

Efficient Algorithms for Computing Risk Parity Portfolio Weights

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Abstract

This paper presents two simple algorithms to calculate the portfolio weights for a risk parity strategy, where asset class covariance information is appropriately taken into consideration to achieve “true” equal risk contribution. Previous implementations of risk parity either (1) used a naïve 1/vol solution, which ignores asset class correlations, or (2) computed “true” risk parity weights using relatively complicated optimizations to solve a quadratic minimization program with non-linear constraints. The two iterative algorithms presented here require only simple computations and quickly converge to the optimal solution. In addition to the technical contribution, we also compute the parity in portfolio “risk allocation” using the Gini coefficient. We confirm that portfolio strategies with parity in “asset class allocation” can actually have high concentration in its “risk allocation”.

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Introduction

Markowitz (1952)'s portfolio optimization has long been the theoretical foundation for traditional strategic asset allocation. However, the difficulties in accurately estimating expected returns, especially given the time-varying nature of asset class risk premiums and their joint covariance, means the MVO approach has been enormously challenging to implement in reality.⁵ In practice, institutional investors often default to a 60/40 equity/bond strategic portfolio without any pretense of mean-variance optimality; the 60/40 portfolio structure, instead, has largely been motivated by the 8~9% expected portfolio return one can attach to the portfolio mix. However, the 60/40 portfolio is dominated by equity risk because stock market volatility is significantly larger than bond market volatility. Even if one adds alternative asset classes to the 60/40 asset allocation, these allocations are generally too small to meaningfully impact the portfolio risk. In this sense, a 60/40 portfolio variant earns much of its return from exposure to equity risk and little from other sources of risk, making this portfolio approach fairly under-diversified.

Risk parity represents a portfolio strategy that attempts to address the equity risk concentration problem in standard 60/40-like balanced portfolios. At the high level, the risk parity concept assigns the same risk budget to each asset component. This way, no asset class can be dominant in driving the portfolio volatility. Many possible interpretations of "risk contribution" exist, however, and there are considerable disagreements, not to mention opacity, in methodologies adopted by different providers. Furthermore, very few of the research articles documenting the benefits of the risk parity approach state the portfolio construction methods clearly.⁶ This makes examining risk parity strategies difficult. We assert that the literature on risk parity stands to benefit from greater congruence in the definition of "risk contribution" and transparency in methodologies.

In this paper, we do not make an attempt to argue that the risk parity approach is superior relative to the traditional 60/40 structure or to mean-variance optimal portfolios. Our focus and, therefore, contribution is technical in nature. First, we define what we hope to be a fairly non-controversial definition of equal risk contribution. This then allows us to present two simple and transparent algorithms for producing risk parity portfolios. We lean heavily on the earlier work of Maillard, Roncalli, and Teiletche (2010), which proposes an approach to

⁵ See Merton (1980) for a discussion on the impact of time-varying volatility on the estimate for expected returns. See Cochrane (2005) for a survey discussion on time-varying equity premium and models for forecasting equity returns. See Campbell (1995) for a survey on time-varying bond premium. See Hansen and Hodrick (1980) and Fama (1984) for evidence on time-varying currency returns. See Bollerslev, Engle and Wooldridge (1987) and Engle, Lilien and Robins (1987) for evidence on time-varying volatility in equity and bond markets.

⁶ See Qian (2005, 2009) and Peters (2009), which are product provider white papers discussing their respective risk parity strategies.

compute an equal risk contribution portfolio. We adopt Maillard, Roncalli and Teiletche (2010)'s equal risk contribution definition as the objective for our risk parity portfolio and develop two simple algorithms to efficiently compute our risk parity asset weights.

Theoretically, if all asset classes have roughly the same Sharpe ratios and the same correlations, the standard naïve risk parity weighting (weight assets by $1/\text{vol}^2$) could be interpreted as optimal under the Markowitz framework.⁸ Maillard, Roncalli, and Teiletche (2010) extend the naïve risk parity weighting approach to account for a more flexible correlation assumption. However, the numerical optimization necessary to identify the optimal portfolio weights can be tricky, time-consuming, and require special software. By contrast, the algorithms proposed here do not involve optimization routines and can output reasonable portfolio weights quickly with simple matrix algebra. We demonstrate that our algorithmic approaches result in reasonable *ex post* “equal risk contribution” and produce attractive portfolio returns.

Defining Risk Parity

This section first presents a rigorous mathematical definition for risk parity using the equal risk contribution notion of Maillard, Roncalli, and Teiletche (2010). We then demonstrate the algebraic solution as well as the required numerical steps to compute portfolios with non-negative weights. While our initial motivation is similar to Maillard, Roncalli, and Teiletche (2010), we specify the problem slightly differently to demonstrate that our approach leads to different and, most importantly, simplified solutions.⁹ Again, our contribution is largely technical; we provide readers with two alternative approaches to compute sensible risk parity portfolio weights. Note, throughout the initial exposition, we also reference the minimum variance portfolio extensively. This is because the algebraic nuances are best illustrated by comparing and contrasting to the minimum variance portfolio calculation, which is well-understood and intuitive.

Using r_i and x_i to denote, respectively, the return and weight of each individual asset i , the portfolio's return and standard deviation can be written as

⁷ Bridgewater Associates promotes implementing risk parity strategies as a passive management, by treating asset classes as uncorrelated (or assuming constant correlations between them).

⁸ For an exact mathematical proof of this statement, see Maillard, Roncalli and Teiletche (2010).

