Fuzzy Description Logic Programs under the Answer Set Semantics for the Semantic Web

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Motivation

Ingredients:

- Expressive description logics behind OWL Lite and OWL DL ($SHIF(D)$ resp. $SHOIN(D)$).
- Rule-based formalism (normal logic programs under the answer set semantics).
- Fuzzy truth functions (for conjunction and negation).

Motivation:

- Fuzzy query language for multimedia databases, containing images and videos (such as Google’s YouTube), along the lines of “R. Fagin. Fuzzy queries in multimedia database systems. In Proceedings PODS-1998”.
- Expressing vague terms in natural language interfaces to the Web / Semantic Web.
\[ PC \sqcup Camera \sqsubseteq Electronics; \quad PC \sqcap Camera \sqsubseteq \bot; \]
\[ Book \sqcup Electronics \sqsubseteq Product; \quad Book \sqcap Electronics \sqsubseteq \bot; \]
\[ Textbook \sqsubseteq Book; \]
\[ Product \sqsubseteq \geq 1 \text{ related}; \]
\[ \geq 1 \text{ related} \sqcup \geq 1 \text{ related} \sqsubseteq Product; \]
\[ Textbook(tb_{ai}); \quad Textbook(tb_{lp}); \]
\[ PC(pc_{ibm}); \quad PC(pc_{hp}); \]
\[ related(tb_{ai}, tb_{lp}); \quad related(pc_{ibm}, pc_{hp}); \]
\[ provides(ibm, pc_{ibm}); \quad provides(hp, pc_{hp}). \]
Finite set of truth values $TV = \{0/n, 1/n, \ldots, n/n\}$ with $n \geq 1$.

A fuzzy atomic concept assertion has the form $C(a) \geq v$, where $C \in A$, $a \in I$, and $v \in TV$. A fuzzy abstract (resp., datatype) role assertion has the form $R(a, b) \geq v$ (resp., $U(a, s) \geq v$), where $R \in RA$ (resp., $U \in RD$), $a, b \in I$ (resp., $a \in I$, and $s$ is a data value), $v \in TV$.

A fuzzy description logic knowledge base $KB = (L, F)$ consists of an ordinary description logic knowledge base $L$ and a finite set of fuzzy atomic concept assertions and fuzzy role assertions $F$.

**Example:** A simple fuzzy description logic knowledge base $KB = (L, F)$ is given by $L$ above and

$$F = \{Inexpensive(pc_ibm) \geq 0.6, Inexpensive(pc_hp) \geq 0.9\}.$$ 

Here, $F$ encodes the different degrees of membership of PCs by IBM and HP to the fuzzy concept *Inexpensive*. 
The **ordinary equivalent** to a set of fuzzy concept and role assertions $F$, denoted $F^*$, is obtained from $F$ by replacing each $C(a) \geq v$ (resp., $R(a, b) \geq v$, $U(a, s) \geq v$) by $C^v(a)$ (resp., $R^v(a, b)$, $U^v(a, s)$).

The $v$-layer of $L$, denoted $L^v$, is obtained from $L$ by replacing every $C \in A$ (resp., $R \in R_A$, $U \in R_D$) by $C^v$ (resp., $R^v$, $U^v$).

The **ordinary equivalent** to a fuzzy description logic knowledge base $KB = (L, F)$, denoted $KB^*$, is defined as

$$
\bigcup_{v \in TV, v > 0} L^v \cup F^* \cup \{ A^v \sqsubseteq A'^v \mid A \in A, v \in TV, v \geq 2/n, v' = v - 1/n \} \cup
\{ R^v \sqsubseteq R'^v \mid R \in R_A, v \in TV, v \geq 2/n, v' = v - 1/n \} \cup
\{ U^v \sqsubseteq U'^v \mid U \in R_D, v \in TV, v \geq 2/n, v' = v - 1/n \} .
$$

$KB$ is **satisfiable** iff $KB^*$ is satisfiable.

$F$ among $C(a) \geq v$, $R(a, b) \geq v$, and $U(a, s) \geq v$ is a **logical consequence** of $KB$, denoted $KB \models F$, iff $C^v(a)$, $R^v(a, b)$, and $U^v(a, s)$, respectively, are logical consequences of $KB^*$.
Finite set of truth values $\mathcal{TV} = \{0_n, \frac{1}{n}, \ldots, \frac{n}{n}\}$ with $n \geq 1$.

Negation strategies $\ominus: \mathcal{TV} \rightarrow \mathcal{TV}$ such that
$\ominus$ is antitonic and satisfies $\ominus 0 = 1$ and $\ominus 1 = 0$.

Example: $\ominus \nu = 1 - \nu$.

Conjunction strategies $\otimes: \mathcal{TV} \times \mathcal{TV} \rightarrow \mathcal{TV}$ such that
$\otimes$ is commutative, associative, monotonic, and satisfies $\nu \otimes 1 = \nu$ and $\nu \otimes 0 = 0$.

