Bipolar fuzzy graph representation of concept lattice

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Abstract
Formal Concept Analysis (FCA) is a mathematical framework for knowledge processing tasks. FCA has been successfully incorporated into fuzzy setting and its extension (interval-valued fuzzy set) for handling vagueness and imprecision in data. However, the analysis in such settings is restricted to unipolar space. Recently, some applications of bipolar information are shown in bipolar fuzzy graph, lattice theory as well as in FCA. The adequate analysis of bipolar information using FCA requires incorporation of bipolar fuzzy set and an appropriate lattice structure. For this purpose, we propose an algorithm for generating the bipolar fuzzy formal concepts, a method for \( (\alpha, \beta) \)-cut of bipolar fuzzy formal context and its implications with illustrative examples.

1. Introduction

Formal Concept Analysis (FCA) was introduced by Wille [52] for knowledge discovery and representation tasks. FCA starts the analysis from a given formal context which comprises a set of formal objects, a set of formal attributes, and a binary relation between them. From a given context FCA investigates formal concepts, which is a pair of extent and intent representing a subset of objects with their common attributes, respectively. Concept lattice provides connection between investigated formal concepts as generalization and specialization, which plays a major role in knowledge processing tasks [44,45]. FCA became more popular in scientific community when its mathematical foundation was established by Ganter and Wille [26].

Burasco and Fuentes-Gonzalez [19] incorporated FCA into the fuzzy setting for handling vagueness and imprecision in linguistics variables, and was further improved by Belohlavek [12–14]. Afterwards, fuzzy formal concepts and their lattice structure were applied by several researchers for knowledge processing tasks [15,44,45]. Very recently, Li and Tsai [37] discussed an application of the fuzzy concept lattice for sentiments (emotions, love, etc.) classification based on the opinion of people. Antoni et al. [6] introduced heterogeneous formal context for representing preference of people to stay at cottage in a given facility. Subsequently, Franco et al. [25] introduced a model for the preference analysis. The word opinion or preference shows two sides: one is the positive (acceptation) and the other is the negative (non-acceptation) side. These two sides coexist simultaneously, and can be represented as an integral of its positive and negative sides. In this case, we cannot apply the existing fuzzy approaches in FCA [15,20–22,24,27,30,31] because it defines the data in the unipolar space \( \{0, 1\} \) or \([0, 1]\). An another extension of fuzzy set (called as bipolar fuzzy set) represents the bipolar information more precisely [58]. Considering the above scenario, it is important to introduce the bipolar fuzzy set into FCA for analyzing the bipolar information using concept lattice.
A bipolar information consists two sides: one is positive, and another is negative side. For example, the relation between two organizations constitute a conflict side and a common interest side [60]. If we assume -1 to represent negative pole true, and 0 as false then 1 represents positive pole true, and 0 for false. This case can be represented adequately through a bipolar fuzzy set defined in a bipolar fuzzy space [-1,0] x [0,1] [58–60]. A bipolar fuzzy set is a pair of mappings, namely a positive membership (0,1], and a negative membership function [-1,0) [32]. The positive membership degree (0,1] of an element indicates that the element somewhat satisfies the corresponding property, and the negative membership degree [-1,0) of an element indicates that the element somewhat satisfies the implicit counter-property [1,2,55]. The zero membership degree of an element means the element is irrelevant to the corresponding property [34]. This representation is necessary because fuzziness is inseparable from bipolar truth [23,29]. Therefore, it is necessary to introduce it into fuzzy concept lattice.

Techniques available in the literature for visualizing the concept lattice are restricted to analyze the data in the unipolar space [4–16,19–22,24–27,29,33,35–37,39,42,46–50,56,57], which lacks in visualizing the bipolarity. Some applications of bipolar information were shown in lattice theory [6,17,18,20,21], information retrieval [38,51–56], as well as in bipolar fuzzy graph [1–3,28,40,53,54]. In this paper our analysis focus on visualizing the bipolar information using the concept lattice. To achieve this goal, we require an appropriate lattice structure, and graph theory for concept lattice representation. Recently, Bloch [17,18] discussed the lattices of bipolar fuzzy set, its properties and applications. Niesink et al. [40] defined the properties of weighted bipolar fuzzy graph. Akram [1–3] discussed several properties of bipolar fuzzy graphs with its applications followed by Yang et al. [54]. In this paper we focus on representation of concept lattice using the properties of bipolar fuzzy graph and lattices of bipolar fuzzy set. The motivation is to represent the positive and negative side of bipolar fuzzy attributes simultaneously in the concept lattice, and provide a more adequate analysis by the connected (bipolar) fuzzy formal concepts, in the form of generalization and specialization. Such that, the theory of concept lattice can commensurate with preference analysis [6,25], mathematical morphology [17,18], decision making [29], customer satisfaction [28,30], opinion classification [37] and handling bipolar queries [55]. In this process, we address the following problems:

(1) How to represent the bipolar information in fuzzy formal context?
(2) How to investigate the formal concepts hidden in given bipolar fuzzy context?
(3) How to visualize the hierarchical order between the generated bipolar fuzzy concepts into the concept lattice?
(4) How to find bipolar fuzzy attribute implications?