⁹ These authors refer to risk parity portfolios as equally weighted risk contribution portfolios, or ERC. We prefer the first denomination, because it has become standard among both practitioners and academics.

$$r_p = \sum_{i=1}^N x_i \cdot r_i \quad (1)$$

and

$$\sigma_p = \sqrt{\sum_{i=1}^N \sum_{j=1}^N x_i \cdot x_j \cdot \sigma_{ij}} \quad (2)$$

where σ_{ij} is the covariance between assets i and j , and $\sigma_{ii} = \sigma_i^2$ is the variance of asset i . We also present two measures of risk contribution that will be useful for defining and understanding risk parity portfolios. The first one is the *marginal* risk contribution,

$$MRC_i = \frac{\partial \sigma_p}{\partial x_i} = \sum_{j=1}^N x_j \cdot \sigma_{ij} = cov(r_i, r_p) \quad (3)$$

which tells us the impact of an infinitesimal increase in an asset's weight on the total portfolio risk, measured here as the standard deviation. The second one is the *total* risk contribution (TRC)

$$TRC_i = x_i \cdot \frac{\partial \sigma_p}{\partial x_i} = \sum_{j=1}^N x_i \cdot x_j \cdot \sigma_{ij} = x_i \cdot cov(r_i, r_p) \quad (4)$$

which gives us a way to break down the total risk of the portfolio into separate components. To see why this is the case, notice that

$$\sum_{i=1}^N TRC_i = \sum_{i=1}^N x_i \cdot cov(r_i, r_p) = \sigma_p^2 \quad (5)$$

Next we characterize the risk parity (RP) portfolios in terms of risk contributions and show how to find their weights without any optimizations.

To illustrate the intuition of the approach, we discuss the application of the MRC measure on the minimum variance portfolio. The results for the minimum variance portfolio are well known and are presented here since they provide an interesting comparison. Notice that the minimum variance portfolio can be obtained by equalizing all the MRCs. The logic behind this requirement is simple: If the MRCs of any two assets were not the same, one could increase the contribution of one asset and decrease that of the other to obtain a slightly lower portfolio variance. The idea behind risk parity is more straightforward: Simply equalize all the TRCs, as these are direct measures of total portfolio risk. Translating these descriptions into equations we get

$$(MV): \quad \frac{\partial \sigma_p}{\partial x_i} = \frac{\partial \sigma_p}{\partial x_j} = k \quad \forall i, j \tag{6}$$

$$(RP): \quad x_i \cdot \frac{\partial \sigma_p}{\partial x_i} = x_j \cdot \frac{\partial \sigma_p}{\partial x_j} = \lambda \quad \forall i, j$$

where k and λ are constants to be found.

In order to find the solution for these two problems, it helps to write them in matrix form. To do this, recall that $\Omega \cdot x$, where Ω is the covariance matrix and x is the vector of weights, is equivalent to a vector of covariances. Thus, using ι to denote a vector of ones and $1/x$ to denote the vector $[1/x_1 \ \cdots \ 1/x_N]'$, we can write

$$(MV): \quad \Omega \cdot x = k \cdot \iota \tag{7}$$

$$(RP): \quad \Omega \cdot x = \lambda \cdot \frac{1}{x} \tag{8}$$

The solution for the minimum variance portfolio is straightforward. Solving the expression in Equation (7) and imposing the constraint $\iota' \cdot x = 1$, we obtain the well-known solution $x = \Omega^{-1} \cdot \iota / (\iota' \cdot \Omega^{-1} \cdot \iota)$. However, if one wants to impose non-negativity constraints ($x \geq 0$) on the weights, a numerical optimizer is required.

Finding the solution for the risk parity portfolio, even without any constraints on the weights, is a different matter. Simple solutions exist only in the special situation where assets

are assumed to have identical pairwise correlation; in which case, the optimal weights are proportional to the inverse of the standard deviation, $1/\sigma$. The weighting by $1/\sigma$ is of course the naïve risk parity solution commonly referenced in the existing literature. We show below, and in the appendix, that this naïve solution is still important and useful for our proposed approaches. However, with the simple algorithms presented here, which can quickly and easily compute risk parity weights that provide true equal risk contribution, there is no longer reason to favor the naïve $1/\sigma$ approach.

To illustrate the challenges in finding the optimal risk parity weights in the general case, notice that dividing Equation (8) by the variance of the portfolio and rearranging gives us $x_i = \lambda/\beta_i$. This expression combined with the fact that $\sum_{i=1}^N x_i = 1$ yields

$$x_i = \frac{1/\beta_i}{\sum_{j=1}^N 1/\beta_j} \quad (9)$$

The difficulty with Equation (9) is that the β_i s on the right-hand side are a function of the x_i s on the left-hand side. In other words, one cannot find the betas without the weights, but the weights depend on the betas. The usual approach to solving this type of problem is to write down the entire expressions and try to isolate the weights, but this approach does not work with these equations.

To deal with this challenge, Maillard, Roncalli, and Teiletche (2010) propose two optimization problems to find the optimal risk parity weights. The first one minimizes

$$\sum_{i=1}^N \sum_{j=1}^N (TRC_i - TRC_j)^2 \quad (10)$$

subject to $\sum_{i=1}^N x_i = 1$ and, if desired, $0 \leq x_i \leq 1$. The second approach is a more complicated optimization that involves nonlinear constraints, and we refer interested readers to their paper.

The direction we follow here is very different. The algorithms we propose do not allow us to specify constraints on the weights, but we show in the numerical examples later that for most applications with real data our algorithms compute the same “optimal” risk parity solution as the original Maillard, Roncalli, and Teiletche (2010), with weights restricted between zero and one. Moreover, they have the advantage of not requiring optimization software, and can be implemented in any programming language or spreadsheet capable of executing simple arithmetic and matrix operations.

A mathematical interpretation of risk parity portfolios gives us some intuition about the problem and helps in understanding the convergence properties of the optimization. Notice that Equation (8) has a striking similarity to an eigenvector-eigenvalue equation, $\Omega \cdot x = \lambda \cdot x$, which is one of the most studied problems in matrix algebra. And although it is nonlinear, it still

retains some of the properties of its linear counterpart. For instance, notice that by multiplying Equation (8) by Ω^{-1} and rearranging we obtain

$$\Omega^{-1} \cdot \frac{1}{x} = \frac{1}{\lambda} \cdot x \quad (11)$$

This expression presents an easy way to obtain a second solution to the risk parity problem: simply apply one of the iterative algorithms presented below to the inverse of the covariance matrix, Ω^{-1} , and then invert the solution found. This alternative approach has the potential to find a solution even when the original one fails.

