BAYESIAN APPROACH FOR OPTIMUM STEP-STRESS ACCELERATED LIFE TESTING

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ABSTRACT

This paper presents optimum plans for simple step-stress tests from a Bayes viewpoint. We obtain the optimum test plans to minimize the asymptotic variance of the maximum likelihood estimator of the mean life at a design (use) stress. The emphasis of this paper is to establishment of new areas of application to step-stress accelerated life testing (ALT) and an improvement of existing procedures and theories. Examples for Type I and Type II censored cases are illustrated. The numerical results indicate that under the same conditions, the proposed Bayesian approach is superior to that of Bai (1989) in the sense that the asymptotic variance is smaller.

Key words and phrases: bayesian; step-stress; mixture prior distribution; type I censored; type II censored; asymptotic variance.

JEL classification:

1. Introduction

Accelerated life tests (ALT) provide information quickly on the life distribution of the materials or products by testing them at higher-than-nominal levels of stress
such as high temperature, voltage, pressure, vibration, or load to induce early failures. The results obtained from ALT are analyzed and then extrapolated to estimate the life distribution at the design (use) stress. Step-stress scheme which allows the stress of a unit to be changed at a pre-specified times or the occurrence of a fixed number of failures. The former is known as type I (time step-stress test) censoring and the latter is type II (failure step-stress test) censoring. The simplest step-stress ALT which uses only 2 stress levels is called the simple step-stress tests. The main advantage of step-stress testing over constant-stress testing is that it can further reduce test time and the variability of the failure times. The problem of modeling data from ALT and making inference from such data has been studied by many authors. Nelsen (1980) considered data from ALT and obtained MLE for the parameters of a Weibull distribution under the inverse power law using the breakdown-time data of electrical insulation. Miller and Nelsen (1983) studied optimum test plans which minimized the asymptotic variance of the MLE of the mean life at a design (use) stress for the simple two-step ALT when all units are run to failure. Bai et al. (1989) further studied optimum two-step ALT where a pre-specified censoring time is involved. Rene et al. (1996) developed a Bayesian model and studied the inferences of data from ALT. Li (2002) studied a Bayes Empirical Bayes approach to estimation of the failure rate in exponential distribution. Lin et al. (2002) studied Bayesian sampling plans for the exponential distribution based on type I censoring data. Basu and Rama (2003) considered Bayesian estimation of system reliability in Brownian stress-strength models.

The purpose of this paper is to extend the result of Rene et al. (1996) to the case where a prescribed censoring time is involved and our goal is to seek an optimum ALT plan with censoring among these Bayes-viewpoint plans. Our analysis applies where the underlying failure distributions at each accelerated stress-level are exponential and incorporate the prior information into the analysis. Inference concerning the life length of the materials or products are based on the posterior distribution.

In section 2, mixture prior distributions are used to show the relation between step-stress and life distributions. The likelihood function and some basic properties are also discussed. In section 3, the optimum criterion is proposed. Some simulation results are illustrated in section 4. Conclusions are given in section 5.
Notation:

- $x_0$: design (use) stress
- $x_i$: $i$-th test-stress, $i = 1, 2; \ x_1 < x_2$
- $n$: total number of test units placed on test
- $n_c$: number of censored units
- $n_i$: number of failed units at stress $x_i$, $i = 1, 2$
- $Y_{i,j}$: $j$-th failure times of test unit at stress $x_i$, $i = 1, 2, j = 1, 2, \ldots, n_i$
- $Y_{i,(j)}$: $j$-th ordered failure times of test unit at stress $x_i$, $i = 1, 2, j = 1, 2, \ldots, n_i$
- $\lambda_i$: $\theta_i^{-1}$, where $\theta_i$ is the mean time to failure (MTTF) at stress $x_i, i = 0, 1, 2$
- $\tau$: time of stress change
- $T$: censoring time
- $\pi$: proportion of stress change
- $\pi_i$: $n_i/n$, proportion of test units to be observed at stress $x_i, i = 1, 2$
- $\pi_c$: $n_c/n$, proportion of test units to be censored
- $\pi_i(\lambda_1, \lambda_2)$: prior distribution, $i = 1, 2$
- $\pi(\lambda_1, \lambda_2|y_{i,j})$: posterior distribution, $i = 1, 2, j = 1, 2, \ldots, n_i$
- $F_i(\cdot)$: cdf of exponential distribution with mean $\lambda_i, i = 1, 2$
- $G(\cdot)$: cdf of a test unit under simple step-stress
- $\text{Asy. Var.}(\log \hat{\theta}_0)$: asymptotic variance of log (mean failure time) at the design (use) stress

Basic Assumptions:

1. Two stress levels $x_1$ and $x_2$ are used; where $x_1 < x_2$.

2. The life of a test unit at each stress is described by an exponential distribution.

3. At stress level $x$, the mean life of a test unit is a log-linear function of stress, i.e. $\log[\lambda(x)] = a + b x$. The $a, b$ are unknown parameters and depend on the nature of the product and the test method. The failure rate is an increasing function of the applied stress levels.