To address these problems, we aim at the following proposals in this paper:

(1) To introduce FCA with bipolar fuzzy setting.
(2) To introduce a method for investigating the bipolar fuzzy formal concepts.
(3) To introduce the concept lattice representation using bipolar fuzzy graph.
(4) To introduce a method for decomposition of bipolar fuzzy context and its implications.
(5) To provide an application of the proposed methods.

Table 1 provides some possible conditions for objects, attributes and fuzzy relation of a given fuzzy formal context. The condition complete (or incomplete) discusses the availability or partial availability (or non-availability) of objects or attributes (or relations). Recently, incomplete data was studied in decision formal context [33], fuzzy formal context [34] as well as interval-valued fuzzy formal context [4,5]. In this paper, we have focused on the conditions in which data is complete.

Rest of the paper is organized as follows: Section 2, contains a brief background about FCA in the fuzzy setting. In Section 3, we provide the proposed method for generating the bipolar fuzzy formal concepts and a method for decomposition of bipolar fuzzy formal context. Section 4 contains the illustration of the proposed methods with example. In Section 5 we provide discussions followed by conclusions, acknowledgements and references.

2. Formal concept analysis in the fuzzy setting

A fuzzy formal context is a triplet $F = (X, Y, \tilde{R})$, where $X$ is a set of objects, $Y$ is a set of attributes and $\tilde{R}$ is an $L$-relation between $X$ and $Y$, $\tilde{R} : X \times Y \to L$ [12,19]. Each relation $\tilde{R}(x, y) \in L$ represents the membership value at which the object $x \in X$ has the attribute $y \in Y$ in $[0,1]$ ($L$ is a support set of some complete residuated lattice $L$) [14].

A residuated lattice $L = (L, \land, \lor, \rightarrow, 0, 1)$ is the basic structure of truth degrees, where $0$ and $1$ represent least and greatest elements respectively. $L$ is a complete residuated lattice iff [13,43]:

1. $(L, \land, \lor, 0, 1)$ is a complete lattice.
2. $(L, \otimes, 1)$ is a commutative monoid.
3. $\otimes$ and $\rightarrow$ are adjoint operators (called as multiplication and residuum, respectively), that is $a \otimes b \leq c$ iff $a \leq b \rightarrow c$, $\forall a, b, c \in L$.

The operators $\otimes$ and $\rightarrow$ are defined distinctly by Lukasiewicz, Godel, and Goguen t-norms and their residua as given below [14,15,27]:

Lukasiewicz:
- $a \otimes b = \max (a + b - 1, 0)$.
- $a \rightarrow b = \min (1 - a + b, 1)$.

Godel:
- $a \otimes b = \min (a, b)$.
- $a \rightarrow b = 1$ if $a \leq b$, otherwise $b$.

Goguen:
- $a \otimes b = a \cdot b$.
- $a \rightarrow b = 1$ if $a \leq b$, otherwise $b/a$.

Classical logic is a special case of complete residuated lattice which is represented as $\langle \{0, 1\}, \land, \lor, \otimes, \rightarrow, 0, 1 \rangle$. For any $L$-set $A \in L^X$ of objects, and $B \in L^Y$ of attributes we can define $L$-set $A' \in L^Y$ of attributes and $L$-set $B' \in L^X$ of objects as follows [13,14]:

<table>
<thead>
<tr>
<th>Conditions</th>
<th>Objects</th>
<th>Attributes</th>
<th>Fuzzy relation</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>Complete</td>
<td>Complete</td>
<td>Incomplete</td>
</tr>
<tr>
<td>b</td>
<td>Incomplete</td>
<td>Complete</td>
<td>Complete</td>
</tr>
<tr>
<td>c</td>
<td>Complete</td>
<td>Incomplete</td>
<td>Complete</td>
</tr>
<tr>
<td>d</td>
<td>Incomplete</td>
<td>Incomplete</td>
<td>Complete</td>
</tr>
<tr>
<td>e</td>
<td>Binary</td>
<td>Binary</td>
<td>Fuzzy (bipolar)</td>
</tr>
<tr>
<td>f</td>
<td>Binary</td>
<td>Fuzzy (bipolar)</td>
<td>Fuzzy (bipolar)</td>
</tr>
<tr>
<td>g</td>
<td>Fuzzy (bipolar)</td>
<td>Binary</td>
<td>Fuzzy (bipolar)</td>
</tr>
<tr>
<td>h</td>
<td>Fuzzy (bipolar)</td>
<td>Fuzzy (bipolar)</td>
<td>Fuzzy (bipolar)</td>
</tr>
</tbody>
</table>
(1) \( A^1(x) = \bigwedge_{y \in Y} (A(x) \rightarrow R(x,y)). \)
(2) \( B^1(x) = \bigwedge_{y \in Y} (B(y) \rightarrow R(x,y)). \)

