RADII AND CENTERS IN ITERATED LINE DIGRAPHS

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Abstract

We show that the out-radius and the radius grow linearly, or “almost” linearly, in iterated line digraphs. Further, iterated line digraphs with a prescribed out-center, or a center, are constructed. It is shown that not every line digraph is admissible as an out-center of line digraph.

Keywords: center, digraph, line digraph, radius.

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1. Introduction

Line digraphs are useful and important in various problems. In this paper we concentrate on iterated line digraphs, and on their radii (in [4] the diameter and average distance of iterated line digraphs are studied). In [1] Aigner proved the following theorem.

Theorem A. If $D$ is a strongly connected digraph, then $r^+(D) \leq r^+(L(D)) \leq r^+(D) + 1$.

Here, $r^+$ denotes the out-radius of a digraph. We extend this result to a larger class of digraphs (see Lemma 3.2). This enables us to examine the behavior of radii of iterated line digraphs (Theorems 3.3, 3.4, and 3.6). In particular, it is shown that if $D$ is strongly connected then the out-radius of $L^i(D)$ increases linearly with $i$ while the radius of $L^i(D)$ differs from $i$ by
at most a constant (depending only on $D$). This should be contrasted with iterated line graphs, the radius of which satisfies inequalities [6]:

$$i - \sqrt{2 \log_2 i + c_G} < r(L^i(G)) < i - \sqrt{2 \log_2 i + c'_G},$$

and hence its growth is slower than linear. (Here, $c_G$ and $c'_G$ are constants depending only on $G$, and $G$ is any graph different from a path, a cycle, and a claw $K_{3,1}$.)

Further, we consider centers in iterated line digraphs. It is easily seen that for every digraph $D$ there is a digraph $H$ having $D$ as its center (for the graph version of this result, see [3, p.41]). We generalize this observation by investigating conditions under which $L^i(D)$ is a center of some $L^i(H)$ for all $i$ (or for all $i$ up to some fixed number). Finally, we show that there exist line digraphs which are not out-centers of any line digraphs. This may again be compared with the case of graphs, where every line graph serves as a center of some line graph [5].

The outline of this paper is as follows. In section 2 we build our basic tool for counting the distances in iterated line digraphs. In section 3 we examine the functions $r^+(L^i(D))$ and $r(L^i(D))$. Finally, section 4 is devoted to out-centers and centers in iterated line digraphs.

2. Preliminaries

Let $D$ be a digraph. As usual, by $V(D)$ we denote the node set of $D$ and by $E(D)$ the arc set of $D$; $id_D(u)$ denotes the input degree and $od_D(u)$ the output degree of a node $u$ in $D$. If $u$ and $v$ are nodes in $D$, then $d_D(u,v)$ denotes the length of the shortest path from $u$ to $v$ in $D$. If there is no path from $u$ to $v$, we set $d_D(u,v) = \infty$. Throughout the paper, by a path (a cycle) we always mean a directed path (a directed cycle).

The line digraph $L(D)$ of a digraph $D$ is a digraph whose nodes are the arcs of $D$, with two nodes $uv$ and $xy$ joined by an arc in $L(D)$ if and only if $v = x$. If $D$ has no arcs, then $L(D)$ is an empty digraph. By $L^0(D)$ we denote the digraph $D$. The $i$-iterated line digraph of $D$, $L^i(D)$, is $L(L^{i-1}(D))$ where $i \geq 1$.

Let $u$ be a node in $D$. Then:

- **out-eccentricity** of $u$ is $e^+_D(u) = \max\{d_D(u,v) : v \in V(D)\}$;
- **in-eccentricity** of $u$ is $e^-_D(u) = \max\{d_D(v,u) : v \in V(D)\}$;
- **eccentricity** of $u$ is $e_D(u) = \max\{e^+_D(u), e^-_D(u)\}$.