Algorithm 1: Newton's Method

Our first algorithm is an application of Newton's method to solving a system of nonlinear equations. Writing the system in general form, one is interested in finding the solution to $F(y) = 0$. Recall that we can write a linear approximation to this system around any point c using a Taylor expansion,

$$F(y) \approx F(c) + J(c) \cdot (y - c) \quad (12)$$

where $J(c)$ represents the Jacobian (matrix) of $F(y)$ evaluated at point c . Now, since we are looking for a root of the system, we set $F(y) = 0$ and solve for y :

$$y = c - [J(c)]^{-1} \cdot F(c) \quad (13)$$

Of course this solution is only an approximation, but the idea behind the method is that repeated iterations of Equation (13) will get us closer and closer to the optimal solution. In other words, given an approximate solution $y^{(n)}$, one can calculate

$$y^{(n+1)} = y^{(n)} - [J(y^{(n)})]^{-1} \cdot F(y^{(n)}) \quad (14)$$

and, if the method converges, then $y^{(n)} \rightarrow y$.

The remaining step is simply to write the risk parity problem as a system of nonlinear equations. This can be easily done by using Equation (8) from the previous section, and by imposing the restriction that the weights add up to one:

$$F(y) = F(x, \lambda) = \begin{bmatrix} \Omega \cdot x - \lambda \cdot \frac{1}{x} \\ \sum_{i=1}^N x_i - 1 \end{bmatrix} = 0 \quad (15)$$

This represents a system with $N + 1$ equations and $N + 1$ variables (x and λ), and its Jacobian is a $N + 1 \times N + 1$ matrix

$$J(y) = J(x, \lambda) = \begin{bmatrix} \Omega + \lambda \cdot \text{diag}\left(\frac{1}{x^2}\right) & -\frac{1}{x} \\ t' & 0 \end{bmatrix} \quad (16)$$

where $\text{diag}\left(\frac{1}{x^2}\right)$ represents a diagonal matrix with elements equal to $1/x_i^2$.

The following steps illustrate the iterative process just described:

1. Start with an initial guess for portfolio weights $x^{(0)}$ (equal weight or the traditional risk parity approximation, $1/\sigma$), $\lambda^{(0)}$ (any number between zero and one) and a stopping criterion or threshold, ε . Define $y^{(0)} = [x^{(0)'} \quad \lambda^{(0)}]'$.¹⁰
2. Calculate $F(y^{(n)})$, $J(y^{(n)})$ and $y^{(n+1)}$ using the formulas above.
3. If the condition

$$\|y^{(n+1)} - y^{(n)}\| < \varepsilon \quad (17)$$

is satisfied, stop. If not, go back to step 2.

When compared with the second algorithm, this method requires slightly more complicated operations, such as a matrix inverse, but usually converges faster. Moreover, we found that it tends to be more robust, reaching the optimal solution even when the second algorithm fails in special situations.

Algorithm 2: Power Method

To understand the second algorithm, notice that, as shown by Equation (9) above, the weights are a function of the betas, which in turn depend on the weights. These dependencies create a circular relationship that precludes us from finding a simple analytical solution for the weights. Fortunately, it also seems to provide a recipe on how to proceed.

Imagine that one would start with an initial guess for the weights and calculate the betas associated with them. Aside from incredible luck, the TRCs from all assets would not be the same. However, Equation (9) tells us what condition the weights have to satisfy to be an optimal solution. Thus, using this equation we can update the weights to satisfy the conditions. But, now that the weights have changed the betas have also changed and the conditions aren't satisfied anymore. This procedure can be repeated many times and one can observe that with each iteration, the conditions are closer and closer to being satisfied.

¹⁰ Different initial guesses might lead to different solutions. Thus, when a particular guess leads to a solution with negative weights, simply trying a different guess might lead to the desired solution with nonnegative weights.

This procedure is inspired by the “power method,” one of the simplest iterative algorithms to find an eigenvector from a general matrix, Ω . By iterating the multiplication many times or, equivalently, raising the matrix to a high power, $\Omega^n \cdot x$, one can find the eigenvector associated with the largest eigenvalue in absolute terms. The algorithm described above does the same, with one extra step: after calculating the covariances (or betas), $\Omega \cdot x$, one needs to invert and renormalize the resulting vector, as shown in Equation (8), before repeating the step.

To formalize the iterative process just described, we define the procedure or algorithm with the following steps:

1. Start with an initial guess for portfolio weights $x^{(0)}$ (equal weight or the traditional risk parity approximation, $1/\sigma$) and a stopping criterion or threshold, ε .
2. Calculate betas for all individual assets, $\beta_i^{(n)}$, with respect to the current portfolio $x^{(n)}$.
3. If the condition

$$\sqrt{\frac{1}{N-1} \sum_{i=1}^N \left(x_i^{(n)} \cdot \beta_i^{(n)} - \frac{1}{N} \right)^2} < \varepsilon \quad (18)$$

is satisfied, stop. If not, calculate the new weights as

$$x_i^{(n+1)} = \frac{1/\beta_i^{(n)}}{\sum_{j=1}^N 1/\beta_j^{(n)}} \quad (19)$$

and go back to step 2.

The only point that requires some explanation is the stopping condition in the final step. One alternative would be to use Equation (10), as all the TRCs have to be equivalent. However, we think that expressing it as a standard deviation is more succinct and easier to program. (Recall that $\sum_{i=1}^N x_i \beta_i = 1$, so the average percentage TRC is just $1/n$.) In the end both criteria work equally well.

We have no mathematical proof that this second algorithm will generate a positive (or in fact any) solution, but in many numerical applications (see next section) this is exactly what we found. The fact that a well-defined covariance matrix is symmetric and positive-definite gives us some confidence in this regard, given the nice mathematical properties of such matrices.