\( A^1(x) \) is interpreted as the L-set of attribute \( y \in Y \) shared by all objects from \( A \). Similarly, \( B^1(x) \) is interpreted as the L-set of all objects \( x \in X \) having the attributes from \( B \) in common. The fuzzy formal concept is a pair of \( (A,B) \in L^X \times L^Y \) satisfy \( A^1 = B \) and \( B^1 = A \), where fuzzy set of objects \( A \) called as extent and fuzzy set of attributes \( B \) called as intent.

The operators \( (\cdot,\cdot) \) are known as Galois connection [26] and extensively studied with fuzzy [14,15,42] as well as with interval-valued fuzzy setting [20–22,30] by the researchers. When the operator \( (\cdot) \) is applied on a fuzzy set of objects, it provides a fuzzy set of attributes with its membership value being maximal with respect to integrating the information from all the objects. Consequently, when the operator \( (\cdot) \) is applied on the fuzzy set constituted by these covered attributes resulting from integrating the membership information between objects and attributes. It takes a fuzzy set of objects with its membership value being maximal with respect to integrating the information from the attributes. After that, we cannot find any fuzzy set of attributes (objects) which can make the membership value of the obtained fuzzy set of attributes (objects) bigger, if the pair of the set of objects and the set constituted by its covered attributes forms a fuzzy formal concept.

Hence, fuzzy formal concept is a maximal rectangle of a given fuzzy context \( F \) filled with membership value between \([0,1]\), which is an ordered pair of two sets \( (A,B) \), where \( A \subseteq X \) called as fuzzy extent, and \( B \subseteq Y \) is called as fuzzy intent. The set of fuzzy formal concepts \( C \) generated from a given fuzzy formal context \( F \), defines the partial ordering principle, i.e. \( (A_1,B_1) \leq (A_2,B_2) \iff A_1 \subseteq A_2 \quad (\leftrightarrow B_2 \subseteq B_1) \) for every fuzzy formal concept. Together with this ordering, in the complete lattice there exist an infimum and a supremum for some formal concepts as follows [25,48]:

- \( \bigwedge_{\mu \in I} (A_j,B_j) = (\bigcap_{\mu \in I} A_j, (\bigcap_{\mu \in I} B_j)^{1-I}), \)
- \( \bigvee_{\mu \in I} (A_j,B_j) = (\bigcup_{\mu \in I} A_j, (\bigcup_{\mu \in I} B_j)^{1-I}), \)

Computing all the formal concepts and their hierarchical order visualization in the concept lattice structure are major concerns for practical applications of FCA [11]. In this paper we focused on introducing an algorithm for generating the bipolar fuzzy formal concepts using the properties of bipolar fuzzy graph and Galois connection with its application. Moreover, we propose a method to compute the \((\alpha,\beta)\)-cut of a given bipolar fuzzy context with its illustration in the next section.

### 3. Proposed method

A link between bipolar fuzzy graph and the concept lattice can be established using the properties of partial ordering, complete graph and lattice theory. To understand this link Singh and Aswani Kumar [48] recently provided a note and discussed that bipolar fuzzy graph can be incorporated for concept lattice representation as other graph theory has been incorporated [16,27,46]. Extending upon the work in this paper we focused on proposing an algorithm for generating the bipolar fuzzy formal concepts, a method for decomposition of bipolar fuzzy formal context using \((\alpha,\beta)\)-cut and its implications with an example.

#### 3.1. Proposed method for generating bipolar fuzzy formal concepts

Let a bipolar fuzzy formal context \( F = (X,Y,R) \). The set of objects \( (A) = (x_i, [\mu^p(x_i), \mu^b(x_i)]) \) having the attributes \( (B) = (y_j, [\mu^p(y_j), \mu^b(y_j)]) \) can be represented as a node in the bipolar fuzzy graph. The pair \( (A,B) \) is called as a bipolar fuzzy formal concept if \( A, [\mu^p(A), \mu^b(A)] = B^1 \) and \( (B, [\mu^p(B), \mu^b(B)]) = A^1 \). The steps for investigating the bipolar fuzzy formal concepts are shown in Table 2.