Using various eccentricities we obtain various radii and various centers. The **out-radius** $r^+(D)$ (in-radius $r^-(D)$, radius $r(D)$) is the minimum value of
$e^+_D(u)$, $e^-_D(u)$ over all nodes $u$ of $D$; and the out-center $C^+(D)$ (in-center $C^-(D)$, center $C(D)$) is the subgraph of $D$ induced by nodes with the minimum out-eccentricity (in-eccentricity, eccentricity).

Let $D'$ arise from $D$ by reversing the orientation of all arcs. Then $e^+_D(u) = e^-_D(u)$ for every node $u$ in $D$, and hence, $r^+(D) = r^-(D')$ and $C^+(D) = C^-(D')$. This observation allows us to restrict our considerations to radii $r^+$ and $r$ and to centers $C^+$ and $C$ only.

Definitions not included here can be found in [2] or [3].

We remark that in Lemma 2.1 the subtrail $x_0, x_1, \ldots, x_j$ determines a trail in $L^{i-j}(D)$, and there is one-to-one correspondence between the $j$-histories (i.e., the trails of length $j$ in $L^{i-j}(D)$) and the nodes in $L^i(D)$. The $j$-history $B^j(u)$ will be abbreviated to a history $B(u)$ if $i = j$. Note that the history allows us to represent a node of $L^i(D)$ in $D$. The following lemma enables us to count distances in $L^i(D)$ using the distances in $D$.

**Lemma 2.1.** Let $x_0, x_1, \ldots, x_n$ be the shortest trail in $D$ (if such exists) such that $(x_0, x_{i+1}, \ldots, x_{n-1}) = B(u)$ and $(x_{i-1}, x_{i+1}, \ldots, x_{n-1}) = B(v)$. Then $d_{L_i(D)}(u,v) = n - i$. Moreover, $d_{L_i(D)}(u,v) = \infty$ if there is no required trail in $D$.

**Proof.** Let $a$ and $b$ be two adjacent nodes in $L^i(D)$, $B(a) = (a_0, \ldots, a_i)$ and $B(b) = (b_0, \ldots, b_i)$. Moreover, let $B^i(a) = (a, y)$ and $B^i(b) = (w, z)$. Then $a$ and $b$ are adjacent in $L^i(D)$ if and only if $y = w$. Thus $a_j = b_{j-1}$, $1 \leq j \leq i$, and $a_0, a_1, \ldots, a_i, b_i$ determines a trail in $D$. This implies $d_{L^i(D)}(u,v) = n - i$ as $x_0, x_1, \ldots, x_n$ is the shortest trail in $D$ satisfying the assumptions of the lemma.

Clearly, if there is no required trail in $L^i(D)$, then $d_{L^i(D)}(u,v) = \infty$.

We remark that in Lemma 2.1 the subtrail $x_i, x_{i+1}, \ldots, x_{n-1}$ is a path and $d_D(x_i, x_{n-i}) = n - 2i$.

**3. Radii in Iterated Line Digraphs**

First we introduce results concerning the out-radius of iterated line digraphs.

**Lemma 3.1.** Let $D$ be a digraph with out-radius $t < \infty$ and let $u$ be a node in the out-center of $D$ such that $id_D(u) = 0$. If $L^i(D)$ is not empty, then either $t - i \leq r^+(L^i(D)) \leq t$ or $r^+(L^i(D)) = \infty$. 
**Proof.** Suppose that \( r^+(L^i(D)) < \infty \). Since \( L^i(D) \) is not empty, there is a node \( x \) in \( L^i(D) \) such that \( B(x) = \langle u, x_1, \ldots, x_i \rangle \). Now \( id_D(u) = 0 \) implies that \( x \) is the unique node in the out-center of \( L^i(D) \).

Let \( z \) be a node in \( L^i(D) \), \( B(z) = \langle z_0, \ldots, z_i \rangle \), and let \( u, y_1, \ldots, y_n \) be the shortest trail in \( D \) such that \( y_{n-i+j} = z_j, 0 \leq j \leq i \). Then \( y_j = x_j, 1 \leq j \leq i \), as \( r^+(L^i(D)) = e^+_{L^i(D)}(x) < \infty \). By Lemma 2.1, \( d_{L^i(D)}(x, z) = d_{L^i(D)}(u, z_0) \), and hence \( r^+(L^i(D)) \leq t \).