An interesting special case occurs when we have one or more uncorrelated assets (or groups of assets). It’s important to emphasize that individual correlations between two assets can be zero, and that a special case occurs only when one asset is uncorrelated with all other

assets, or when all assets in one group are uncorrelated with all assets in another group. These cases are unlikely to be found in actual datasets, but could arise in artificial examples or through restrictions imposed on the covariance matrix. In our simulations we found that in such cases, this second algorithm tends to cycle between two different partial solutions, but never converges. More specifically, the algorithm finds the optimal weights within each group of assets, but is not able to find the optimal combination between the two uncorrelated groups. Fortunately, there is a very simple strategy to fix this problem and still obtain an optimal solution. In the interest of space we present this solution and a numerical example in the Appendix.

Note, while the second algorithm is simpler to implement, the first algorithm almost guarantees convergence and can be more efficient in terms of computation time; as such algorithm 1 is our preferred method for calculating risk parity weights. Generally, both algorithms compute the same “optimal” risk parity solution as the original Maillard, Roncalli, and Teiletche (2010) equal risk contribution solution using non-linear optimization.

Applications

The proposed algorithms in this paper allow us to efficiently compute the portfolio weights for risk parity portfolios, based on the equal risk contribution and more realistic assumptions on asset class covariances. We refer to this risk parity portfolio as optimal risk parity, as compared to naïve ($1/\sigma$) risk parity, which ignores the covariance information and therefore does not achieve true “parity” in asset class contribution to portfolio volatility. The improved computational efficiency relative to the traditional optimization approach using quadratic programming with non-linear constraints makes it possible to backtest a large number of optimal risk parity portfolios constructed on different universes for careful in-sample and out-of-sample examinations. In this section, using three different datasets, we illustrate the property of risk parity portfolios. Our primary goal is to evaluate various portfolio strategies relative to optimal risk parity in terms of asset weight concentration and asset class risk contribution concentration. Our secondary goal is to compare the performance of these strategies over long horizons.

We compare four annually-rebalanced strategies: equal-weight, minimum variance, $1/\sigma$ (naïve) risk parity and equal risk contribution (optimal) risk parity. We restrict all portfolios to have non-negative weights. These strategies have in common that they require no estimates of expected returns. Michaud (1989) and Best and Grauer (1991), among others, show that portfolio optimization is highly sensitive to errors or uncertainty in expected returns. DeMiguel, Garleppi and Uppal (2008) show that equal weighting is generally superior to MVO portfolios

due to estimation errors in expected returns. Thus we choose to use strategies that require only estimates of second moments (variances and covariances). For a comparison between naïve risk parity and maximum Sharpe ratio (MVO) approaches we refer readers to Chaves, Hsu, Li, and Shakernia (2011).

When evaluating asset class weight and risk concentration, we report the “Gini” coefficient. This measure is widely used by economists to measure income inequality (or wealth concentration) and then compare inequity across countries.¹¹ The Gini coefficient ranges between zero (perfect equality) and one (perfect inequality). More recently, Maillard, Roncalli, and Teiletche (2010) have applied the Gini coefficient as a measure of portfolio concentration by computing risk contribution inequality (or risk contribution concentration). The Gini coefficient has an advantage over the more popular Herfindahl measure of portfolio concentration in that it is always unitized between 0 and 1 and, for our specific study, is more variable across different portfolio strategies, which allows for more useful comparisons.

Asset Classes

The first dataset comes from Morningstar EnCorr and contains 10 asset classes from January 1991 through March 2012: BarCap Agg (BarCap US Agg Bond TR), investment grade bonds (BarCap US Corp IG TR), high-yield bonds (BarCap US Corporate High Yield TR), long-term Treasuries (BarCap US Treasury Long TR), long-term credit (BarCap US Long Credit TR), commodities (DJ UBS Commodity TR), REITs (FTSE NAREIT All REITs TR), S&P 500 (IA SBBI S&P 500 TR), MSCI EAFE (MSCI EAFE GR) and MSCI emerging markets (MSCI EM GR). It is meant to replicate a well-diversified investment with exposure to different types of risk. Excess returns and Sharpe ratios are calculated relative to the 30-day Treasury Bill (IA SBBI US 30 Day TBill TR).

We start by reporting some summary statistics—annualized excess return, annualized volatility and correlations—over the entire sample. As **Table 1** shows, there is significant variation across assets in terms of returns and volatilities. Commodities have the lowest return at 1.81%, whereas the best performers are REITs (8.46%), MSCI EM (7.81%) and HY bonds (6.67%). The most volatile asset classes are MSCI EM (23.96%), REITs (18.85%) and MSCI EAFE (16.97%), while the least volatile ones are the BarCap Agg (3.71%) and IG bonds (5.51%). This dataset is interesting because of these differences and because of some potential for diversification as seen in the correlations across asset classes.

¹¹ The Gini coefficient for a set of allocations or risk contributions, y_i , is calculated by first sorting them in non-decreasing order ($y_i \leq y_{i+1}$) and then computing: $G = \frac{2}{N} \sum_{i=1}^N i (y_i - \bar{y})$.

Figure 1 shows, in the top graph, the portfolio weights assigned to each asset class assuming that we have perfect knowledge of the steady-state asset class risk parameters—that is we used the variance and covariance estimates computed from the entire sample for constructing the backtested portfolios. Note, that since forward-looking information was used for computing portfolio weights, the strategies are not real-time investable. The perfect foresight portfolio solution addresses the efficacy of various portfolios in creating “risk” parity when the risk parameters are known. Later, we report results using only non-forward-looking data to test the impact of noisy estimation of the risk parameters on these methodologies.

Minimum variance is heavily concentrated in the BarCap Agg Index with a weight of roughly 90%. HY bonds and commodities have small allocations and most other asset classes have a zero weight. The two risk parity strategies have very similar weights, showing that we can achieve excellent risk diversification with the naïve $1/\sigma$ method. These results are confirmed in the lower graph of Figure 1, which shows the risk contribution of each asset class as a function of the total portfolio variance. The optimal risk parity solution has perfectly distributed risk allocation. However, the naïve risk parity is not very far behind, displaying only minor deviations from the optimal solution. The minimum variance portfolio is heavily concentrated in its allocation to the BarCap Agg Index, which is intuitive given its low volatility and correlations with other asset classes. The equal-weight portfolio has a relatively larger exposure to the more volatile asset classes (equities) and a relatively small exposure to less volatile ones (bonds).

