conditions achieve the same bounds on \(\overline{d}\) and \(\rho\). However, the numerical complexity of condition [24, Theorem 5] calculated as indicated in Section 3.3 is about 67.8% greater than the numerical complexity of Corollary 2. Thus, in this example, the information achieved by [24, Theorem 5] can be obtained by Corollary 2 in a less expensive way. Also note that, due to the polytopic representation of the uncertainties, norm-bounded based approaches, such as [9,10], does not apply to this case.

**Example 3 (Input delay and controller fragility analysis).** In this example, it is considered a physically motivated model of a cold rolling mill system found in [5], where a sample
period of $T_s = 0.1$ s has been used. The nominal delay-free model is assumed for this system with matrices $\tilde{A}_0$ and $\tilde{B}_0$ given by

$$
\tilde{A}_0 = \begin{bmatrix} 0_{1\times 9} & \zeta_0 \\ I_{9\times 9} & 0_{9\times 1} \end{bmatrix} \quad \text{and} \quad \tilde{B}_0 = \begin{bmatrix} 2.76 & -1.35 & -0.46 \\ \frac{1}{C_2} & \frac{1}{C_2} & \frac{1}{C_2} \end{bmatrix}
$$

with $\zeta_0 = 0.112$. Suppose there is a time-varying delay, $d_k$, in the feedback loop and that $\zeta$ is uncertain and given by $\zeta = \zeta_0 \pm 5\%$. Thus, it is possible to obtain a polytopic representation of the corresponding $A(z)$ and $B(z)$ given by

$$
\tilde{A}_i = [\tilde{A}_i | \tilde{B}_i], \quad i = 1, 2
$$

where $\tilde{A}_1$ and $\tilde{A}_2$ are computed using the extremum values of $\zeta$, $\tilde{B}_1 = \tilde{B}_2 = B_0$, $n = 10$ and $p = 3$. Suppose that this system is controlled by a static feedback gain

$$
K_d = \begin{bmatrix} -5.1784 & 0.6114 & -1.6754 & 4.1266 & -6.8142 \\ 0 & 0 & 0 & 0 & 0 \\ -4.0239 & -1.9537 & -0.8677 & -0.4145 & -0.2618 \\ -1.6566 & 3.4657 & 4.3534 & 1.9838 & -0.6713 \\ 0 & 0 & 0 & 0 & 0 \\ -0.2294 & -0.2557 & -0.3091 & -0.3361 & 7.0238 \end{bmatrix} \times 10^{-2}
$$

(55)

By using Theorem 1 the closed-loop system is robustly stable for $1 \leq d_k \leq 3$. That is, the state used in the feedback action can be delayed up to 0.3 s from the actual instant.

Consider that there is an implementation error on the value of $K_d$ up to $\zeta\%$. In this case, the gain $K_d$ is replaced by an uncertain gain $\tilde{K}_d$ given by $\tilde{K}_d = K_d(1 + \varepsilon)$, with $0 \leq \varepsilon \leq \zeta/100\%$. This yields a 4 vertex polytope, with vertices computed by combining the extremum values of $\tilde{K}_d$ and $\tilde{A}$. By means of conditions of Theorem 1 it is made a search on the maximum value of $p$ such that the closed-loop system remains stable. It is found that $\zeta_{T_1} = 0.3\%$. If $\tilde{d}$ is reduced from 3 to 2, then it is found that the uncertainty allowed in $\tilde{K}_d$ can be admitted up to $\zeta_{T_1} = 22.9\%$. This shows that, in this example, the fragility of the controller is very sensitive to the maximum value of $d_k$.

5. Conclusions

Delay-dependent and delay-independent convex conditions for both robust stability and robust stabilizability of uncertain discrete-time systems with time-varying delay in the state are given. All system matrices are supposed to be affected by polytopic-type uncertainties. The proposed conditions employ parameter-dependent Lyapunov–Krasovskii functions and extra-variables yielding less conservative results than other conditions available in the literature. Also, these slack variables decouple the system matrices from the function matrices, allowing a convex formulation in the synthesis case. All conditions encompass quadratic stability based ones as special cases. Relaxed LMI conditions have been presented for the robust stability case, as a way to reduce the conservatism of the approach. It has been shown that robust stabilizability conditions can be easily used to cope with decentralized control, static output feedback, input delay and control fragility analysis. Numerical examples are presented, including models based on two distinct real processes, to illustrate the efficiency of the proposed conditions.
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References


