Shape changing and accelerating solitons in integrable variable mass sine-Gordon model

Anjan Kundu

Theory Group, Saha Institute of Nuclear Physics

Calcutta, INDIA

anjan.kundu@saha.ac.in

March 27, 2007

Abstract

Sine-Gordon (SG) models with variable mass or perturbed soliton appear in many physical situations, which however breaks the integrability of the model. A class of such inhomogeneous models with accelerating and shape changing solitons is constructed, which are integrable both at the classical and the quantum level with exact solutions.

Among the exclusive families of nonlinear integrable systems the sine-Gordon (SG) model enjoys a special status and continues receiving attention till today, for its inherent richness and wide range of applications in different fields [1, 2, 3, 4]. Apart from the fascinating properties of integrable systems in general, e.g., Lax pair, infinite number of independent conserved charges, exact N-soliton solution obtainable by inverse scattering (IS) and Hirota's bilinear method etc, [6], the SG model possesses special properties, like relativistic invariance, integer-valued topological charge represented by solutions like kink, antikink, breather etc. [7]. Ultralocality with r-matrix formulation and leads to its quantum integrability described by the quantum Yang-Baxter equation (QYBE) $R(\frac{\lambda}{\mu})U_j(\lambda) \otimes U_j(\mu) = U_j(\mu) \otimes U_j(\lambda)R(\frac{\lambda}{\mu}), \quad j = 1, 2, ..., N,$ which for the SG model results to the well known quantum $su_q(2)$ algebra [8, 9].

Since a physical oscillator going beyond the simple harmonic motion is described by nonlinear equation: $\ddot{x} + \sin x = 0$, in a chain of such coupled oscillators the SG equation appears naturally in the continuum. This is the generic reason why the SG model can describe many physical events like current through Josephson junction (JJ), spin-wave in ferromagnet, charge-density wave, DNA transcription etc. [1, 2], apart from playing an important role in nonperturbative QFT [10]. Solitons in the SG model, as in other integrable systems, move with constant velocity and shape. In the realistic situations however under the influence of external forces or inhomogeneities soliton velocity may change [2, 3], which can be used as a desirable effect for fast transport, fast communication, or even for the possibility of a soliton gun [4]. Inhomogeneities can appear due to impurity, dislocation, defect or incommensuration in the media, producing additional terms in the SG equation or modifying the existing ones, with diverse consequences [2, 3, 11]. In a Josephson junction dissipation of fluxons or local enhancing of the Josephson current may occur, or the variable mass SG (VMSG) equation can also appear in modeling the propagation of domain walls, dislocations, fluxons etc. in the presence of noise, defect of the order parameter etc. [2]. Inhomogeneous xxz spin chains can arise in the Cooper-pair pumping in a linear array of JJ [5] or when the interaction strength varies periodically along the spin-chain. If the periods are incommensurating such inhomogeneities can lead to specific

form of VMSG model in the continuum [11]. Soliton velocity of the SG model can change under the influence of force or stepwise defect [3, 4].

However the inhomogeneities as well as variable soliton velocity tend to destroy the integrability of the SG model and hence its exact solutions, which are the most cherishable properties of this model and the related results can at best be perturbative [2, 3]. Therefore to meet the challenge of building SG model with accelerated soliton or variable mass and keeping its integrability preserved both at the classical and the quantum level, we observe that, the spoiling effect of variable velocity can be compensated for by a variable mass. The solitons of such integrable VMSG model can exhibit intriguing properties as shown in Fig. 1i-iii). The quantum integrability of the model can also be made valid, since the modifications do not affect its quantum R-matrix.