We also simulate the performance of annually rebalanced, no-look-ahead, strategies. These portfolios give us a more nuanced understanding of the strategies when we estimate the asset class risk parameters with noise and when these risk parameters are time-varying rather than constant over the full sample period. Note, since these strategies are investable, they are, therefore, of greater relevance for assessing the usefulness of the different methods. The dataset is limited to monthly frequencies; at each annual rebalance, we use the previous five years of data (60 observations) to estimate the covariance matrix. We omit an asset class if it does not have five years of data at the rebalance date.

Table 2 shows the results; we include the traditional 60/40 equities/bonds portfolio for comparison. Unsurprisingly, minimum variance has the lowest volatility amongst the various portfolio strategies, at only 6%. The two risk parity strategies (naïve and optimal), respectively, have volatilities of 7.4 and 7.5%. In terms of Sharpe ratio, equal-weight and the two risk parity strategies have similar SR of roughly 0.65, outperforming minimum variance and the 60/40 portfolio, which has SR of 0.49 and 0.48, respectively.

The most interesting insight, however, is provided by the Gini coefficients for the various methodologies. First we examine the Gini coefficient associated with the allocation to

asset classes (or asset class weight concentration). For each strategy, we calculate the Gini coefficient for each month and then report the average coefficient over the entire sample. Equal-weight, for instance, has a value of zero at each rebalance date, but then the portfolio weights drift with price movements (and the Gini coefficient drifts upward away from 0) during the year until they are rebalanced again at yearend. This explains why equal-weight has a non-zero value of 0.01 instead of a zero Gini coefficient as one might expect. Minimum variance and the 60/40 portfolio have very high coefficients of 0.81 and 0.82, respectively, highlighting their concentration in a small number of asset classes. Both risk parity strategies have lower coefficients of 0.29 (optimal) and 0.30 (naive), confirming their superior asset class weight parity.

In terms of risk allocation to each asset class, the Gini coefficients comparisons are more extreme and more illustrative.¹² Both risk parity strategies have significantly lower Gini coefficients when compared to the other strategies. While the minimum variance, 60/40 and equally weighted portfolio have coefficients of 0.72, 0.90 and 0.39, the naïve and optimal strategies have coefficients of 0.14 and 0.12, respectively. The 60/40 result is clearly unsurprising since it is well known that the 60/40 portfolio volatility is largely dominated by equity risk. Perhaps more surprising is the observation that the minimum variance portfolio, which is generally considered a low risk portfolio, is significantly more concentrated in asset class risk; specifically, its volatility dominated by duration risk, arising from its 80%+ allocation to the BarCap Agg index. We note that the naïve risk parity portfolio does a surprisingly commendable job of sourcing volatility risk from each asset class evenly; it is only slightly more concentrated in its asset class risk allocation relative to the optimal risk parity portfolio. However, since we can compute the optimal risk parity weight exactly using the algorithms proposed in this paper, there is little reason to favor the naïve solution, which can be non-robust (see Chaves, Hsu, Li and Shakernia (2011)). Note, the asset class risk allocation Gini coefficient for the optimal risk parity is not zero, because we compute risk contribution using *ex post* realizations. At each rebalance date, the optimal portfolio does have an *ex ante* risk allocation Gini coefficient of zero with respect to the historical covariance matrix. However, as correlations and volatilities might be time-varying and /or imprecisely measured, the *ex post* Gini could deviate significantly from the *ex ante* expectation.

We also note that asset class risk contribution as defined by Equation (4) is different from the portfolio's risk factor exposure as defined by the emerging literature on risk-factor-based risk parity strategies (see Bhansali (2011), Chaves, Hsu, Li and Shakernia (2012) and

¹² The percentage of risk allocation to each asset is its total risk contribution (*TRC*, defined in Equation (4)) divided by the total portfolio variance. When computing the Gini coefficient for the risk allocations, we use the full-sample to calculate variances and covariances.

Lohre, Opfer, and Orszag (2012)). Therefore, parity in asset class contribution to the portfolio's total volatility should not be confused with parity in the true underlying risk factor exposure.

Commodities

The second dataset comes from Bloomberg and includes 28 commodity futures sub-indices calculated by Dow Jones from January 1991 through June 2012: Aluminum (DJUBSAL), Brent Crude (DJUBSCO), Cocoa (DJUBSCC), Coffee (DJUBSKC), Copper (DJUBSHG), Corn (DJUBSCN), Cotton (DJUBSCT), Cattle (DJUBSFC), Gas Oil (DJUBSGO), Gold (DJUBSGC), Heating Oil (DJUBSHO), Lead (DJUBSPB), Lean Hogs (DJUBSLH), Live Cattle (DJUBSLC), Natural gas (DJUBSNG), Nickel (DJUBSNI), Orange Juice (DJUBSOJ), Platinum (DJUBSPL), Silver (DJUBSSI), Soybean Meal (DJUBSSM), Soybean Oil (DJUBSBO), Soybeans (DJUBSSY), Sugar (DJUBSSB), Tin (DJUBSSN), Gasoline (DJUBSRB), Wheat (DJUBSWH), WTI Crude (DJUBSCL) and Zinc (DJUBSZS).

In the interest of space we do not report the summary statistics for the commodity components and jump directly to the portfolio simulations. Historical returns, correlations and variances for commodity futures can be found easily in Erb and Harvey (2006), Gorton and Rouwenhorst (2006), or Chaves, Kalesnik, and Little (2012), among others. Similar to before, the portfolios are also rebalanced annually. However, we estimate the covariance matrix using *daily* data and a backward-looking three-year window; the availability of daily data allows us to use a shorter rolling window to capture time-varying covariances while simultaneously improve estimation reliability statistically. The commodity dataset is interesting because of the high diversity in risk characteristics in commodities (including acyclical, counter-cyclical and pro-cyclical commodities).