Let \( v \) be a node in \( D \) with \( d_D(u, v) = t \). Suppose that \( t \geq i \). Then there is a node \( z \) in \( L^i(D) \) with \( B(z) = \langle z_0, \ldots, z_i-1, v \rangle \). Now \( d_{L^i(D)}(x, z) = d_D(u, z_0) \geq t - i \), and hence \( r^+(L^i(D)) \geq t - i \).

By Lemma 3.1 if a digraph \( D \) with out-radius \( t < \infty \) contains no cycle and \( L^i(D) \) is not empty, then either \( t - i \leq r^+(L^i(D)) \leq t \) or \( r^+(L^i(D)) = \infty \).

**Lemma 3.2.** Let \( D \) be a digraph with out-radius \( t < \infty \) and let \( u \) be a node in the out-center of \( D \) such that \( id_D(u) \geq 1 \). Then \( t \leq r^+(L^i(D)) \leq t + i \).

**Proof.** As \( id_D(u) \geq 1 \), there is a node \( v \) in \( D \) with \( vu \in E(D) \). Since \( r^+(L^i(D)) < \infty \), we have \( d_D(u, v) < \infty \), and thus \( u \) lies in a cycle in \( D \). Hence there is a node \( x \) in \( L^i(D) \) with \( B(x) = \langle x_0, \ldots, x_{i-1}, u \rangle \). Since \( e^+_{L^i(D)}(u) = t \), we have \( r^+(L^i(D)) \leq e^+_{L^i(D)}(x) \leq e^+_{D}(u) + i = t + i \), by Lemma 2.1.

Let \( z \) be a node in the out-center of \( L^i(D) \), \( B(z) = \langle z_0, \ldots, z_i \rangle \), and let \( w \) be a node in \( D \) such that \( d_D(z_i, w) = e^+_{D}(z_i) \). Clearly, there is a node, say \( y \), in \( L^i(D) \) such that \( B(y) = \langle y_0, \ldots, y_{i-1}, w \rangle \) for some \( y_0, \ldots, y_{i-1} \). Now \( r^+(L^i(D)) \geq d_{L^i(D)}(z, y) \geq d_D(z_i, w) = t \).

**Theorem 3.3.** Let \( D \) be a digraph no two of whose cycles are joined by a path, and assume that \( D \) contains at least one cycle. Then there are numbers \( k \) and \( t \) such that for every \( i \geq k \) either \( r^+(L^i(D)) = t \) or \( r^+(L^i(D)) = \infty \).

**Proof.** Since \( D \) contains cycles, no two of which are joined by a path, there are numbers \( j_D \) and \( i_D \) such that \( L^i(D) \) is isomorphic to \( L^{i+j_D}(D) \) for every \( i \geq i_D \) (see e.g. [2, Theorem 10.9.1]). Suppose that there is \( k \geq i_D \) such that \( r^+(L^k(D)) = t < \infty \). Distinguish two cases:

(i) There is a node \( u \) in \( L^k(D) \) with \( id_{L^k(D)}(u) = 0 \). Then \( u \) is in the out-center of \( L^k(D) \). By Lemma 3.1 either \( r^+(L^{k+i}(D)) = \infty \) or \( t = r^+(L^{k+j_D}(D)) \leq r^+(L^{k+i}(D)) \leq r^+(L^k(D)) = t \) for all \( i \), \( 0 \leq i \leq j_D \).

(ii) \( id_{L^k(D)}(u) \geq 1 \) for every node \( u \) in \( L^k(D) \). By Lemma 3.2 we have \( t = r^+(L^k(D)) \leq r^+(L^{k+i}(D)) \leq r^+(L^{k+j_D}(D)) = t \) for all \( i \), \( 0 \leq i \leq j_D \). Thus, \( r^+(L^i(D)) = t \) or \( r^+(L^i(D)) = \infty \) for every \( i \geq k \), as required.
Now we prove the main result concerning the out-radius.