The proposed algorithm starts investigating the bipolar fuzzy formal concepts using the subset \( 2^m \) of given attributes and their acceptance membership value \((0,0.1,0.1)\) (through Steps 1–3). The operator \( (\cdot) \) is applied on fuzzy set constituted by each subset of attributes using the membership value \((0,0.1,0.1)\) in Step 5 and 6. The operator \( (\cdot) \) provides the fuzzy set of maximal objects to integrate the information from the constituted attributes. The (bipolar) fuzzy membership values of covered objects are computed using the min and max operators defined in Step 7. Consequently, the operator \( (\cdot) \) is applied on the bipolar fuzzy set of obtained objects to find the maximal fuzzy set of attributes with respect to integrating the information from all these objects (through Step 8). If the obtained fuzzy set of attributes is equivalent to initially considered subset of attributes, then the corresponding formal concept is a bipolar fuzzy formal concept (Step 9). The investigated bipolar fuzzy formal concept is represented in a set \( C \) (Step 10). Otherwise, the extra attributes are added to the obtained fuzzy set of attributes to find the maximal set of attributes to integrate the information from the constituted extent (Steps 9 and 10). Similarly, we can investigate the remaining bipolar fuzzy formal concepts for each subset of attributes. Since the proposed algorithm uses subset of attributes for generating the bipolar fuzzy formal concepts, it is easier to construct their hierarchical order in a lattice structure. The generated bipolar fuzzy formal concepts can be connected to each other using the bipolar fuzzy graph as represented by other graph theories [15,16,27,42,46,48]. The bipolar fuzzy concept lattice structure provides the connection between each of the generated bipolar fuzzy formal concepts in form of specialization and generalization. Generalized concepts contain more objects while specialized concepts contain more attributes. This connection between bipolar fuzzy formal concepts is helpful in the practical applications of FCA as shown in the next section.
Complexity: Let, the number of objects (|X|) = n and number of attributes (|Y|) = m in the given fuzzy formal context. Then proposed algorithm finds the subset of attribute (P) for generating bipolar fuzzy formal concepts which takes $2^m$ time. Thereafter algorithm combines all the objects for the given relation and its visibility [40]. Similarly, the $\alpha$-cut can be defined for the bipolar fuzzy formal context as given below:

If $F = (X, Y, R)$ be a bipolar fuzzy formal context which concept lattice is represented through a bipolar fuzzy graph $G = (I, J)$. Then the given bipolar fuzzy formal context can be decomposed with help of $(\alpha, \beta)$-cut denoted by ordered set $G_{\alpha, \beta} = (I_{\alpha, \beta}, J_{\alpha, \beta})$ where:

1. $V_{\alpha, \beta} = \{ (\mu^p_{v_1}, \mu^N_{v_1}), (\mu^p_{v_2}, \mu^N_{v_2}), (\mu^p_{v_3}, \mu^N_{v_3}), \ldots, (\mu^p_{v_m}, \mu^N_{v_m}) \} = V$, 
2. $E_{\beta}(\alpha) = \{ (\mu^p_{v_j}, \mu^N_{v_j}) | \mu^p_{v_j}(v_i) \geq \alpha, \mu^N_{v_j}(v_i) \leq \beta, j = 1, 2, 3, \ldots, m \}$, 
3. $E_{m+1}(\alpha, \beta) = \{ v_i | \mu^p_{v_i}(v_i) < \alpha, \mu^N_{v_i}(v_i) > \beta, j = 1, 2, 3, \ldots, m \}$, where $0 \leq \alpha \leq 1$ and $-1 \leq \beta \leq 0$.

The edge $E_{m+1}(\alpha, \beta)$ is added to the group of elements which are not contained in any edge $E_{\beta}(\alpha)$ of $G_{\alpha, \beta}$. The $(\alpha, \beta)$ can be based on the given context or user defined. For this method we will provide a illustrative example in the Section 4.3. In the next section we provide an illustrative example for the proposed link.

### 4. Illustrations

#### 4.1. Illustrations of the proposed algorithm

To illustrate the bipolarity in formal context, we have considered a fuzzy formal context shown in Table 3 as well as its negation shown in Table 4. These two contexts are an integral part of the two sides (positive and negative) of an information as discussed by Djouadi [20]. It can be represented more adequately through bipolar fuzzy set as shown in Table 5. Then this representation can be transformed into a bipolar fuzzy formal context as shown in Table 6. It is useful when we can generate bipolar fuzzy formal concepts and visualize them in the concept lattice. For this purpose, we have proposed an algorithm in Section 3.1.
In the next section we provide an application of the proposed link and algorithm. Fig. 1 reflects the generalization and specialization between bipolar fuzzy formal concepts. We believe that this current study the objects are inherited from the most specific minimum node. We can observe that, the concept lattice structure shown in Every concept lattice structure contains two special nodes at their top and bottom boundaries representing the most general and the most specific concepts, respectively. Generalized concepts contain more objects while Specialized concepts contain more attributes [26,52]. The attributes of each formal concepts are inherited from the most general maximum node, while the objects are inherited from the most specific minimum node. We can observe that, the concept lattice structure shown in Fig. 1 reflects the generalization and specialization between bipolar fuzzy formal concepts. We believe that this current study will be helpful for the practical applications of FCA in the various fields for knowledge discovery and representation tasks [7–32,48–51]. In the next section we provide an application of the proposed link and algorithm.
4.2. Application of the proposed algorithm