Example: $\nu_1 \otimes \nu_2 = \min(\nu_1, \nu_2)$ and $\nu_1 \otimes \nu_2 = \nu_1 \cdot \nu_2$. 
A normal fuzzy rule $r$ is of form (with atoms $a, b_1, \ldots, b_m$):

$$a \leftarrow \otimes_0 b_1 \land \otimes_1 b_2 \land \otimes_2 \cdots \land \otimes_{k-1} b_k \land \not \otimes_k \not b_{k+1} \land \otimes_{k+1} \cdots \land \otimes_{m-1} \not \otimes_m b_m \geq v,$$

(1)

A normal fuzzy program $P$ is a finite set of normal fuzzy rules.

A dl-query $Q(t)$ is of one of the following forms:

- a concept inclusion axiom $F$ or its negation $\neg F$;
- $C(t)$ or $\neg C(t)$, with a concept $C$ and a term $t$;
- $R(t_1, t_2)$ or $\neg R(t_1, t_2)$, with a role $R$ and terms $t_1, t_2$.

A fuzzy dl-rule $r$ is of form (1), where any $b \in B(r)$ may be a dl-atom, which is of form $DL[S_1 op_1 p_1, \ldots, S_m op_m p_m; Q](t)$.

A fuzzy dl-program $KB = (L, P)$ consists of a description logic knowledge base $L$ and a finite set of fuzzy dl-rules $P$. 
(1) \( pc(pc_1) \geq 1; \) \( pc(pc_2) \geq 1; \) \( pc(pc_3) \geq 1; \)

(2) \( brand\_new(pc_1) \geq 1; \) \( brand\_new(pc_2) \geq 1; \)

(3) \( offer(X) \leftarrow \otimes DL[PC \cup pc; Electronics](X) \land \otimes \neg \oplus brand\_new(X) \geq 1; \)

(4) \( buy(C, X) \leftarrow \otimes needs(C, X) \land \otimes offer(X) \geq 0.7; \)

(5) \( buy(C, X) \leftarrow \otimes needs(C, X) \land \otimes DL[Inexpensive](X) \geq 0.3. \)

(4) A customer who needs a product on offer buys this product with degree of truth of at least 0.7.

(5) A customer who needs an inexpensive product buys this product with degree of truth of at least 0.3.

\( \oplus \) and \( \otimes \) are given by \( \oplus v = 1 - v \) and \( v_1 \otimes v_2 = \min(v_1, v_2). \)
An interpretation \( I \) (relative to \( P \)) is a mapping \( I: HB_P \rightarrow TV \).

The truth value of \( a = DL[S_1 \cup p_1, \ldots, S_m \cup p_m; Q](c) \) under \( L \), denoted \( I_L(a) \), is defined as the maximal truth value \( v \in TV \) such that \( L \cup \bigcup_{i=1}^{m} A_i(I) \models Q(c) \geq v \), where

\[
A_i(I) = \{ S_i(e) \geq I(p_i(e)) \mid I(p_i(e)) > 0, p_i(e) \in HB_P \}.
\]

\( I \) is a model of a ground fuzzy dl-rule \( r \) of the form (1) under \( L \), denoted \( I \models_L r \), iff

\[
I_L(a) \geq v \otimes_0 I_L(b_1) \otimes_1 I_L(b_2) \otimes_2 \cdots \otimes_{k-1} I_L(b_k) \otimes_k \otimes_{k+1} I_L(b_{k+1}) \otimes_{k+1} \cdots \otimes_{m-1} \ominus_m I_L(b_m),
\]

\( I \) is a model of a fuzzy dl-program \( KB = (L, P) \), denoted \( I \models KB \), iff \( I \models_L r \) for all \( r \in \text{ground}(P) \).
A fuzzy dl-program $KB = (L, P)$ is positive iff $P$ is “not”-free.

**Theorem:** Positive fuzzy dl-programs $KB$ are satisfiable and have a unique least model, denoted $M_{KB}$, as a natural semantics.

**Example:** Consider the fuzzy dl-program $KB$ consisting of the above fuzzy description logic knowledge base and the fuzzy dl-rules

1. $needs(john, pc_{ibm}) \geq 1$;
2. $pc(pc_{1}) \geq 1$; $pc(pc_{2}) \geq 1$; $pc(pc_{3}) \geq 1$;
3. $brand\_new(pc_{1}) \geq 1$; $brand\_new(pc_{2}) \geq 1$;
4. $buy(C, X) \leftarrow \otimes needs(C, X) \land \otimes offer(X) \geq 0.7$;
5. $buy(C, X) \leftarrow \otimes needs(C, X) \land \otimes DL[Inexpensive](X) \geq 0.3$.