**Theorem 3.4.** Let $D$ be a digraph containing two cycles joined by a path (possibly of length 0). Then either there are $k$ and $t$ such that $r^+(L^i(D)) = i + t$ for every $i \geq k$ or there is $k$ such that $r^+(L^i(D)) = \infty$ for every $i \geq k$.

**Proof.** Let $C_1 = (a_1, a_2, \ldots, a_l, a_1)$ and $C_2 = (b_1, b_2, \ldots, b_l, b_1)$ be two cycles in $D$ joined by a path. Suppose that there is $j \geq 0$ such that $r^+(L^j(D)) < \infty$. Distinguish two cases:

(i) There is a node $u$ in $L^j(D)$ with $d_{L^j(D)}(u) = 0$. Then $B(u) = (u, l, u_1, \ldots, u_j)$ and $d_D(u_0) = 0$. Since $r^+(L^j(D)) = e_{L^j(D)}^+(u) < \infty$, there are paths $P_1$ and $P_2$ from $u_j$ to $C_1$ and $C_2$, respectively. For $k = 1, 2$ denote by $B_k$ the trail starting at $u_0$, traversing $B(u)$, $P_k$, and then continuing once around the cycle $C_k$. Clearly, both $B_1$ and $B_2$ can be completed to $i$-histories, say $B^i(x')$ and $B^i(y')$, for every sufficiently large $i$. As $d_{L^i(D)}(x') = d_{L^i(D)}(y') = 0$ and $x' \neq y'$, we have $r^+(L^i(D)) = \infty$.

(ii) $d_{L^i(D)}(u) \geq 1$ for every node $u$ in $L^j(D)$. Then $r^+(L^j(D)) \leq r^+(L^j(D)) + i - j$ for every $i \geq j$, by Lemma 3.3. Let $x$ and $y$ be nodes in $L^j(D)$, $i \geq j$, such that $B(x) = (a_1, a_2, \ldots, a_l, a_1, a_2, \ldots)$ and $B(y) = (b_1, b_2, \ldots, b_l, b_1, b_2, \ldots)$. The cycles $C_1$ and $C_2$ may have some common paths. Let $l$ be the maximum length of a path common to $C_1$ and $C_2$. Let $z$ be a node in the out-center of $L^j(D)$ and $B(z) = (z_0, \ldots, z_l)$. Suppose that $d_{L^j(D)}(z, x) < i - l$. By Lemma 2.1 $z_{i-l-1}, z_{i-l}, \ldots, z_l$ lies on $C_1$, and hence $d_{L^j(D)}(z, y) \geq i - l$. Thus, $i - l \leq r^+(L^j(D)) \leq r^+(L^j(D)) + i - j$ for every $i \geq j$.

By Lemma 3.2 $r^+(L^{i+1}(D)) \leq r^+(L^i(D)) + 1$ for every $i \geq j$, so that $r^+(L^{i+1}(D)) = r^+(L^i(D)) + 1$ with finitely many (at most $r^+(L^j(D)) - j + l$) exceptions. Hence, there are numbers $k$ and $t$ such that $r^+(L^i(D)) = i + t$ for every $i \geq k$, as required.

From now on we consider the radius (as opposed to the out-radius) of iterated line digraphs. Since $r(D) < \infty$ if and only if $D$ is strongly connected, we consider only nontrivial strongly connected digraphs.

**Lemma 3.5.** For each nontrivial strongly connected digraph $D$ we have $r(L(D)) \geq r(D)$.

**Proof.** Let $x$ be a node in $L(D)$ with $B(x) = (u, v)$. We show that $e_{L(D)}(x) \geq e_D(u)$. 


Let $z$ be a node in $D$ for which $e_D^-(u) = d_D(z, u)$, and let $y$ be a node in $L(D)$ such that $B(y) = (z, w)$ for some $w$. By Lemma 2.1 $d_{L(D)}(y, x) \geq d_D(z, u)$, and hence $e_{L(D)}^-(x) \geq e_D^-(u)$.