4. We suppose that a cumulative exposure model (CEM) holds, that is, the remaining life of a test unit depends only on its present cumulative exposure.

5. It is possible to find some prior information on the failure rate at both the use-stress level and the accelerated stress levels.
2. Simple step-stress ALT censored data

For Bayesian analysis, one needs to specify prior distribution for the unknown parameters of the model. A conjugate type prior is commonly used. Problems related to Bayesian approach have been studied by Rene et al. (1996) in connection with the step-stress ALT, by Li (2002) in connection with the estimation of failure rate in exponential distribution and by many others in various contexts.

Carlin and Louis (2000) provide that while a single conjugate prior may not accurately reflect available prior knowledge, a finite mixture of conjugate priors may be sufficiently flexible while still enabling simplified posterior calculations. It is assumed that the parameter \( \lambda \) is a realization of a positive random variable \( \Lambda \) having a prior density \( \pi(\lambda) \) over \((0, \infty)\). Given assumption 2 and 5, we choose a conjugate prior distribution for each stress with gamma density function given by

\[
\pi_i(\lambda_1, \lambda_2) = \frac{1}{\Gamma(\alpha_i)\beta_i} \lambda_1^{\alpha_i-1} e^{-\lambda_1/\beta_i}, \quad \lambda_i > 0, \quad i = 1, 2
\]

and consider the mixture prior distribution

\[
\pi(\lambda_1, \lambda_2) = r \cdot \pi_1(\lambda_1, \lambda_2) + (1 - r) \cdot \pi_2(\lambda_1, \lambda_2) = r \cdot \frac{\lambda_1^{\alpha_1-1} e^{-\lambda_1/\beta_1}}{\Gamma(\alpha_1)\beta_1^{\alpha_1}} + (1 - r) \cdot \frac{\lambda_2^{\alpha_2-1} e^{-\lambda_2/\beta_2}}{\Gamma(\alpha_2)\beta_2^{\alpha_2}},
\]

where \( r, 0 \leq r \leq 1 \) is pre-specified by the experimenter. Since each of the mixture components should specify a joint prior for \( \lambda_i, i = 1, 2 \) and the above prior specification implies that \( \lambda_2 \) has an proper prior. That is, \( \lambda_2 \) is uniform over \((0,1)\) in the first mixture component and \( \lambda_1 \) has a similar prior for the second mixture component.

Let us consider the simple step-stress ALT with censoring. Suppose the lifetimes of the test units \( Y_{i,j}, i = 1, 2, j = 1, \cdots, n_i \), follow an exponential distribution having expected lifetime \( \lambda_i \) for stress \( x_i, i = 1, 2 \), respectively. That is, \( Y_{i,j} \sim \text{Exp}(\lambda_i) \) and \( f_i(y_{i,j}) = \lambda_i e^{-\lambda_i y_{i,j}}, i = 1, 2, j = 1, \cdots, n_i \). Therefore under the assumption of cumulative exposure model, the cdf of a test unit under simple step-stress is given by

\[
G(y) = \begin{cases} 
F_1(y), & 0 \leq y < \tau \\
F_2(y + s - \tau), & \tau \leq y < T 
\end{cases}, \quad s \text{ is the solution to } F_2(s) = F_1(\tau).
\]

(3)
Thus the likelihood function from observations \( Y_{i,j}, \ i = 1, 2, \ j = 1, \ldots, n_i \) under Type I censoring is formally given by

\[
L(\lambda_1, \lambda_2) = \prod_{j=1}^{n_1} \left[ \lambda_1 e^{-\lambda_1 y_{1,(j)}} \right] \cdot \prod_{j=1}^{n_2} \left[ \lambda_2 e^{-\lambda_2 (y_{2,(j)} - \tau) - \lambda_1 \tau} \right] \cdot \prod_{j=1}^{n_c} \left[ e^{-(\lambda_1 \tau - \lambda_2 (T - \tau))} \right],
\]

where \( n_c = n_1 - n_1 - n_2, \tau \) is time of stress-change and \( T \) is the censoring time.