Since our strategy is to respect integrability, we start from the linear spectral problem $\Phi_x(x,\lambda) = U(\lambda,x)\Phi(x,\lambda)$, $\Phi_t(x,\lambda) = V(\lambda,x)\Phi(x,\lambda)$, with the Lax pair of the SG model: $U = \frac{i}{4}\left(-u_t\sigma^3 + mk_1\cos\frac{u}{2}\sigma^2 - mk_0\sin\frac{u}{2}\sigma^1\right)$, $V = \frac{i}{4}\left(-u_x\sigma^3 - mk_0\cos\frac{u}{2}\sigma^2 + mk_1\sin\frac{u}{2}\sigma^1\right)$, where $k_0 = 2\lambda + \frac{1}{2\lambda}$, $k_1 = 2\lambda - \frac{1}{2\lambda}$, with λ as the spectral parameter. Compatibility $\Phi_{xt} = \Phi_{tx}$ leads to the flatness condition $U_t - V_x + [U,V] = 0$, yielding the SG equation for constant mass m and λ. Recall that in the IS method applied to integrable systems the solitons are obtained as a reflection-less potential with discrete spectrum λ_n, $n = 1, 2, \ldots N$, representing poles of the transmission coefficient $\frac{1}{a(\lambda)}$, $a(\lambda = \lambda_n) = 0$ and the velocity of SG soliton (kink) is linked to $\lambda_1 = \frac{i}{2}e^\theta$ as $v_s = \tanh\theta$. Therefore for accommodating variable soliton velocity one should have a variable λ , which however violates in general the flatness condition. Making m also variable, we get on the other hand the constraint: $(k_0 m)_t + (mk_1)_x = 0$, $(k_1 m)_t + (mk_0)_x = 0$, with a solution

$$m(x,t) = mf_+f_-, \ \ k_0(x,t) = \cosh(\theta - \rho(x,t)), \ \ k_1(x,t) = \sinh(\theta - \rho(x,t)), \ \ \rho(x,t) = \ln\frac{f_+}{f_-}, \quad (1)$$

where m, θ are constants and $f_\pm$ are arbitrary smooth functions of $x \pm t$, respectively.

This constructs an integrable VMSG equation

$$u_{tt} - u_{xx} + m^2(x,t)\sin u = 0, \quad m(x,t) = mf_+f_-, \quad (2)$$

the relativistic invariance of which is lost in general. Nevertheless under a Lorenz transformation $(x,t) \to (x',t')$ the form of the equation (2) remains the same with the replacement $f_+f_- \to f'_+f'_-$ and therefore choosing the functions $f_\pm$ we can control the variable mass, as suitable to physical situations. Fig. 1 i-iv) show the dynamics of soliton with different mass functions.

For the exact solution of the VMSG model, let us apply both Hirota's bilinearization and the IS formalism, the former being a direct method, while the later is an indirect one for more general solutions. Hirota's solution for the SG equation may be expressed as $u = -2i\ln\frac{g_+}{g_-}$, where $g_\pm$ are conjugate functions with expansion in plane-wave solutions for its soliton solution. For the VMSG model (2) the same ansatz seems to work, only the plane waves should be replaced by their generalized form: $g^{(n)} = \frac{c_n}{\lambda_n}e^{\frac{i}{2}(X(\lambda_n,x,t)-T(\lambda_n,x,t))}$, where

$$X(\lambda_n,x,t) = \int^x dx' m(x',t)k_{1n}(x',t), \ \ T(\lambda_n,x,t) = \int^t dt' m(x,t')k_{0n}(x,t'). \quad (3)$$

This gives the soliton solutions through the expansion:

$$g_\pm = 1 \pm g^{(1)}, \quad \text{for the kink solution .}$$
$$g_\pm = 1 \pm (g^{(1)} + g^{(2)}) + s(\frac{\theta_1 - \theta_2}{2})g^{(1)}g^{(2)}, \quad \text{for 2-soliton solutions} \quad (4)$$

etc. with the scattering matrix $s(\theta) = tanh^2\theta$ and $\lambda_n = \frac{i}{2}e^{\theta_n}, n = 1, 2$, for the kink-kink and $\lambda_2 = -\lambda_1^* = \eta e^{i\theta}$, for the kink-antikink bound state (breather solution).