Similarly, if $z'$ satisfies $e_D^+(u) = d_D(u, z')$, choose $y'$ from $V(L(D))$ such that $B(y') = (z', w')$. Then $e_{L(D)}^+(x) \geq d_{L(D)}(x, y') \geq d_D(u, z') = e_D^+(u)$.

Thus, $e_{L(D)}(x) \geq e_D(u)$ for each node $x$ in $L(D)$, $B(x) = (u, v)$, and hence $r(L(D)) \geq r(D)$.

\[\text{Theorem 3.6.} \text{ Let } D \text{ be a nontrivial strongly connected digraph different from a cycle. Then there are } t \text{ and } t', \text{ such that } i+t \leq r(L^i(D)) \leq i+t' \text{ for every } i \geq 0.\]

\[\text{Proof.} \text{ Clearly, } D \text{ contains two cycles joined by a path (possibly of length 0). Moreover, the digraph } L^i(D) \text{ is not empty and } r(L^i(D)) < \infty \text{ for every } i \geq 0. \text{ By Theorem 3.4. there are } k' \text{ and } t' \text{ such that } r^+(L^i(D)) = i+t' \text{ for every } i \geq k', \text{ and hence there is } t \leq t' \text{ such that } i+t \leq r(L^i(D)) \text{ for every } i \geq 0.\]

Let $u$ be a central node in $D$, and let $C$ be the shortest cycle in $D$ containing $u$, $C = (u, a_2, \ldots, a_l, u)$. Then there is a node $x$ in $L^i(D)$ such that $B(x) = (u, a_2, \ldots, a_l, u)$. We have $r(L^i(D)) \leq e_{L^i(D)}(x) \leq e_D(u) + l = r(D) + l$, by Lemma 2.1. Analogously, if $j > 1$, then $r(L^j(D)) \leq jl + r(D)$ (take a trail going $j$ times around $C$ for $B(x)$). By Lemma 3.5 we have $r(L^i(D)) \leq r(L^{i+1}(D)) \leq (j+1)l + r(D)$ for all $i$, $jl < i \leq (j+1)l$, and hence $r(L^i(D)) \leq i+r(D)+l$ for every $i \geq 0$. \[\square\]

4. Centers in Iterated Line Digraphs

First we introduce results concerning the out-centers. In what follows, by $H \supseteq D$ we mean that $D$ is a subgraph of $H$, and if $D$ and $H$ are isomorphic, we write $D \cong H$.

\[\text{Theorem 4.1.} \text{ Let } D \text{ be a nontrivial strongly connected digraph. Then there is a digraph } H, H \supseteq D, \text{ such that } C^+(L^i(H)) = L^i(D) \text{ for every } i \geq 0.\]

\[\text{Proof.} \text{ Let } d = \max\{2, d(D)\}, \text{ where } d(D) \text{ denotes the diameter of } D. \text{ As } D \text{ is strongly connected, we have } d < \infty. \text{ Let } V(H) = V(D) \cup \{a_1, b_1, \ldots, a_d, b_d\} \text{ and } E(H) = E(D) \cup \{ua_1, ub_1 : u \in V(D)\} \cup \{a_ja_{j+1}, b_jb_{j+1} : 1 \leq j \leq d-1\} \cup \{a_da_{d-1}, b_db_{d-1}\} \text{ (see Figure 1). Let } i \geq 0. \text{ We show that } C^+(L^i(H)) = L^i(D).\]
Let $u$ be a node in $L^i(D)$. For every node $x$ in $D$, it holds that $e^+_H(x) = d_H(x,a_d) = d$. Since there is a node $v$ in $L^i(H)$ with $B(v) = (a_d,a_{d-1},a_d,\ldots)$, we have $e^+_L(H)(u) = d + i$, by Lemma 2.1.