The mixture posterior distribution of \( Y_{i,j}, \ i = 1, 2, \ j = 1, 2, \ldots, n_i \), is obtained as the product of (2) and (4) given by

\[
\pi(\lambda_1, \lambda_2|y_{i,j}, \ i = 1, 2, \ j = 1, 2, \ldots, n_i) = \frac{L(\lambda_1, \lambda_2) \times \pi(\lambda_1, \lambda_2)}{(1 - r) \cdot \lambda_1^{n_1+\alpha_1-1} e^{-\lambda_1 \left( \sum_{j=1}^{n_1} y_{1,(j)} + \frac{1}{\beta_1} \right)} \cdot \lambda_2^{n_2+\alpha_2-1} e^{-\lambda_2 \left( \sum_{j=1}^{n_2} y_{2,(j)} + \frac{1}{\beta_2} \right)}}
\]

\[
\Gamma(\alpha_1) \beta_1^{\alpha_1} \Gamma(\alpha_2) \beta_2^{\alpha_2}
\]

We integrate \( \pi(\lambda_1, \lambda_2|y_{i,j}, \ i = 1, 2, \ j = 1, 2, \ldots, n_i) \) w.r.t. \( \lambda_i, \ i = 1, 2 \). Then the mixture marginal posterior distributions of \( \lambda_i, \ i = 1, 2 \) are given by (6) and (7).

\[
\pi(\lambda_1|y_{i,j}, \ i = 1, 2, \ j = 1, 2, \ldots, n_i) \propto r \cdot \Gamma \left( n_1 + \alpha_1, \left( U_1 + \frac{1}{\beta_1} \right)^{-1} \right) + (1 - r) \cdot \Gamma \left( n_1 + 1, (U_1)^{-1} \right)
\]

\[
\pi(\lambda_2|y_{i,j}, \ i = 1, 2, \ j = 1, 2, \ldots, n_i) \propto r \cdot \Gamma \left( n_2 + 1, (U_2)^{-1} \right) + (1 - r) \cdot \Gamma \left( n_2 + \alpha_2, \left( U_2 + \frac{1}{\beta_2} \right)^{-1} \right)
\]

where \( U_1 = \sum_{j=1}^{n_1} y_{1,(j)} + (n_2 + n_e)\tau, (n - n_1)\tau \leq U_1 \leq n\tau \) and \( U_2 = \sum_{j=1}^{n_2} (y_{2,(j)} - \tau) + (T - \tau)n_e, n_e T - (n - n_1)\tau \leq U_2 \leq (n - n_1)T \) are the total time on test at stress \( x_i, i = 1, 2 \), respectively.

When presenting a statistical estimator, it is usually necessary to indicate the accuracy of the estimator. The customary Bayesian measure of an estimator is to deal with the squared error loss. Therefore, we use (6) and (7) which are the mixture marginal posterior distribution of \( \lambda_i \) to estimate \( \lambda_i \) under stress \( x_i, i = 1, 2 \), respectively. Then
the Bayesian estimator of \( \lambda_i, \ i = 1, 2 \) under the squared error loss is the mixture marginal posterior mean of (6) and (7) given by

\[
\hat{\lambda}_1 = r \cdot \frac{n_1 + \alpha_1}{U_1 + 1/\beta_1} + (1 - r) \cdot \frac{n_1 + 1}{U_1},
\]

and

\[
\hat{\lambda}_2 = r \cdot \frac{n_2 + 1}{U_2} + (1 - r) \cdot \frac{n_2 + \alpha_2}{U_2 + 1/\beta_2}.
\]

Other common Bayesian estimates of \( \lambda_i, \ i = 1, 2 \) include the mode and the median of (6) and (7). It is probably worthwhile to calculate and compare all three in a Bayesian study, especially with regard to their robustness to changes in the prior.

3. Optimum Criterion

The optimum test plans under Type I and Type II censoring are defined to be the optimum values of \( \tau^* \) and \( \pi_i^* \) that minimize the asymptotic sampling variance of \( \log \hat{\lambda}_0 \), where \( \hat{\lambda}_0 \) is the estimated mean life time under the design (use) stress \( x_0 \) which is non-accelerated.

3.1 Optimum Test Plan for Type I censoring

By assumption 3, we substitute \( \log \lambda_i = a + bx_i \) for \( \lambda_i, \ i = 1, 2 \) in (5). Here we see \( \lambda_i, \ i = 1, 2 \) as the realization of random variables \( \Lambda_i, \ i = 1, 2 \) of the prior parameters.