Similarly we can apply the IS formalism to the inhomogeneous SG model, for which the crucial step is to analyze and use the analytic properties of the solutions for Jost function Φ, identified by their asymptotic at space infinities. The required analyticity based on the behavior at $\lambda \to \infty$, should hold equally for the inhomogeneous extension, replacing again the asymptotic plane waves by their generalized form. Therefore, going parallel to the standard SG model [6], we get for N-soliton $(r(\lambda) = 0)$ with discrete spectrum λ_n , the solution for the Jost function component $\psi_n^{(1)}$ as

$$\psi = \frac{1}{2}[(1+V)^{-1} + (1-V)^{-1}]e^{\frac{i}{4}X}, \tag{5}$$

denoting column vectors as $\psi = \{\psi_1(\lambda_1), \ldots, \psi_N(\lambda_N)\}$, $e^X = \{e^{X(\lambda_1, x, t)}, \ldots, e^{X(\lambda_N, x, t)}\}$ and $V = \{V_{nm} = (\frac{c_m(t)}{\lambda_n + \lambda_m})e^{\frac{i}{4}(X(\lambda_n, x, t) + X(\lambda_m, x, t))}\}$, with $c_m(t) = c_m(0)e^{-\frac{i}{2}T(\lambda_m, x, t)}$, where the generalized X, T are as defined in (3). On the other hand comparing the leading term of $\psi^{(1)}(\lambda)$ at $\lambda \to 0$ we can connect it with the field: $\sin \frac{u}{2} = i \sum_n \frac{c_n(t)}{\lambda_n} \psi_n^{(1)} e^{\frac{i}{4}X(\lambda_n, x, t)}$, which derives using (5) the exact N-soliton solution for the SG field linked to the inhomogeneous model (2). We can get $N = 1$-soliton (kink) solution with $\lambda_1 = i\eta$ explicitly, either from the Hirota's or from the IS method presented above, as

$$\sin \frac{u}{2} = \frac{1}{cosh\zeta}, \quad u = 4\tan^{-1}(e^{\pm\zeta}), \quad \zeta = \frac{i}{2}(X(i\eta, x, t) - T(i\eta, x, t)). \tag{6}$$

with variable soliton velocity $v_s(x, t) = -\frac{dx}{dt} = \frac{k_1(\eta, x.t)}{k_0(\eta, x.t)}$.

To see the effect of different inhomogeneities on the properties of soliton, we consider some concrete cases. Notice that the choice of inhomogeneous functions as $f_+ = f_- = f$ with $f(x) = x^n$ leads to the variable mass $m(x^2 - t^2)^n$, preserved under relativistic motion. For the simplest case $n = 1$ we get the soliton and the kink solution (6) where $\zeta = \frac{m}{3}(2\eta(x-t)^3 + \frac{1}{2\eta}(x+t)^3)$. The evolution of this soliton is depicted in Fig 1i), showing clearly the intriguing change in its velocity and shape during the propagation. Since we have here $\zeta(x \to \pm\infty) \to \pm\infty$, the kink (antikink) solution corresponds to the usual topological charge $Q = \frac{1}{2\pi}(u(\infty) - u(-\infty)) = \pm 1$.

With the choice of inhomogeneity $f_- = 1$, $f_+ = \sqrt{2}\cos(x+t)$, or similarly $f_+ = 1, f_- = \sqrt{2}\cos q(x-t)$, we can get an integrable SG with variable mass $\sqrt{2}m\cos q(x \pm t)$ and therefore may conclude that the nonintegrable physical model with mass $m(x) = \cos qx$ found in [11] can be tuned to an integrable one by making coupling strength to oscillate also periodically in time. For mass $\sqrt{2}m\cos q(x+t)$ we can get soliton solution (6) with $\zeta = \frac{m}{2}(k_0 x - k_1 t + \frac{1}{2q}(k_0 - k_1)\sin 2q(x+t))$, $k_a = k_a(\eta), a = 1, 2$ with velocity v_s and width d of the soliton changing periodically as $v_s = md(k_1 + \frac{1}{\eta q}\cos 2q(x+t))$, $d = \frac{1}{m(k_0 + \frac{1}{\eta q}\cos 2q(x+t))}$. The behavior of the solution is shown in Fig. 1ii).