Then, $KB$ is positive, and $M_{KB}(buy(john, pc_{ibm})) = 0.3$. 
Stratified fuzzy dl-programs are composed of hierarchic layers of positive fuzzy dl-programs linked via default negation:

A stratification of $KB = (L, P)$ with respect to $DL_P$ is a mapping $\lambda: HB_P \cup DL_P \rightarrow \{0, 1, \ldots, k\}$ such that

- $\lambda(H(r)) \geq \lambda(a)$ (resp., $\lambda(H(r)) > \lambda(a)$) for each $r \in ground(P)$ and $a \in B^+(r)$ (resp., $a \in B^-(r)$), and
- $\lambda(a) \geq \lambda(a')$ for each input atom $a'$ of each $a \in DL_P$, where $k \geq 0$ is the length of $\lambda$. A fuzzy dl-program $KB = (L, P)$ is stratified iff it has a stratification $\lambda$ of some length $k \geq 0$.

**Theorem:** Every stratified fuzzy dl-program $KB$ is satisfiable and has a canonical minimal model via a finite number of iterative least models (which does not depend on the stratification of $KB$).
Example: Consider the fuzzy dl-program $KB$ consisting of the above fuzzy description logic knowledge base and the fuzzy dl-rules

(0) $needs(john, pc_{ibm}) \geq 1$;
(1) $pc(pc_{1}) \geq 1; \ pc(pc_{2}) \geq 1; \ pc(pc_{3}) \geq 1$;
(2) $brand\_new(pc_{1}) \geq 1; \ brand\_new(pc_{2}) \geq 1$;
(3) $offer(X) \leftarrow_D \ DL[PC \cup pc; Electronics](X) \wedge \not\otimes brand\_new(X) \geq 1$;
(4) $buy(C, X) \leftarrow_D needs(C, X) \wedge_D offer(X) \geq 0.7$;
(5) $buy(C, X) \leftarrow_D needs(C, X) \wedge_D DL[Inexpensive](X) \geq 0.3$.

Then, $KB$ is stratified, and it holds in particular $M_{KB}(offer(pc_{ibm})) = 1$ and $M_{KB}(buy(john, pc_{ibm})) = 0.7$. 
Let $KB = (L, P)$ be a fuzzy dl-program. The fuzzy dl-transform of $P$ relative to $L$ and an interpretation $I \subseteq HB_P$, denoted $P^I_L$, is the set of all fuzzy dl-rules obtained from $ground(P)$ by replacing all default-negated atoms $not \ominus_j a$ by the truth value $\ominus_j I_L(a)$.

An answer set of $KB$ is an interpretation $I \subseteq HB_P$ such that $I$ is the least model of $(L, P^I_L)$.

**Theorem:** Let $KB$ be a fuzzy dl-program, and let $M$ be an answer set of $KB$. Then, $M$ is a minimal model of $KB$.

**Theorem:** Let $KB$ be a positive (resp., stratified) fuzzy dl-program. Then, $M_{KB}$ is its only answer set.
For a fuzzy dl-program $KB = (L, P)$, define the operator $T_{KB}$ as follows. For every $I \subseteq HB_P$ and $a \in HB_P$, let $T_{KB}(I)(a)$ be defined as the maximum of $v$ subject to $r \in \text{ground}(P)$, $H(r) = a$, and $v$ being the truth value of $r$’s body under $I$ and $L$. Note that if there is no such rule $r$, then $T_{KB}(I)(a) = 0$.

**Lemma:** Let $KB = (L, P)$ be a positive fuzzy dl-program. Then, the operator $T_{KB}$ is monotonic.

**Theorem:** Let $KB = (L, P)$ be a positive fuzzy dl-program. Then, $lfp(T_{KB}) = M_{KB}$. Furthermore,

$$lfp(T_{KB}) = \bigcup_{i=0}^{n} T_{KB}^i(\emptyset) = T_{KB}^n(\emptyset), \text{ for some } n \geq 0.$$
$M_{KB}$ of a stratified fuzzy dl-program $KB$ can be characterized by a sequence of fixpoint iterations along a stratification:

Let $\hat{T}^i_{KB}(I) = T^i_{KB}(I) \cup I$, for all $i \geq 0$.

**Theorem:** Let $KB = (L, P)$ be a fuzzy dl-program with stratification $\lambda$ of length $k \geq 0$. Let $M_i \subseteq HB_P$, $i \in \{-1, 0, \ldots, k\}$, be defined by $M_{-1} = \emptyset$, and $M_i = \hat{T}^{n_i}_{KB_i}(M_{i-1})$ for every $i \geq 0$, where $n_i$ such that $\hat{T}^{n_i}_{KB_i}(M_{i-1}) = \hat{T}^{n_i+1}_{KB_i}(M_{i-1})$. Then, $M_k = M_{KB}$. 
Summary:

- Simple fuzzy extensions of $\text{SHIF}(\mathcal{D})$ and $\text{SHOIN}(\mathcal{D})$.
- Unique least model and iterative least model semantics of positive resp. stratified fuzzy dl-programs.
- Answer set semantics of general fuzzy dl-programs. Coincides with the canonical semantics in the positive and stratified case.
- Fixpoint and iterative fixpoint characterization of the canonical semantics of positive resp. stratified fuzzy dl-programs.

Outlook:

- Computational complexity, efficient algorithms (especially for general fuzzy dl-programs), and implementation.
- Integration of more expressive fuzzy description logics.