Then the log likelihood function for \( Y_{i,j} = y_{i,j}, \ i = 1, 2, \ j = 1, 2, \ldots, n_i \) which is a function of unknown parameter \( a \) and \( b \) given by

\[
\ell \ln\pi(\lambda_1, \lambda_2|y_{i,j}, \ i = 1, 2, \ j = 1, 2, \ldots, n_i)
\]

\[
= (2n_1 + 2n_2 + \alpha_1 + \alpha_2 - 2)a + [(2n_1 + \alpha_1 - 1)x_1 + (2n_2 + \alpha_2 - 1)x_2]b
\]

\[
- \left( 2U_1 + \frac{1}{\beta_1} \right) e^{a+bx_1} - \left( 2U_2 + \frac{1}{\beta_2} \right) e^{a+bx_2} + \ell n \left[ \frac{r(1 - r)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\beta_1^{\alpha_1}\beta_2^{\alpha_2}} \right].
\]

Then the Fisher’s Information Matrix \( I \) is obtained by taking the expectation of the second partial and mixed partial derivatives of \( \ell = \ell \ln\pi(\lambda_1, \lambda_2|y_{i,j}, \ i = 1, 2, \ j = \ldots, n_i) \).
\[ E(U_1 \lambda_1) = E[E(U_1 \lambda_1 | \lambda_1, \lambda_2)] = E \left\{ \lambda_1 E \left[ \sum_{j=1}^{N_1} Y_{1,(j)} + (n - N_1) \tau \right] \left| \lambda_1, \lambda_2 \right. \right\} 
= n \left\{ \left[ 1 - r(1 + \beta_1 \tau)^{-\alpha_1} + (1 - r) \left( \tau^{-1}(1 - e^{-\tau}) \right) \right] \right\}. \tag{14} \]

\[ E(U_2 \lambda_2) = E[E(U_2 \lambda_2 | \lambda_1, \lambda_2)] = E \left\{ \lambda_2 E \left[ \sum_{j=1}^{N_2} (Y_{2,(j)} - \tau) + (T - \tau)n_e \right] \left| \lambda_1, \lambda_2 \right. \right\} 
= n \left\{ \left[ 1 - (1 + \beta_2 (T - \tau))^{-\alpha_2} + (1 - (1 + \beta_2 (T - \tau))^{-\alpha_2}) \right] \right\}. \tag{15} \]

where

\[ E \left( \frac{N_1}{n} \bigg| \lambda_1 \right) = \int_0^\tau \lambda_1 e^{-\lambda_1 y} dy = 1 - e^{-\lambda_1 \tau}, \]
\[ E \left( \frac{N_2}{n} \bigg| \lambda_2 \right) = \int_0^T \lambda_2 e^{-\lambda_2 (y - \frac{\lambda_2 - \lambda_1}{\lambda_2 - \lambda_1})} dy = e^{-\lambda_1 \tau} \left( 1 - e^{-\lambda_2 (T - \tau)} \right) \]

and

\[ E \left[ \sum_{j=1}^{N_1} Y_{1,(j)} \bigg| N_1, \lambda_1 \right) = N_1 E \left( Y_{1,j}^* \bigg| N_1, \lambda_1 \right) = N_1 \left( \frac{1}{\lambda_1} - \frac{\tau e^{-\lambda_1 \tau}}{1 - e^{-\lambda_1 \tau}} \right), \]
\[ E \left[ \sum_{j=1}^{N_2} Y_{2,(j)} \bigg| N_2, \lambda_2 \right) = N_2 E \left( Y_{2,j}^* \bigg| N_2, \lambda_2 \right) = N_2 \left( \frac{(\tau + \lambda_2^{-1}) - e^{-\lambda_2 (T - \tau)} (T + \lambda_2^{-1})}{1 - e^{-\lambda_2 (T - \tau)}} \right). \]

Here \( Y_{1,j}^* \) is truncated at \((0, \tau)\), \( Y_{2,j}^* \) is truncated at \((\tau, T)\), \( N_1 | \lambda_1 \sim \text{Binominal} \left( n, 1 - e^{-\lambda_1 \tau} \right) \)

and \( N_2 | \lambda_2 \sim \text{Binominal} \left( n, e^{-\lambda_1 \tau} \left( 1 - e^{-\lambda_2 (T - \tau)} \right) \right) \).