Bipolar fuzzy set has been applied in several research areas like fuzzy graph [1–3,40], mathematical morphology [17,18], computing similarity between customer satisfaction [31,32], decision analysis [28,29], information retrieval [51,53–56]. In this section, we discuss an application of the proposed link and algorithm for solving a decision making problem as described below:

Let us suppose, a company manufactures set of cars (objects) = \( \{c_1, c_2, c_3, c_4\} \) considering some parameters \( (e_1 = \text{Costly}, e_2 = \text{Beautiful}, e_3 = \text{Fuel efficient}, e_4 = \text{Modern technology}, e_5 = \text{Luxurious}) \). Suppose, a customer wants to purchase the car based on his/her desired parameters as given below [28]:

(a) \( e_1 = \{(c_1, 0.4, -0.5), (c_2, 0.6, -0.3), (c_3, 0.8, -0.2), (c_4, 0.5, -0.2)\} \).

(b) \( e_2 = \{(c_1, 0.5, -0.5), (c_2, 0.3, -0.1), (c_3, 0.4, -0.4), (c_4, 0.7, -0.3)\} \).

(c) \( e_5 = \{(c_1, 0.7, 0.0), (c_2, 0.5, -0.3), (c_3, 0.6, -0.3), (c_4, 0.4, -0.4)\} \).

The company wants to analyze the preference of the customer. It can be analyzed using the proposed link and the algorithm as follows:

We can observe that the preference of the customer represents the bipolar fuzzy relation between the set of cars and its parameters. We can represent them in the form of a bipolar fuzzy formal context shown in Table 7, where the set of cars \( \{c_1, c_2, c_3, c_4\} \) is considered as objects (X) and its constituent parameters \( (e_1 = \text{Costly}, e_2 = \text{Beautiful}, e_5 = \text{Luxurious}) \) are considered as attributes (Y). We can analyze it through the generated bipolar fuzzy formal concepts as well as its lattice structure as given below:

The following bipolar fuzzy formal concepts are generated from Table 7 (as per the proposed algorithm shown in Table 2):

1. \( \{(0.4, 0.0)/c_1 + (0.3, -0.1)/c_2 + (0.4, -0.2)/c_3 + (0.4, -0.2)/c_4, (0.0, 1.0)/e_1 + (0.0, 1.0)/e_2 + (0.0, 1.0)/e_3\} \).

2. \( \{(0.4, -0.5)/c_1 + (0.3, -0.1)/c_2 + (0.4, -0.2)/c_3 + (0.5, -0.2)/c_4, (0.0, 1.0)/e_1 + (0.0, 1.0)/e_2\} \).

3. \( \{(0.5, 0.0)/c_1 + (0.5, -0.3)/c_2 + (0.6, -0.2)/c_3 + (0.4, -0.2)/c_4, (0.0, 1.0)/e_1 + (0.0, 1.0)/e_5\} \).

4. \( \{(0.4, 0.0)/c_1 + (0.3, -0.1)/c_2 + (0.4, -0.2)/c_3 + (0.4, -0.2)/c_4, (0.0, 1.0)/e_1 + (0.0, 1.0)/e_5\} \).

5. \( \{(0.4, -0.5)/c_1 + (0.6, -0.3)/c_2 + (0.8, -0.2)/c_3 + (0.5, -0.2)/c_4, (0.1, 0.0)/e_1\} \).

6. \( \{(0.5, -0.5)/c_1 + (0.5, -0.1)/c_2 + (0.4, -0.4)/c_3 + (0.7, -0.3)/c_4, (0.1, 0.0)/e_2\} \).

7. \( \{(0.7, 0.0)/c_1 + (0.5, -0.3)/c_2 + (0.6, -0.3)/c_3 + (0.4, -0.4)/c_4, (0.0, 1.0)/e_3\} \).

8. \( \{(1.0, -1.0)/c_1 + (1.0, -1.0)/c_2 + (1.0, -1.0)/c_3 + (1.0, -1.0)/c_4, \emptyset\} \), where \( \emptyset \) represents null set.

The above generated bipolar fuzzy formal concepts are shown in Fig. 2, which represent generalization and specialization between the car and its properties. The generalized concepts 5, 6, and 7 shown in Fig. 2, represents following information:

- Concepts number 5 represents \( \{(0.4, -0.5)/c_1 + (0.6, -0.3)/c_2 + (0.8, -0.2)/c_3 + (0.5, -0.2)/c_4, (0.0, 1.0)/e_1\} \). We can observe that the car-c_3 shows maximum positive membership value for the acceptance of the parameter (attribute)-\( e_1 \) (Costly). Hence, it is more suitable car while considering attribute-\( e_1 \).