The choice of inhohomogeneity through exponential functions $f_+ = f_- = f = \exp(\frac{\rho}{2}x)$, leads to the only possible x-dependent mass as $m(x) = me^{\rho(x-x_0)}$ for an integrable SG equation . This demonstrates also why the VMSG with a different $m(x)$ derived in [11] turned out to be nonintegrable. The soliton solution for this integrable case is obtained from (6) with $\zeta = \frac{1}{\rho}k_0(t)m(x)$, $m(x) = \exp(\rho(x-x_0))$, $k_0(t) = \cosh(\theta - \rho(t-t_0))$. Fig. 1iii) shows the soliton evolution with changing width $d = \frac{1}{m(x)k_0(t)}$, shape and velocity $v_s = tanh(\theta - \rho(t-t_0))$, with acceleration and a bumeron [16] like property due to change in the direction of the velocity; ihe soliton speed however always remains less than the velocity of light: $|v_s| \leq 1$. For $\rho > 0$ we have $\zeta(x = \infty) = \infty$, but $\zeta(x = -\infty) = 0$ and that makes the soliton to loose its usual localized form and the finite-energy solution needs proper normalization. The corresponding kink solution will have a topological charge $Q = \pm\frac{1}{2}$ and it requires 2-soliton to regain $Q = \pm 1$ and the localized form. Similar unusual properties can be observed for inhomogeneous mass $m(x^2 - t^2)^n$ with even n.

At $\rho \to 0$: $\zeta = \frac{1}{\rho}k_0(t)m(x) \to \zeta = m(k_0(x-x_0) - k_1(t-t_0)$ and we recover the standard SG soliton and kink with $m = const, v_s = const$. This standard case is shown in Fig. 1iv) for comparison.
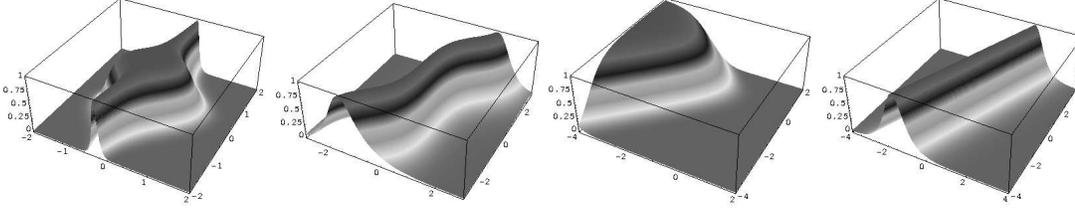
Figure 1: Exact soliton solutions for $\sin\frac{u}{2}$ of the SG equation with variable mass i) $m(x^2 - t^2)^2$, ii) $2m\cos q(x+t)$ iii) $m\exp(\rho x)$ and iv) $m = $ const.

Now we switch over to explore the quantum integrability of our inhomogeneous SG model and follow the algebraic Bethe ansatz method developed for the standard model applicable to its exact lattice version [8]. Quantum lattice SG Lax operator $U_j(\lambda, \mathbf{S}_j, m)$, $j = 1, 2, \ldots, L$ involves operators $S_j^3(u_j), S_j^{\pm}(u_j, p_j, m)$ with canonical momentum $p_j = \dot{u}_j$ and mass parameter m, which should be considered now as site dependent: m_j [12]. Recall that QYBE with the quantum R-matrix ensures the quantum integrability, which for the SG model becomes equivalent to the quantum algebra. Find that, the trigonometric R-matrix associated with the SG model remains unchanged under our inhomogeneous extension, since this $R(\frac{\lambda}{\mu})$-matrix depends on the ratio of two spectral parameters, in which x, t-dependence enters as multiplicative functions (1) and therefore cancels out. Moreover, QYBE being a local algebra (at each lattice site j) is not affected by inhomogeneity and yields the same quantum algebra $su_q(2)$; only with the replacement of m by a site-dependent function m_j in its structure constant: $[S_j^+, S_k^-] = \delta_{jk} m_j \frac{\sin\alpha 2 S_j^3}{\sin\alpha}$.