Table 3 shows that the equal-weight portfolio outperforms the other portfolios with a Sharpe ratio of 0.35, closely followed by the naïve and optimal risk parity portfolios with SR of 0.33 and 0.29, respectively. The minimum variance portfolio of commodities again has a relatively high asset class risk contribution Gini coefficient at 0.57. The Gini coefficient for the equally weighted commodities is 0.25. By comparison, again, the risk parity strategies are much less concentrated in their asset class risk contribution; the asset class risk contribution Gini coefficients are 0.10 (optimal) and 0.18 (naïve). Again, the optimal risk parity portfolio provides a better *ex post* “parity” risk contribution from the underlying investment than naïve risk parity.

Equities

For equities we consider three different equity universes obtained from Ken French’s data library. The data sample extends from January of 1965 through to December of 2011. The first data universe contains 10 industry portfolios (non-durables, durables, manufacturing, energy, high-tech, telecom, retail, health, utilities and others); the second contains 49 sub-industry portfolios; the third contains 25 portfolios sorted on size and book-to-market. These universes are interesting because they represent relatively more homogeneous universes—the “securities” in each of the universes are relatively more similar in returns, variances and pairwise correlations than the previous two examples using universes of asset classes and commodities.

Table 4 shows the summary statistics for the 10 industry portfolios. We do not include the summary statistics for the other universes due to space constraint. There is some variation in returns and variances, but less than in the first (asset classes) and second (commodities) example. Further, notice that the correlations are positive and the majority of them are in excess of 0.5. The result of this is seen in **Figure 2**. Using the entire sample, all four strategies look strikingly similar, with only minimum variance showing some variability in portfolio weights or risk contribution.

Tables 5, 6 and 7 show the portfolio characteristics for equity portfolio strategies constructed from the three universes. Again, portfolios are rebalanced annually; similar to what is done in the commodities section, we use a three-year window of daily returns to estimate the covariance matrix. We note that in each case, the different strategies have very similar performance in terms of Sharpe ratios; the minimum variance portfolio constructed from the 25 size x book-to-market portfolios is the only exception, where it displayed a significantly better SR relative to equal-weighting and the two risk parity strategies. Again, we observe the same pattern in the Gini coefficient for asset class risk contribution. In each case, minimum variance strategy has the highest Gini. For the 10- and 49-industry universe, the optimal risk parity is more risk balanced than naïve risk parity than the equally weighted portfolio. However, for the 25 size x book-to-market portfolio, the Gini coefficient for the two risk parity portfolios and the equally weighted portfolio are the same. This suggests that the pairwise correlations and the volatilities are similar for the universe, which we confirm with our calculations not reported here.

From our portfolio backtests using various datasets, we confirm that naïve risk parity based on $1/\sigma$ is generally effective at improving “parity” in asset class contribution to the total portfolio risk relative to other standard portfolio strategies. However, we also show that in all cases, the more sophisticated risk parity approach, which uses the full covariance matrix information to ensure “true” risk parity, always results in better “parity” in asset class risk contribution.

Conclusion

In this paper we propose two algorithms for computing portfolio weights for risk parity portfolios, in which each asset component equally contributes to the total portfolio variance. Our algorithms represent a significant simplification relative to the traditional optimization methods for solving non-linear minimization, because they do not involve optimization and are based on simple matrix algebra. The improved computational efficiency allows us to more effectively calculate the optimal risk parity portfolio weights and simulate portfolio backtests for examination.

We compare the optimal risk parity portfolio to the naïve risk parity solution based on $1/\sigma$ and to other popular portfolio solutions like equal weighting and minimum variance weighting. We find that risk parity portfolios, both naïve and optimal, do provide superior diversification in asset class risk contribution. That is, each of the included asset classes does provide relatively more equal contribution to the total portfolio volatility versus other portfolio strategies, as measured by the risk allocation Gini coefficient. The optimal risk parity portfolio, which we can now compute easily with the proposed algorithms, in all situations, provides the best *ex ante* and *ex post* “parity” in asset class risk contribution.

Appendix

What about Uncorrelated Assets?

As discussed in the text, uncorrelated assets provide a challenge to the second algorithm, because the algorithm cycles indefinitely and never converges. This issue is not very important in practice, since it is rare to find an asset that is perfectly uncorrelated with all the other assets in the selection universe. However, we could still argue that in some special situations one might want to impose certain restrictions or views on the correlation matrix and, as a result, introduce zero correlation assumption. Fortunately, there is a simple solution to this problem.

We introduce here the simple case of two blocks of assets uncorrelated with each other, but the solution can be easily extended to more complex situations. The example in the next section makes the point clear.

Consider a universe of $N = N_1 + N_2$ assets, in which the first N_1 are uncorrelated with the last N_2 , i.e., all the assets in the first group have zero correlation with all the assets in the second group. Applying the algorithm proposed in the text, one can find the optimal risk parity weights, $x_{(N_1)} = [x_1 \ \cdots \ x_{N_1}]'$, for the portfolio composed only by the first N_1 assets, and the optimal risk parity weights, $x_{(N_2)} = [x_{N_1+1} \ \cdots \ x_{N_1+N_2}]'$, for the portfolio composed only by the last N_2 assets. Denote the standard deviations of these two individual risk parity portfolios by $\sigma_{(N_1)}^2$ and $\sigma_{(N_2)}^2$, respectively, so that the two-asset covariance matrix is

$$\begin{bmatrix} \sigma_{(N_1)}^2 & 0 \\ 0 & \sigma_{(N_2)}^2 \end{bmatrix} \quad (20)$$

The next step involves finding the optimal risk parity portfolio between these two assets: the optimal portfolio $x_{(N_1)}$ and the optimal portfolio $x_{(N_2)}$. Denote the weights of this portfolio by w_1 and $w_2 = 1 - w_1$. The total variance of this portfolio can be written as

$$\sigma_{(N_1+N_2)}^2 = w_1^2 \cdot \sigma_{(N_1)}^2 + w_2^2 \cdot \sigma_{(N_2)}^2 \quad (21)$$

Because the total risk contributions are already equalized within each block, the only requirement imposed on this portfolio is that the TRCs are *inversely* related to the number of assets in each portfolio, or that the average TRCs are the same:

$$\frac{w_1^2 \cdot \sigma_{(N_1)}^2}{N_1} = \frac{w_2^2 \cdot \sigma_{(N_2)}^2}{N_2} \quad (22)$$