Let $u$ be a node from $V(L^i(H)) - V(L^i(D))$ with $B(u) = (x_0,\ldots,x_i)$. Then either $d_H(x_i,a_d) = \infty$ or $d_H(x_i,b_d) = \infty$ as $x_i$ is not in $D$.

A digraph $D$ is antisymmetric if and only if $uv / \in E(D)$ for every $vu \in E(D)$.

We remark that if $D$ is an antisymmetric strongly connected digraph, then there is an antisymmetric digraph $H$, $H \supseteq D$, such that $C^+(L^i(H)) = L^i(D)$ for every $i \geq 0$. (Just replace the two-cycles at $a_d-1$ and $b_d-1$ by three-cycles.) In contrast with Theorem 4.1, if $D$ is not strongly connected we cannot guarantee the existence of $H$ for which $C^+(L^i(H)) \cong L^i(D)$, even if $i = 1$.

![Diagram](image)

**Figure 1**

**Theorem 4.2.** Let $D$ be a digraph that is not strongly connected with $C^+(D) \neq D$ and $id_D(u) \geq 2$ for every node $u$ in $D$. Then there is no digraph $H$ for which $C^+(L(H)) \cong D$.

**Proof.** Suppose that $H$ satisfies $C^+(L(H)) \cong D$. Let $F = C^+(L(H))$, and let $t = r^+(L(H))$. Clearly, $e^+_L(H)(u) = t < \infty$ for every node $u$ in $F$. As $D$ is not strongly connected, there are nodes $x$ and $y$ in $F$ such that $d_F(x,y) = \infty$ while $d_L(H)(x,y) \leq t$. Denote by $P$ a path from $x$ to $y$ in $L(H)$. Let $w$ be the last node on $P$ that is not in $F$ and let $v$ be a successor of $w$ in $P$. Since $id_F(v) \geq 2$, there are nodes $v_1$ and $v_2$ in $F$ such that $v_1v, v_2v \in E(F)$. Then $e^+_L(H)(v_i) = t$, $i = 1,2$, and $e^+_L(H)(w) > t$. 

Theorem 4.3. It is well-known that if \( x_1y_1, x_1y_2, \) and \( x_2y_1 \) are arcs in a line digraph, then so is \( x_2y_2 \) (see e.g. [2, Theorem 10.8.4]). Thus, \( uu \) is an arc in \( L(H) \) if and only if so are \( v_1u, i = 1, 2 \). Now if \( u \) is from \( V(L(H)) - \{v_1, u\} \) we have \( d_{L(H)}(w,u) = d_{L(H)}(v_1,u) \), while \( d_{L(H)}(w,v_1) = d_{L(H)}(v_2,v_1) \leq t \). Hence \( e_{L(H)}^+(w) \leq t \), a contradiction.

Let \( D_1 \) be a digraph on \( n \geq 5 \) nodes, say \( u_1, u_2, \ldots, u_n \), and let \( E(D_1) = \{u_iu_{i+1}, u_iu_{i+2} : 1 \leq i \leq n\} \) (the addition is modulo \( n \)). Let \( D'_1 \) be isomorphic to and distinct from \( D_1 \), and let \( D \) consist of \( D_1, D'_1 \), and all arcs joining the nodes of \( D_1 \) to nodes in \( D'_1 \). Then \( L(D) \) satisfies the assumptions of Theorem 4.2, and hence there is an infinite number of antisymmetric iterated line digraphs \( D \) such that \( L(D) \not\sim C^+(L(H)) \) for every digraph \( H \).

From now on we consider the centers (instead of the out-centers) in iterated line digraphs. Here, the situation is different since each \( i \)-iterated line digraph is admissible as a center of \( i \)-iterated line digraph, \( i \geq 0 \).

**Theorem 4.3.** Let \( D \) be a digraph and let \( j \geq 0 \). If \( L^j(D) \) is not empty then there is a digraph \( H, H \supseteq D, \) such that \( C(L^i(H)) = L^i(D) \) for every \( i \leq j \).