Let \( \frac{E(U_1 \lambda_1)}{n} = A_1(\tau) \) and \( \frac{E(U_2 \lambda_2)}{n} = A_2(\tau) \), where \( A_i(\tau) \) is the probability that a
test unit fails while testing at stress \( x_i, i = 1, 2 \). Then the Fisher’s Information Matrix
Therefore of Type I censored case, only the time changing point

3.2 Optimum Test Plan for Type II censoring

deivative with respect to \( \tau \) and the time changing point \( \lambda \). The Asymptotic variance of \( \log \hat{\lambda}_0 \) which is the log mean life time of the design (use) stress \( x_0 \) is given by

\[
\text{Asy. Var}_\tau(\log \hat{\lambda}_0) = \text{Asy. Var}_\tau(\hat{\alpha}) + 2x_0 \text{Asy. Cov}_\tau(\hat{\alpha}, \hat{b}) + x_0^2 \text{Asy. Var}_\tau(\hat{b}),
\]

\[
(17)
\]

(18)

where \( \text{Var}_\tau(\hat{\alpha}), \text{Cov}_\tau(\hat{\alpha}, \hat{b}) \) and \( \text{Var}_\tau(\hat{b}) \) are obtained explicitly in entries of (17).

From (16)-(18), it is clear that the asymptotic variances depend only on \( x_i, i = 1, 2 \) and the time changing point \( \tau \). In order to minimize \( \text{Asy. Var}_\tau(\log \hat{\lambda}_0) \), we take partial derivative with respect to \( \tau \) to get the optimum time changing point \( \tau^* \) to attain the optimum plan.

\[
\text{i.e. let } \frac{\partial \text{Asy. Var}_\tau(\log \hat{\lambda}_0)}{\partial \tau} = 0.
\]

3.2 Optimum Test Plan for Type II censoring

In Type II censored plan, the likelihood function is analogous to the situation of Type I censored case, only the time changing point \( \tau \) and censored time \( T \) turn to the ordered observation \( Y_{1,(n_1)} \) and \( Y_{2,(n_2)} \). Applying the analogous computation given in section 3.1 for Type I optimum plan, we can immediately obtain the Fisher's Information Matrix \( I \) for Type II censored case by taking the expectation of the second partial and mixed partial derivatives of \( \ell' = \ell n \pi(\lambda_1, \lambda_2 | y_{i,(j)}, i = 1, 2, j = 1, 2, \cdots, n_i) \).
with respect to \( a \) and \( b \) given by

\[
E \left( \frac{\partial^2 \ell'}{\partial a^2} \right) = -E \left[ \left( 2V_1 + \frac{1}{\beta_1} \right) \lambda_1 + \left( 2V_2 + \frac{1}{\beta_2} \right) \lambda_2 \right],
\]

(19)

\[
E \left( \frac{\partial^2 \ell'}{\partial b^2} \right) = -E \left[ \left( 2V_1 + \frac{1}{\beta_1} \right) x_1^2 \lambda_1 + \left( 2V_2 + \frac{1}{\beta_2} \right) x_2^2 \lambda_2 \right],
\]

(20)

\[
E \left( \frac{\partial^2 \ell'}{\partial a \partial b} \right) = -E \left[ \left( 2V_1 + \frac{1}{\beta_1} \right) x_1 \lambda_1 + \left( 2V_2 + \frac{1}{\beta_2} \right) x_2 \lambda_2 \right],
\]

(21)

where \( V_1 = \sum_{j=1}^{n_1} Y_{1,(j)} + (n_2 + n_c)Y_{1,(n_1)}, (n - n_1)\pi \leq V_1 \leq n\pi \) and \( V_2 = \sum_{j=1}^{n_2} (Y_{2,(j)} - Y_{1,(n_1)})Y_{1,(n_1)}\) are the total time on test at stress \( x_i, i = 1, 2 \) respectively. We have

\[
E(V_1 \lambda_1)
\]

\[
= E[E(V_1 \lambda_1 | \lambda_1, \lambda_2)]
\]

\[
= E \left[ \sum_{j=1}^{n_1} Y_{1,(j)} + (n_2 + n_c)Y_{1,(n_1)} | \lambda_1, \lambda_2 \right]
\]

\[
= n_1 \alpha_1 \beta_1
\]

(22)

\[
E(V_2 \lambda_2)
\]

\[
= E[E(V_2 \lambda_2 | \lambda_2)]
\]

\[
= E \left\{ \sum_{j=1}^{n_2} \left[ Y_{2,(j)} - Y_{1,(n_1)} \right] + n_c \left[ Y_{2,(n_2)} - Y_{1,(n_1)} \right] \right| \lambda_1, \lambda_2 \right\}
\]

\[
= n_2 \alpha_2 \beta_2
\]

(23)