The reason for this requirement is straightforward: we want the TRC of each asset in the first block to be the same as the TRC of each asset in the second block. Finally, Equation (23) is easily solved by:

$$\begin{aligned} w_1 &= \frac{\frac{\sqrt{N_1}}{\sigma_{(N_1)}^2}}{\frac{\sqrt{N_1}}{\sigma_{(N_1)}^2} + \frac{\sqrt{N_2}}{\sigma_{(N_2)}^2}} \\ w_2 &= \frac{\frac{\sqrt{N_2}}{\sigma_{(N_2)}^2}}{\frac{\sqrt{N_1}}{\sigma_{(N_1)}^2} + \frac{\sqrt{N_2}}{\sigma_{(N_2)}^2}} \end{aligned} \quad (23)$$

The complete vector of $N_1 + N_2$ weights is finally calculated with

$$x_{(N_1+N_2)} = [w_1 \cdot x_{(N_1)}' \quad w_2 \cdot x_{(N_2)}']' \quad (24)$$

Numerical Example with Uncorrelated Assets

As an illustration of the procedure, consider the numerical example with four assets obtained from Maillard, Roncalli, and Teiletche (2010). The standard deviations are equal to 10%, 20%, 30% and 40%, respectively. The correlation matrix is given by

$$\rho = \begin{bmatrix} 1.0 & & & \\ 0.8 & 1.0 & & \\ 0.0 & 0.0 & 1.0 & \\ 0.0 & 0.0 & -0.5 & 1.0 \end{bmatrix} \quad (25)$$

Clearly, the first two assets are independent from the last two assets. Since each of the two blocks contains only two assets, the $1/\sigma$ approximation is the actual solution for each block. This gives us

$$\begin{aligned} x_1 &= \frac{1/0.1}{1/0.1 + 1/0.2} = 66.66\% \\ x_2 &= \frac{1/0.2}{1/0.1 + 1/0.2} = 33.33\% \end{aligned} \quad (26)$$

and

$$\begin{aligned} x_3 &= \frac{1/0.3}{1/0.3 + 1/0.4} = 57.14\% \\ x_4 &= \frac{1/0.2}{1/0.3 + 1/0.4} = 42.86\% \end{aligned} \quad (27)$$

Using the weights in the equations above, one obtains a standard deviation of 12.65% for the first block and 17.14% for the second block. Because both blocks have the same number of assets, Equation (18) simplifies to the $1/\sigma$ rule, yielding a weight of 57.54% for block one and 42.46% for the second block. Finally, multiplying the weight of each block by the weight of each asset inside the block, we obtained the solution

$$\begin{aligned} x_1 &= 66.66\% \cdot 57.54\% = 38.36\% \\ x_2 &= 33.33\% \cdot 57.54\% = 19.18\% \\ x_3 &= 57.14\% \cdot 42.46\% = 24.26\% \\ x_4 &= 42.86\% \cdot 42.46\% = 18.20\% \end{aligned} \quad (28)$$

which is the same solution obtained.

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Table 1 – Asset Classes

	Excess Return	Volatility	Correlations									
BarCap Agg	3.62%	3.71%	1	0.88	0.21	0.86	0.85	0.03	0.15	0.09	0.07	-0.01
IG Bonds	4.28%	5.51%		1	0.53	0.65	0.96	0.19	0.33	0.29	0.30	0.24
HY Bonds	6.67%	9.19%			1	-0.09	0.48	0.31	0.61	0.59	0.58	0.62
Long Treasuries	5.51%	9.60%				1	0.73	-0.11	-0.05	-0.12	-0.14	-0.20
Long Credit	5.41%	8.53%					1	0.13	0.33	0.27	0.28	0.22
Commodities	1.81%	15.00%						1	0.27	0.30	0.43	0.41
REITS	8.46%	18.85%							1	0.57	0.53	0.48
SP 500	6.05%	15.04%								1	0.78	0.72
MSCI EAFE	2.50%	16.97%									1	0.74
MSCI EM	7.81%	23.96%										1

Table 2 – Performance (10 Asset Classes)

Strategy	Excess Return	Volatility	Sharpe Ratio	Gini Coef. Portfolio Allocation	Gini Coef. Risk Allocation
60/40 (SP500/BarCap Agg)	4.7%	9.8%	0.48	0.82	0.90
Equal-Weight	5.6%	8.8%	0.63	0.04	0.39
Min Var	2.9%	6.0%	0.49	0.83	0.72
Naïve RP	4.7%	7.4%	0.64	0.30	0.14
Optimal RP	4.7%	7.5%	0.63	0.29	0.12

Table 3 – Performance (Commodities)

Strategy	Excess Return	Volatility	Sharpe Ratio	Gini Coef. Portfolio Allocation	Gini Coef. Risk Allocation
Equal-Weight	5.0%	14.3%	0.35	0.09	0.25
Min Var	1.0%	9.5%	0.11	0.73	0.57
Naïve RP	4.3%	12.9%	0.33	0.19	0.18
Optimal RP	3.5%	11.8%	0.29	0.25	0.10

Table 4 – 10 Industries

	Excess Returns	Volatility	Correlations									
Non-Durables	7.07%	15.17%	1.00	0.67	0.82	0.49	0.59	0.61	0.83	0.77	0.61	0.83
Durables	1.98%	22.11%		1.00	0.84	0.47	0.68	0.60	0.76	0.52	0.45	0.79
Manufacturing	4.95%	17.48%			1.00	0.63	0.77	0.63	0.83	0.70	0.54	0.89
Energy	6.54%	18.88%				1.00	0.45	0.41	0.43	0.43	0.58	0.58
Hi-Tech	3.56%	23.06%					1.00	0.61	0.71	0.61	0.32	0.71
Telecomm	3.33%	16.42%						1.00	0.62	0.53	0.51	0.66
Retail	5.57%	18.40%							1.00	0.67	0.47	0.83
Health	6.09%	17.18%								1.00	0.47	0.71
Utilities	4.07%	14.17%									1.00	0.59
Other	3.88%	18.75%										1.00