- Concepts number 6 represents that car-c_4 is more suitable while considering parameter (attribute)-\( e_2 \) (Beautiful).

Table 7
A bipolar fuzzy formal context representation of car and its properties.

<table>
<thead>
<tr>
<th></th>
<th>( e_1 )</th>
<th>( e_2 )</th>
<th>( e_5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>c_1</td>
<td>(0.4, -0.5)</td>
<td>(0.5, -0.5)</td>
<td>(0.7, 0.0)</td>
</tr>
<tr>
<td>c_2</td>
<td>(0.6, -0.3)</td>
<td>(0.3, -0.1)</td>
<td>(0.5, -0.3)</td>
</tr>
<tr>
<td>c_3</td>
<td>(0.8, -0.2)</td>
<td>(0.4, -0.4)</td>
<td>(0.6, -0.3)</td>
</tr>
<tr>
<td>c_4</td>
<td>(0.5, -0.2)</td>
<td>(0.7, -0.3)</td>
<td>(0.4, -0.4)</td>
</tr>
</tbody>
</table>
• Concepts number 7 represents that car-c is more suitable while considering parameter (attribute)-e (Luxurious).

From the above observations, the company can conclude that car-c has more desired parameters for the customer. Hence, car-c will be the customer’s first preference and car c will be the customer’s second preference. These conclusions are in good agreement with [28]. However, the proposed link in this study provides a rigorous analysis to process the preference of the customer, to purchase the car based on its constituent parameters using all the generated bipolar fuzzy formal concepts. The company can also analyze the hierarchical ordering between the car and its constituent parameters using the concept lattice as shown in Fig. 2. In the next section the (x, β)-cut of bipolar fuzzy formal context will be demonstrated.

4.3. (x, β)-cut of a bipolar fuzzy formal context

To demonstrate the (x, β)-cut, we have considered the bipolar fuzzy formal context shown in Table 7. We can observe that the considered formal context represents the preference of a customer to purchase the car based on his/her desired parameters. Generally, the customer will purchase the car if it satisfies the acceptation of his desired parameters (positive membership value more than 0.5) and non-acceptation of his desired parameters (negative membership value near to 0.1). We can consider μ^1 = 0.6, μ^2 = 0.2, as (0.6, 0.2) for (x, β)-cut of the given bipolar fuzzy formal context shown in Table 7. Then we can compute its (x, β)-cut as follows (as per the proposed method in the Section 3.2):

1. μ(c1, e1) = (0.4, 0.5) means μ^1(c1, e1) = 0.4 and μ^2(c1, e1) = 0.5. The positive membership-μ^1(c1, e1) = 0.4 ≤ 0.6. Hence, (0.6, 0.2)-cut of (c1, e1) = 0.
2. μ(c1, e2) = (0.5, 0.5) means μ^1(c1, e2) = 0.5 and μ^2(c1, e2) = 0.5. The positive membership-μ^1(c1, e2) = 0.5 ≤ 0.6. Hence, (0.6, 0.2)-cut of (c1, e2) = 0.
3. μ(c1, e3) = (0.7, 0.5) means μ^1(c1, e3) = 0.7 and μ^2(c1, e3) = 0.0. The positive membership-μ^1(c1, e3) = 0.7 ≥ 0.6. However, negative membership-μ^2(c1, e3) = 0.0 ≥ 0.2. Hence, (0.6, 0.2)-cut of (c1, e3) = 0.
4. μ(c1, e4) = (0.6, 0.3) means μ^1(c1, e4) = 0.6 and μ^2(c1, e4) = 0.3. The positive membership-μ^1(c1, e4) = 0.6 ≥ 0.6 also the negative membership-μ^2(c1, e4) = 0.3 ≤ 0.2. Hence, (0.6, 0.2)-cut of (c1, e4) = 1.
5. μ(c1, e5) = (0.3, 0.1) means μ^1(c1, e5) = 0.3 and μ^2(c1, e5) = 0.1. The positive membership-μ^1(c1, e5) = 0.3 ≤ 0.6. Hence, (0.6, 0.2)-cut of (c1, e5) = 0.
6. μ(c2, e1) = (0.5, 0.3) means μ^1(c2, e1) = 0.5 and μ^2(c2, e1) = 0.3. The positive membership-μ^1(c2, e1) = 0.5 ≤ 0.6. Hence, (0.6, 0.2)-cut of (c2, e1) = 0.
7. μ(c2, e2) = (0.8, 0.2) means μ^1(c2, e2) = 0.8 and μ^2(c2, e2) = 0.2. The positive membership-μ^1(c2, e2) = 0.8 ≥ 0.6 also negative membership-μ^2(c2, e2) = 0.2 ≤ 0.2. Hence, (0.6, 0.2)-cut of (c2, e2) = 1.
8. μ(c2, e3) = (0.7, 0.4) means μ^1(c2, e3) = 0.7 and μ^2(c2, e3) = 0.4. The positive membership-μ^1(c2, e3) = 0.4 ≤ 0.6. Hence, (0.6, 0.2)-cut of (c2, e3) = 0.
9. μ(c2, e4) = (0.6, 0.3) means μ^1(c2, e4) = 0.6 and μ^2(c2, e4) = 0.3. The positive membership-μ^1(c2, e4) = 0.6 ≥ 0.6 also negative membership-μ^2(c2, e4) = 0.3 ≤ 0.2. Hence, (0.6, 0.2)-cut of (c2, e4) = 1.
10. μ(c2, e5) = (0.5, 0.2) means μ^1(c2, e5) = 0.5 and μ^2(c2, e5) = 0.2. The positive membership-μ^1(c2, e5) = 0.5 ≥ 0.6. Hence, (0.6, 0.2)-cut of (c2, e5) = 0.
11. μ(c3, e1) = (0.7, 0.3) means μ^1(c3, e1) = 0.7 and μ^2(c3, e1) = 0.3. The positive membership-μ^1(c3, e1) = 0.7 ≥ 0.6 also negative membership-μ^2(c3, e1) = 0.3 ≤ 0.2. Hence, (0.6, 0.2)-cut of (c3, e1) = 1.
12. μ(c3, e2) = (0.6, 0.4) means μ^1(c3, e2) = 0.6 and μ^2(c3, e2) = 0.4. The positive membership-μ^1(c3, e2) = 0.4 ≤ 0.6. Hence, (0.6, 0.2)-cut of (c3, e2) = 0.