Let \( \frac{E(V_1 \lambda_1)}{n_1 \alpha_1 \beta_1} = \frac{n_1}{n} = \pi_1 \) and \( \frac{E(V_2 \lambda_2)}{n_2 \alpha_2 \beta_2} = \frac{n_2}{n} = \pi_2 \), where \( \pi_i, i = 1, 2 \), are the proportion of test units to be observed at stress \( x_i, i = 1, 2 \) respectively. Then the Fisher’s Information Matrix \( I \) is given by

\[
I = \begin{bmatrix}
2n(\pi_1 + \pi_2) + a_1(r) + a_2(r) & 2n(\pi_1 x_1 + \pi_2 x_2) + a_1(r)x_1 + a_2(r)x_2 \\
2n(\pi_1 x_1 + \pi_2 x_2) + a_1(r)x_1 + a_2(r)x_2 & 2n(\pi_1 x_1^2 + \pi_2 x_2^2) + a_1(r)x_1^2 + a_2(r)x_2^2
\end{bmatrix}
\]

(24)

where \( a_1(r) = E\left( \frac{\lambda_1}{\beta_1} \right) = \beta_1^{-1} \left[ r \cdot \alpha_1 \beta_1 + (1-r) \cdot \frac{1}{2} \right] \) and \( a_2(r) = E\left( \frac{\lambda_2}{\beta_2} \right) = \beta_2^{-1} \left[ r \cdot \frac{1}{2} + (1-r) \cdot \alpha_2 \beta_2 \right] \).

Therefore

\[
I^{-1} = \frac{1}{\det(I)} \times
\]

\[
\begin{bmatrix}
2n \left( \pi_1 x_1^2 + \pi_2 x_2^2 \right) + a_1(r)x_1^2 + a_2(r)x_2^2 & -2n(\pi_1 x_1 + \pi_2 x_2) + a_1(r)x_1 + a_2(r)x_2 \\
-2n(\pi_1 x_1 + \pi_2 x_2) + a_1(r)x_1 + a_2(r)x_2 & 2n(\pi_1 + \pi_2) + a_1(r) + a_2(r)
\end{bmatrix}
\]

(25)
The Asymptotic variance of \( \log \hat{\lambda}_0 \) which is the log mean life time of the use stress \( x_0 \) is given by

\[
\text{Asy. Var}_{\pi_1,\pi_2} \left( \log \hat{\lambda}_0 \right) = \text{Asy. Var}_{\pi_1,\pi_2}(\hat{a}) + 2x_0\text{Asy. Cov}_{\pi_1,\pi_2}(\hat{a},\hat{b}) + x_0^2\text{Asy. Var}_{\pi_1,\pi_2}(\hat{b}),
\]

(26)

where \( \text{Var}_{\pi_1,\pi_2}(\hat{a}) \), \( \text{Cov}_{\pi_1,\pi_2}(\hat{a},\hat{b}) \) and \( \text{Var}_{\pi_1,\pi_2}(\hat{b}) \) are obtained explicitly in entries of (25).

From (24)-(26), it is clear that the asymptotic variance depends only on \( x_i, i = 1, 2 \) and failure changing proportion \( \pi_i, i = 1, 2 \). In order to minimize \( \text{Asy. Var}_{\pi_1,\pi_2}(\log \hat{\lambda}_0) \), we take partial derivative with respect to \( \pi_i, i = 1, 2 \) to get the optimum time changing proportion \( \pi_i^* \), \( i = 1, 2 \) to attain the optimum plan.

i.e. let \( \frac{\partial \text{Asy. Var}_{\pi_1,\pi_2}(\log \hat{\lambda}_0)}{\partial \pi_i} = 0 \), \( i = 1, 2 \).

4. Example comparison

To illustrate the use of the proposed method in this paper, a real data set reported in Nelson (1990) is investigated. A step-stress test of fluid insulation was run to estimate insulation life at a design (use) stress of 20Kv. Each test units was at 26Kv (low stress) before it went into 32Kv (high stress). For the purpose of illustrating the method discussed in this paper, we grouped the lifetime data according to consecutive stress-change times and the test system runs to a specified time \( T \) or proportion \( \pi_c \), the experiment stops. It is clear that the inverse power law model can be transformed into a log-linear function by taking the natural logarithm. That is, the design stress \( x_0 = \log(20) = 2.9957 \), the low stress level \( x_1 = \log(26) = 3.2581 \), the high stress level \( x_2 = \log(32) = 3.4657 \). The amount of stress extrapolation is \( \xi = (26 - 20)/(32 - 26) = 1 \). By using exact 76 test failure times, Nelson (1990) fitted the Weibull model to the step-stress data and presented the MLE of model on Nelson (1990, Table 2.2 of Chapter 10). The MLE estimate for the Weibull shape parameter is 0.756 with an asymptotic 95% confidence interval as from 0.18 to 1.33. Because the confidence interval contains
the value 1, there is no significant evidence against the hypothesis that the failure times of these specimens follow an exponential distribution when tested against the larger family of Weibull distributions based on the standard normal test at a significance level of 5%, although it could be due to the lack of statistical power. We choose to base our analysis on exponential failure time in the step-stress test. We give some examples to illustrate the performance with Bai’s (1989) result. Some Monte Carlo studies are also given. This test is used to illustrate the optimum tests. Following section 3, the ML estimates of the model parameters and the means at the high and low stresses those data are \( \hat{a} = 64.912, \hat{b} = -17.704, \hat{\lambda}_1 = 1.67\text{min.} \) and \( \lambda_2 = 1380\text{min.} \), respectively. All failure rate estimates in this table are in failures per \( 10^{-5} \) minutes. We use (8) and (9) to estimate \( \lambda_1 \) and \( \lambda_2 \) and use \( \hat{\lambda}_0 \) to generate 76 test units with 500 replications. Table 1 gives the prior failure estimates in each step for type I and type II censored cases.