**Proof.** Let \( k = \left\lceil \frac{j+1}{2} \right\rceil \). Add to \( D \) \( 2(k+1) \) new nodes \( a_1, a_2, \ldots, a_{k+1}, b_1, b_2, \ldots, b_{k+1} \), and \( 4k \) arcs \( a_ia_{i+1}, a_{i+1}a_i, b_ib_{i+1}, b_{i+1}b_i, 1 \leq i \leq k \). Moreover, subdivide each of the added arcs not incident with \( a_{k+1} \) or \( b_{k+1} \) (see Figure 2 for the case \( k = 2 \)). Finally, join every node \( u \) of \( D \) to \( a_1 \) and \( b_1 \) by paths of length \( k \), and join \( a_1 \) and \( b_1 \) to every node \( u \) in \( D \) by another pair of paths of length \( k \). Do this so that all \( 4|V(D)| \) paths are pairwise internally node disjoint (that is, we have appended \( 4(k-1)|V(D)| \) new vertices).

Denote by \( H \) the resulting digraph. Then \( e_H^+(x) = e_H^-(x) = 3k-1 \) for every node \( x \) of \( D \) (although \( D \) can be disconnected), and both \( e_H^+(x) \) and \( e_H^-(x) \) are greater than \( 3k-1 \) for every node \( x \) from \( V(H)-V(D) \). Let \( i \leq j \). We show \( C(L^j(H)) = L^i(D) \).

Let \( u \) be a node in \( L^j(D) \). As \( e_H(x) = 3k-1 \) for every node \( x \) of \( D \), we have \( e_{L^j(H)}(u) \leq 3k-1+i \), by Lemma 2.1. Let \( v \) be a node in \( L^j(H) \), for which \( B(v) = (\ldots, a_k, a_{k+1}) \). Then \( d_{L^j(H)}(v,u) = 3k-1+i \), and hence \( e_{L^j(H)}(u) = 3k-1+i \).

Let \( u \) be a node from \( V(L^j(H))-V(L^i(D)) \), \( B(u) = (x_0, x_1, \ldots, x_i) \). Since \( i < 2k \) either \( x_0 \) or \( x_i \) is not in \( D \). Suppose that \( x_0 \notin V(D) \). Let \( v \) and \( z \) be nodes in \( L^j(H) \) such that \( B(v) = (\ldots, a_k, a_{k+1}) \) and \( B(z) = (\ldots, b_k, b_{k+1}) \). Then, by Lemma 2.1, \( d_{L^j(H)}(v,u) \geq 3k+i \) or \( d_{L^j(H)}(z,u) \geq 3k+i \), and hence \( e_{L^j(H)}^-(u) \geq 3k+i \). Similarly, \( e_{L^j(H)}^+(u) \geq 3k+i \) if \( x_i \notin V(D) \), and hence \( e_{L^j(H)}(u) \geq 3k+i \). ■
Suppose that there is a node, say $x$, which is contained in a cycle of length at most $C$ of $C$ strongly connected. Thus, there is a cycle, say $H$, in a sense, as shown by Corollary 4.5 and Corollary 4.6 for digraphs and antisymmetric digraphs, respectively. Theorem 4.4 is best possible in a sense, as shown by Corollary 4.5 and Corollary 4.6 for digraphs and antisymmetric digraphs, respectively.

**Theorem 4.4.** Let $D$ be a digraph with a nonempty arc set, each arc of which contains a cycle of length at most $l$. Then there is no $H$, $H \supset D$, such that $r(L(H)) \geq l$ and $C(L'(D)) = L'(D)$ for every $i \geq 0$.

**Proof.** Suppose that there is $H \supset D$ satisfying the assumptions of theorem. Since $H \supset D$, we have $E(H) \supset E(D)$ and $r(L(H)) = t < \infty$.