The edge E_m,x(β) is added for the object c1 as vertex e6(0.6, 0.2).

The above computed (x, β)-cut is shown in Table 8 respectively. We can observe that Table 8 represents a binary formal context of given bipolar fuzzy formal context shown in Table 7.

The formal concepts generated from Table 8 are:
1. {∅, {e1, e2, e4, e6}}.
2. {{c1}, {e6}}.
Table 8
A (0.6, –0.2)-cut of the bipolar fuzzy formal context shown in Table 7.

<table>
<thead>
<tr>
<th>e₁(0.6, –0.2)</th>
<th>e₂(0.6, –0.2)</th>
<th>e₃(0.6, –0.2)</th>
<th>e₄(0.6, –0.2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>c₁</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>c₂</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>c₃</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>c₄</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

3. \(\{(c₃),(e₂)\}\).
4. \(\{(c₃),(e₁,e₅)\}\).
5. \(\{(c₂,c₃),(e₁)\}\).
6. \(\{(c₁,c₂,c₃,c₄),\emptyset\}\).

The concept lattice for above generated concepts is shown in Fig. 3 (http://conexp.sourceforge.net/). The attribute implications generated from Table 8 are:

1. \(e₅ \rightarrow e₁\).
2. \(e₁e₆ \rightarrow e₂e₆\).
3. \(e₂e₆ \rightarrow e₁e₅\).
4. \(e₁e₂ \rightarrow e₁e₅\).

The formal concept (number 4)-\(\{(c₃),(e₁,e₅)\}\) and the attributes implication (number 3)-\(\{e₂e₆ \rightarrow e₁e₅\}\) represents that, car-c₃ covers more desired parameters. Hence, it will be the first preference for the customer to purchase the car.

Similarly, we can conclude that:

• car-c₄ and car-c₂ covers one–one desired parameters. The customer can consider any one of them as second preference at the time of the purchasing of car.

We can observe that (0.6, –0.2)-cut of the bipolar fuzzy formal context (shown in Table 7) and its lattice structure (shown in Fig. 3) are helpful for the company in preference analysis of the customer for purchasing the car. The obtained conclusions using \((x,β)\)-cut are similar to those obtained by its bipolar fuzzy graph representation of concept lattice (shown in Fig. 2 discussed in Section 4.2) as well as in [28].

5. Discussions

FCA is a well established mathematical model based on lattice theory, for knowledge processing tasks [8,26,44,45]. The outputs of FCA are formal concepts, concept lattice and attribute implications. These outputs plays a major role in the practical applications of FCA with binary [7–11,26,52] as well as in fuzzy attributes [12–16,19,27]. FCA was incorporated with the interval-valued fuzzy setting for handling partial ignorance and uncertainty in data [20–22,24,30]. Recently, Singh and Aswani Kumar [46,48] discussed concept lattice representation through interval-valued fuzzy graph and its extension where as some researchers discussed its applications [56,57]. However, these representations of concept lattice were restricted to the unipolar space. Few research papers are focused on knowledge processing of bipolar information [23,25,29,32,38,55], in lattice theory [6,17,18,37,48] as well as in bipolar fuzzy graph [1,2,6,21,28,40,54]. Bipolarity is important to distinguish between positive information (which represents what is guaranteed to be possible) and negative information (which represents what is impossible or surely false). Hence, in this study we have focused on hierarchical order visualization of bipolar information through concept lattice. The motivation for this study can be understood from some important literature on FCA as shown in Table 9.
Table 9
Some important literature on FCA and its emerging research trends.