Atwood (2005) gave the estimation of \( \alpha_i \) and \( \beta_i, i = 1, 2 \), prior distribution for the mixture model. Four selected censored time \( T \) for type I censored were considered in the simulation of data. The results compared with Bai’s (1989) are shown in table 2.

In contrast, for the Bai’s optimum plan, when \( T \) is 400 minute, the minimum asymptotic variance is 32.38, larger by a factor of about 2.27 than our case 14.26. That is, the Bai’s optimum plan requires about 2.27 times as many test units as our optimum plan for the same accuracy. We observe that the minimum Asy. \( \text{Var}_T(\log \hat{\lambda}_0) \) decreasing as the censored time \( T \) increasing. The reason is obviously that the more censored time \( T \) we use, the more failure test units we observe and obtain more information from the failure test units.

For type II censored case, five selected censored proportion \( \pi_c \) were considered in the simulation of data. The results compared with Bai’s (1989) are shown in table 3.

In contrast, for the Bai’s optimum plan, when \( \pi_c \) is 0.20, the minimum asymptotic variance is 18.900, large by a factor about 1.74 times than our case 10.861. That is, the Bai’s optimum plan requires about 1.74 times as many test units as our optimum plan for the same accuracy. We observe that the minimum Asy. \( \text{Var}_{\pi_1,\pi_2}(\log \hat{\lambda}_0) \) increasing as the censored proportion \( \pi_c \) increasing. The reason is obviously that the more censored proportion \( \pi_c \) we use, the less failure test units we observe and obtain less information from the failure test units.
The most basic tool for investigating model uncertainty is the sensitivity analysis. That is, we simply make reasonable modifications to the assumption in question, re-compute the posterior quantities of interest and see whether they have changed in a way that has practical impact on interpretations or decision. Table 4 illustrates the most commonly used technique for investigating robustness; simple try different reasonable choice of $r$ for mixture prior distribution. We find that the data were strongly informative with respect to this assumption and the mixture prior distribution provides a degree of robustness to the selection of $r$.

As expected, the pre-specified values of the mixture prior distribution $r$ plays an important role in determining the optimal plan. Therefore, a sensitivity analysis is conducted to assess how robust the minimum Asy. Var($\log \hat{\lambda}_0$) against the values $r$. The results suggest that the ratio of Bai’s and our Asy. Var($\log \hat{\lambda}_0$) has minimum at $r = 1/2$. The ratio increases for small deviation from $r = 1/2$.

5. Conclusions

We reconsider the Bai (1989) model under a non-Bayesian setup. The disadvantage of non-Bayesian techniques is that they often require large sample size due to their inability to incorporate available prior information into the analysis. Due to the small sample size and to the lack of failures in the test results, a non-Bayesian approach for inference would have poor performance, and less precise. Each of these tests discussed in section 3 is optimum in the sense that it minimizes the asymptotic variance of the maximum likelihood estimate of the mean at a specified design (use) stress. By Bayesian approach and through the optimum criterion, we can save much time and more efficiency when we undertaking accelerated life testing experiment. Furthermore, Bai’s (1989) results are our special case and our result is more generalized. In section 4, a numerical study is carried out to find the optimal plans for various combinations of model parameter values. In essence, for given values of censoring time $T$ or proportion $\pi_c$, the corresponding optimal plan can be determined. Usually, the test under high stress level would end earlier than that under low stress level because the failure rate at the high level is higher than that at the low level.
Finally, to extend the life model under consideration, it is natural to consider Weibull or more general life distribution. Furthermore, take the cost of entire ALT experiment into consideration can be analogously applied through some computations. These questions are also the direction of future research in this area. Another important and interesting variation for future research associated with grouped and censored data from a step-stress ALT is to study the statistical analysis and the optimum design when both the stress change times and the censoring time are random variables. These may become our next goal.