Write $H' = L(H)$ and $D' = L(D)$. Then $V(H') - V(D') \neq \emptyset$ and $H'$ is strongly connected. Thus, there is a cycle, say $C_1$, in $H'$ passing through a node, say $x$, of $D'$ such that $C_1 \not\subset D'$. Let $C_2$ be a cycle of length $l_2 \leq l$ in $D'$ passing through the node $x$ (i.e.; the arc $x$ of $D$). Denote by $l_1$ the length of $C_1$. There are $a \in V(L^{l_1}(H')) - V(L^{l_1}(D'))$ and $b \in V(L^{l_1}(D'))$ such that $V(B(a)) = V(C_1)$, $V(B(b)) = V(C_2)$, and $x$ is the initial and terminal node of both $B(a)$ and $B(b)$. (The $B(a)$ circles $l_2$ times around $C_1$ and $B(b)$ circles $l_1$ times around $C_2$.) In what follows we show that $e_{L^{l_1}L^{l_2}(H')}(a) \leq e_{L^{l_1}L^{l_2}(H')}(b)$.

We have $e_{H'}(x) = r(H') = t$. Suppose that $e_{H'}(x) = e_{H'}^+(x)$ (the case $e_{H'}(x) = e_{H'}^-(x)$ can be proved similarly). Then there is a node $y$ in $H'$ such that $d_{H'}(x, y) = t$. Since $H'$ is strongly connected, there is a node $u$ in $L^{l_1}(H')$ with $B(u) = (y, \ldots )$. As $t \geq l \geq l_2$, $y$ is not in $C_2$, and hence $e_{L^{l_1}L^{l_2}(H')}(b) \geq d_{L^{l_1}L^{l_2}(H')}(b, u) = t + l_1l_2$, by Lemma 2.1.
Now let \( v \) be a node in \( L^{1+2}(H') \) with \( B(v) = (v_0, \ldots, v_{l1+2}) \). Then \( d_{L^{1+2}(H')}(a, v) \leq d_{H'}(x, v_0) + l_1 l_2 \leq t + l_1 l_2 \) and \( d_{L^{1+2}(H')}(v, a) \leq d_{H'}(v_{l1+2}, x) + l_1 l_2 \leq t + l_1 l_2 \), and hence \( e_{L^{1+2}(H')}(a) \leq r + l_1 l_2 \leq e_{L^{1+2}(H')}(b) \), a contradiction.

**Corollary 4.5.** Let \( D \) be a digraph, each arc of which lies in a two-cycle and assume that \( C(D) \neq D \). Then there is no \( H, H \supseteq D \), such that \( C(L^i(H)) = L^i(D) \) for every \( i \geq 0 \).

**Proof.** Suppose that \( H \) satisfies \( C(L^i(H)) = L^i(D) \) for every \( i \geq 0 \). Let \( e = u_1u_2 \) be an arc in \( D \). Then also \( f = u_2u_1 \in E(D) \), and since \( C(D) \neq D \) there is another arc, say \( g = v_1v_2 \), in \( D \). Now \( v_1 \neq u_2 \) or \( v_1 \neq u_1 \), and hence \( d_{L(H)}(e, g) \geq 2 \) or \( d_{L(H)}(f, g) \geq 2 \), respectively. Thus \( r(L(H)) \geq 2 \), contradicting Theorem 4.4.

**Corollary 4.6.** Let \( D \) be an antisymmetric digraph, each arc of which lies in a triangle and assume that \( C(D) \neq D \). Then there is no antisymmetric digraph \( H, H \supseteq D \), such that \( C(L^i(H)) = L^i(D) \) for every \( i \geq 0 \).

**Proof.** Suppose that \( H \) satisfies \( C(L^i(H)) = L^i(D) \) for every \( i \geq 0 \). Let \( e = uv \) be an arc in \( D \) and let \( T \) be a triangle in \( D \) containing \( e \). Since \( C(T) = T \), there is an arc \( f \) in \( D \) such that \( f \notin E(T) \). As \( D \) is strongly connected, we may assume that \( f = uz, z \neq v \). Since \( H \) is antisymmetric, \( d_{H}(v, u) \geq 2 \), and hence \( d_{L(H)}(e, f) \geq 3 \). Thus \( r(L(H)) \geq 3 \), which contradicts Theorem 4.4.

**References**


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