The aim of the algebraic Bethe ansatz is to solve exactly the eigenvalue problem of $trT(\lambda)$, $T(\lambda) = \prod_j U_j$, giving all conserved operators including the Hamiltonian with the eigenstates given as $|\lambda_1, \ldots, \lambda n > = \prod_j^n B(\lambda_j)|0 >$. $T_{12} = B(\lambda)$ acts as creation operator, while $T_{21} = C(\lambda)$ as destruction operator annihilating the pseudovaccum: $C(\lambda)|0 > = 0$. A crucial step in this formalism is to construct the pseudovaccum state $|0 >$, which is achieved for the SG model by combining the actions of consequitive pair of Lax operators: $U_j U_{j+1}|0 > [8]$. Following closely the procedure of [8] but generalizing it due to the site-dependent mass m_j, we can solve for the local pseudovaccum as $\Omega_j^{(2)} = \tilde{f}_{m_{2j} m_{2j+1}}(q_{2j}, q_{2j+1})$, $\tilde{f}_{m_1 m_2}(q_1, q_2) = (1 + \delta^2 g_{m_1 m_2}(q_1, q_2)) f_{m_1 m_2}(q_1, q_2)$, where g and f are generalizations

$$f_{m_1 m_2} = \left(\frac{m_2^{\frac{q_2}{\alpha}}}{m_1^{\frac{q_1}{\alpha}}}\right) f_m, \; g_{m_1 m_2} = \frac{m_2^2}{m} g_m \tag{7}$$

over their known solutions f_m, g_m for constant m [8]. Consequently the vacuum eigenvalues are generalized for the inhomogeneous SG model as $A(\lambda)|0 > = \alpha_m|0 > = \prod_j a(\theta, \frac{m_j}{m_{j+1}})|0 >$, $D(\lambda)|0 > = \beta_m|0 > = \prod_j a^*(\theta, \frac{m_{j+1}}{m_j})|0 >$, where $a(\theta, \frac{m_1}{m_2}) = (\frac{m_1}{m_2} + \delta^2 m_1 m_2(\cosh(2\theta + i\alpha)))$, yielding the exact eigenvalue for the conserved quantities: $trT(\lambda)$ as

$$\Lambda(\lambda; \lambda_1, \ldots, \lambda n) = \alpha \prod_j^n f(\frac{\lambda_j}{\lambda}) + \beta \prod_j^n f(\frac{\lambda}{\lambda_j}), \tag{8}$$

where $f(\frac{\lambda}{\mu})$ are expressed through the elements of the quantum $R(\frac{\lambda}{\mu})$-matrix for the SG model, which remains unchanged. The Bethe equations for determining the parametrs $\{\lambda_j\}$ are generalized similarly. The details on the exact spectrum and quantum soliton or breather solutions should follow [8], with $m_j \to m$ reducing to the standard result.