Table 1  \( \lambda \) estimates for type I and type II censored case with \( r = 1/2 \) (all failures rates are in “failure/10^{-5} \) minutes.”)

<table>
<thead>
<tr>
<th>Stress</th>
<th>Prior estimate ( \lambda )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_0 = 20Kv ) (use)</td>
<td>0.696</td>
</tr>
<tr>
<td>( x_1 = 26Kv ) (low)</td>
<td>72.404</td>
</tr>
<tr>
<td>( x_2 = 32Kv ) (high)</td>
<td>2859.248</td>
</tr>
</tbody>
</table>

Table 2  Simulation results of type I censored case under four-selected censored time \( T \) with \( r = 1/2 \)

<table>
<thead>
<tr>
<th>( T ) (censored time)</th>
<th>400</th>
<th>600</th>
<th>800</th>
<th>1000</th>
<th>1200</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bai’s optimum ( \tau^* )</td>
<td>279.00</td>
<td>465.00</td>
<td>558.01</td>
<td>744.01</td>
<td>930.01</td>
</tr>
<tr>
<td>Our optimum ( \tau^* )</td>
<td>249.18</td>
<td>415.30</td>
<td>498.37</td>
<td>664.49</td>
<td>830.61</td>
</tr>
<tr>
<td>(A)</td>
<td>32.38</td>
<td>24.69</td>
<td>20.34</td>
<td>18.10</td>
<td>17.97</td>
</tr>
<tr>
<td>(B)</td>
<td>14.26</td>
<td>13.57</td>
<td>13.20</td>
<td>13.00</td>
<td>12.74</td>
</tr>
<tr>
<td>(A)/(B)</td>
<td>2.27</td>
<td>1.82</td>
<td>1.54</td>
<td>1.39</td>
<td>1.41</td>
</tr>
</tbody>
</table>

(A) is the Bai’s minimum Asy. Var, \( \tau^* (\log \hat{\lambda}_0) \) and (B) is Our minimum Asy. Var, \( \tau^* (\log \hat{\lambda}_0) \)
Table 3  Simulation results of type II censored case under six-selected censored proportion $\pi_c$ with $r = 1/2$

<table>
<thead>
<tr>
<th>$\pi_c$ (censored proportion)</th>
<th>0.00</th>
<th>0.05</th>
<th>0.10</th>
<th>0.15</th>
<th>0.20</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bai’s optimum $\pi^*_1$</td>
<td>0.571</td>
<td>0.543</td>
<td>0.517</td>
<td>0.496</td>
<td>0.461</td>
</tr>
<tr>
<td>Bai’s optimum $\pi^*_2$</td>
<td>0.429</td>
<td>0.407</td>
<td>0.383</td>
<td>0.354</td>
<td>0.339</td>
</tr>
<tr>
<td>Our optimum $\pi^*_1$</td>
<td>0.537</td>
<td>0.517</td>
<td>0.495</td>
<td>0.437</td>
<td>0.394</td>
</tr>
<tr>
<td>Our optimum $\pi^*_2$</td>
<td>0.463</td>
<td>0.433</td>
<td>0.405</td>
<td>0.413</td>
<td>0.406</td>
</tr>
<tr>
<td>(C)</td>
<td>15.120</td>
<td>15.914</td>
<td>16.783</td>
<td>17.785</td>
<td>18.900</td>
</tr>
<tr>
<td>(D)</td>
<td>10.278</td>
<td>10.378</td>
<td>10.550</td>
<td>10.691</td>
<td>10.861</td>
</tr>
<tr>
<td>(C)/(D)</td>
<td>1.471</td>
<td>1.533</td>
<td>1.591</td>
<td>1.664</td>
<td>1.740</td>
</tr>
</tbody>
</table>

(C) is the Bai’s minimum $\text{Var}_{\pi_1, \pi_2}(\log 0)$ and (D) is Our minimum $\text{Var}_{\pi_1, \pi_2}(\log 0)$.

Table 4  Sensitivity and robustness of different choice in $r$

<table>
<thead>
<tr>
<th>Type of censored</th>
<th>Type I censored ($T=400$ hr)</th>
<th>Type II censored ($\pi_c=0.20$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r$</td>
<td>0/4 1/2 3/4 1</td>
<td>0/4 1/2 3/4 1</td>
</tr>
<tr>
<td>(E)/(F)</td>
<td>2.53 2.37 2.27 2.45 2.60</td>
<td>1.91 1.80 1.74 1.86 1.96</td>
</tr>
</tbody>
</table>

(E) is the Bai’s minimum $\text{Var}(\log 0)$ and (F) is Our minimum $\text{Var}(\log 0)$.

References


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