Table 5 – Performance (10 Industries)

Strategy	Excess Return	Volatility	Sharpe Ratio	Gini Coef. Portfolio Allocation	Gini Coef. Risk Allocation
Equal-Weight	5.7%	15.2%	0.38	0.04	0.11
Min Var	5.1%	13.1%	0.39	0.82	0.75
Naïve RP	5.8%	14.7%	0.40	0.12	0.08
Optimal RP	5.9%	14.5%	0.41	0.14	0.04

Table 6 – Performance (49 Industries)

Strategy	Excess Return	Volatility	Sharpe Ratio	Gini Coef. Portfolio Allocation	Gini Coef. Risk Allocation
Equal-Weight	5.9%	17.6%	0.34	0.05	0.11
Min Var	5.4%	13.1%	0.41	0.92	0.82
Naïve RP	6.1%	16.9%	0.36	0.14	0.10
Optimal RP	6.2%	16.7%	0.37	0.16	0.09

Table 7 – Performance (25 Size and Book-to-Market Portfolios)

Strategy	Excess Return	Volatility	Sharpe Ratio	Gini Coef. Portfolio Allocation	Gini Coef. Risk Allocation
Equal-Weight	7.3%	18.3%	0.40	0.03	0.10
Min Var	9.0%	16.9%	0.53	0.89	0.71
Naïve RP	7.6%	18.0%	0.42	0.11	0.10
Optimal RP	7.6%	18.0%	0.42	0.11	0.10

Figure 1: Portfolio Weights and Risk Allocations (10 Asset Classes)

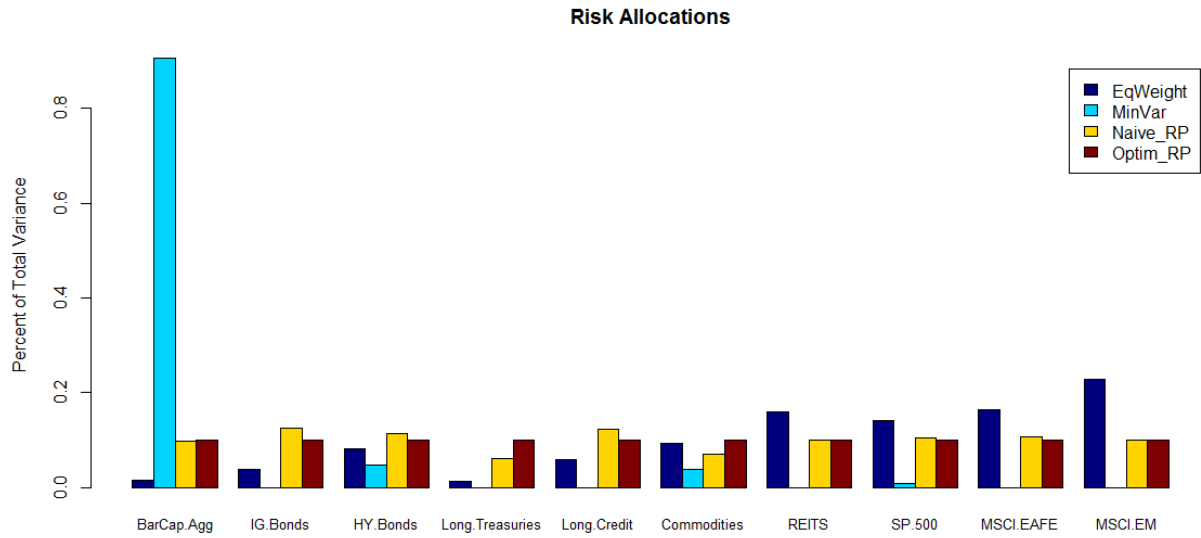
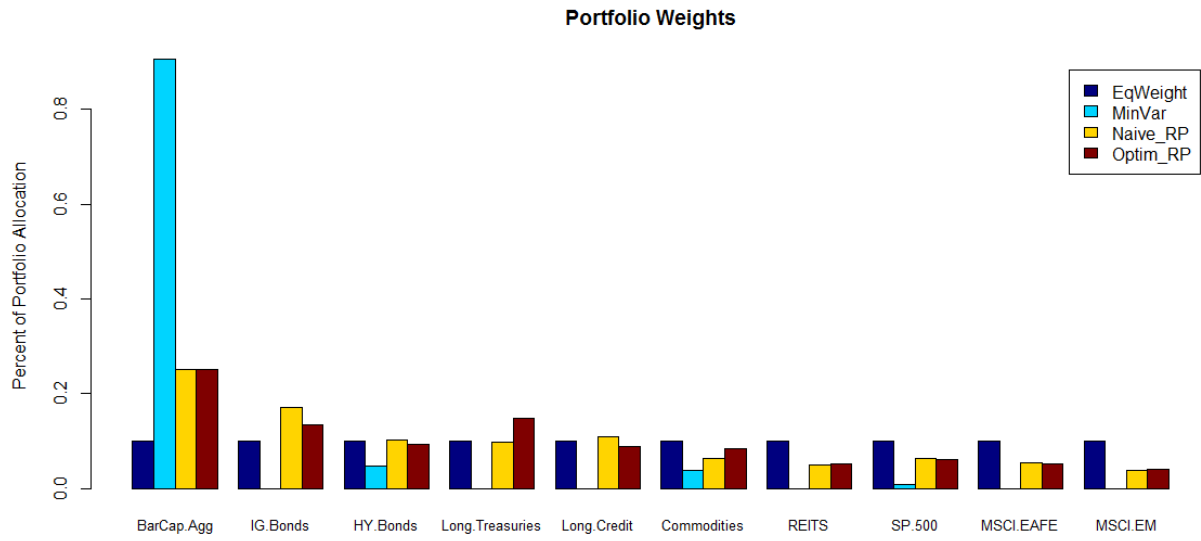


Figure 2: Portfolio Weights and Risk Allocations (10 Industry Portfolios)