Thus we have shown the classical and quantum integrability of a class of variable mass SG models and constructed their exact solutions. The solitons exhibit changing velocity, shape, amplitude, depending on the nature of inhomogeneity. It may be noted that the inhomogeneous SG model constructed here can be transformed formally to a homogeneous model by moving to a noninertial frame $(x,t) \rightarrow (X,T)$ through nonlinear transformation $X = \int dx F_+ + \int dt F_-$, $T = \int dx F_- + \int dt F_+$, $F_\pm = \frac{1}{2}(f_+^2 \pm f_-^2)$. However for investigating the physical effect of accelerating and shape changing solitons propagating through given inhomogeneous media, one has to analyze the model in the original form with variable mass. Similar situation arises also in other inhomogeneous systems with integrable non-isospectral flow, e.g. in the study of accelerated solitons in plasma and other models governed by the nonlinear Schrödinger (NLS) equation [13], in analyzing discrete NLS with inhomogeneities exhibiting trapped solitons in an oscillating potential [14], in exactly solvable inhomogeneous Toda chain [15] or in matrix Schrödinger problem with the velocity of bumeron solution changing its direction [16]. Quantum generalization of related models was considered recently [17]. In most of these systems though the inhomogeneities could be removed by tricky nonlinear transformations, the investigations were carried out in the original systems due to their physical relevance.

The author likes to thank Profs. P. Mitra (SINP) and D. Sen (IISC) for helpful discussions.

References

[1] A Barone, G. Palerno, *Physics and applications of Josephson effect*, (Wiley, NY, 1982); V. G. Baryakhtar, *Dynamics of topological magnetic solitons*, (Springer, Berlin, 1994); G.L. Lamb, Rev. Mod. Phys. **43**, 99 (1971); M. J. Rice et al, Phys. Rev. Lett. , **36**, 432 (1976); L. V. Yukeshevich, *Nonlinear Physics of DNA*, (Chicheler, 2004)

[2] A. R. Bishop, Solitons in Action , (Acad. Press, NY, 1978), ch. 4, p 72; R. D. Parmentier, *ibid*, ch. 8, p 186; D. W. McLaughlin and A. C. Scott, , *ibid*, ch. 9, p 186

[3] J. C. Fernandez et al, Phys. Rev. Lett. **46**, 753 (1981); N. F. Pedersen et al, Phys. Rev. **B 30**, 4057 (1984); N. Theodorakopoulos and E. W. Weller, Phys. Rev. **B 38**, 2749 (1988); M. Cadoni et al, arXiv:hep-th/0702213 (2007)

[4] N. Riazi and A. Sheykhi, Phys. Rev. **D 74**, 025003 (2006)

[5] M. Cholascinski and R. W. Chhajlany, Phys. Rev. Lett., **98**, 127001 (2007)

[6] Theory of Soliton

[7] D. Finkelstein and C. N. Misner, Ann. Phys. **6**, 230 (1969)

[8] E. Sklyanin et al, Teor.Mat.Fiz. **40**, 194 (1979)

[9] L. Faddeev, Int. J. Mod. Phys. **10**, 1845 (1995)

[10] R. Rajaraman, *Solitons and Instantons* (N.Holland, Amsterdam, 1982); A. B. Zamolodchikov, Comm. Math. Phys. **55**, 183 (1977); M. Karowski et al, Phys. Lett. **B 67**, 321 (1977)

[11] D. Sen and S. Lal, Eur. Phys. Lett. **52**, 337 (2000)

[12] Anjan Kundu, Phys. Rev. Lett. **82**, 3936 (1999)

[13] H. H. Chen and C. S. Liu, Phys. Rev. Lett. **37**, 693 (1976); R. Balakrishnan, Phys. Rev. **A 32**, 1144 (1985); K. Porsezian and M. Lakshmanan, J. math. Phys. **32**, 2923 (1991)

[14] R. Scharf and A. R. Bishop, Phys. Rev. **A 43**, 6535 (1991); V.V. Konotop et al, Phys. Rev. **E 48**, 563 (1993)

[15] O. Ranisco and R. Levi, Univ. Roma preprint No. 762 (1990)

[16] F. Calogero and A. Degasperis, in *Solitons* (Ed. R. Bullough and P. Caudrey, Springer, Berlin, 1980), p 323

[17] Anjan Kundu, Phys. Lett. **B 633**, 657 (2006)