<table>
<thead>
<tr>
<th></th>
<th>Binary</th>
<th>Fuzzy</th>
<th>Interval-valued</th>
<th>Bipolar</th>
</tr>
</thead>
<tbody>
<tr>
<td>Concept lattice with graph</td>
<td>Berry and Sigayret [16]</td>
<td>Ghosh et al. [27]</td>
<td>Prem and Aswani Kumar [46,48]</td>
<td></td>
</tr>
<tr>
<td>FCA application</td>
<td>Poelmans et al. [44,45]</td>
<td>Antoni et al. [6]</td>
<td>Zerarga and Djouadi [56]</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Li and Tsai [37]</td>
<td></td>
</tr>
</tbody>
</table>

This study has focused on the following problems (marked as * in Table 9):

1. How to represent the bipolar information in fuzzy formal context?
2. How to investigate the formal concepts hidden in the given bipolar fuzzy formal context?
3. How to visualize the hierarchical order between the generated bipolar fuzzy concepts (into the concept lattice)?
4. How to find bipolar fuzzy attribute implications?

To address these problems following proposals have been made:

1. Fuzzy formal context is represented using bipolar fuzzy set with an example in Section 4.1.
2. A method is proposed for investigating the bipolar fuzzy formal concepts (as shown in Table 2) with its illustration in Section 4.1.
3. An application of the proposed method is discussed with an example in Section 4.2 and analysis compared with [28].
4. A method is proposed for decomposition of bipolar fuzzy formal context (in Section 3.2) with illustration (in Section 4.3).

Bipolarity meets five conditions: (1) Coexistence, (2) Equilibrium, (3) Negation, (4) Linearity and (5) Integrity [58–60]. In this paper we have focused on negation condition to introduce bipolarity in fuzzy formal context [34]. It was already discussed by Djouadi and Prade [20,21], that concept lattice can be used for IR (information retrieval) in order to express disjunction (shown in Table 3) and negation in queries (shown in Table 4). In this paper we have analyzed these two contexts through bipolar fuzzy set (shown in Table 5) and further by a bipolar fuzzy formal context (shown in Table 6). For the analysis of bipolar fuzzy context we need to generate the bipolar fuzzy formal concepts and their visualization in the concept lattice. For this purpose, we have incorporated the bipolar fuzzy graph into concept lattice and have proposed an algorithm for generating the bipolar fuzzy formal concepts (as given in Section 3). We can observe that the concept lattice shown in Fig. 1 represents the positive and negative side simultaneously. Moreover, the generated bipolar fuzzy formal concepts represents the formal concepts more adequately than the existing fuzzy approaches [15,16,20–22,27,30,39,42,46–50]. For finding the bipolar fuzzy attribute implications, we need a bipolar complement (¬) and implication (⇒). However, bipolar complement ( ¬ ) and implication (⇒) are needed more explanation for bipolar inference [29,30,56]. Hence, the last problem is still remaining to solve.

An application of the proposed method is discussed for analyzing the preference of a customer to buy the car (given in Table 7) in Section 4.2 as well as its (α, β)-cut (as shown in Table 8) in Section 4.3. We can observe that the proposed link provides hierarchical ordering between the car and its constituent parameters (as shown in Fig. 2). The hierarchical ordering representation of given bipolar information provides a way to analyze the preference (opinion) of the customer (to purchase the car) using the properties of concept lattice.

We believe that the analysis discussed in this paper will be helpful for the researchers in practical applications of FCA in various fields like:

- Knowledge representation and processing tasks [5–15,19,20,25,30,31,36].
- Handling bipolar data and their representations [1–3,17,18,23–25,29,32,40].
- Concept lattice representation [11–22,24,25,27,30,46–50].
- Handling incomplete data [4,21,33,35].

6. Conclusions

This paper focused on visualizing positive and negative side of bipolar information simultaneously in the concept lattice. However, the available graphical representation of concept lattice defines the data in the unipolar space and lacks in visualizing bipolarity. For this purpose, we have aimed at establishing the link between bipolar fuzzy graph and concept lattice. The outline of this study is as follows:
Bipolar fuzzy set is introduced into the fuzzy formal context for handling bipolar information.

For generation of bipolar fuzzy formal concepts an algorithm has been proposed with its application.

For decomposition of a bipolar fuzzy formal context a method is proposed based on \((\alpha, \beta)\)-cut with its application.

Problem of bipolar fuzzy attribute implications is also discussed.

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