Mathematics and Pictures.
Some popular examples

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Abstract

Mathematics is a science that is traditionally known as a highly abstract discipline. Due to its apparent possibility of deducing abstract formulas without the necessity of back-linking to the outside reality, pure mathematics' status is often experienced as being isolated from and superior to the dubious reality and our evenly ambiguous perception of it. Despite this attitude, several examples can be given of the usefulness of this back-linking. Moreover, since the commercialisation of the computer, new possibilities for mathematical research became available. These possibilities though can only be reached through experimenting. One of the aspects of this experimental approach to mathematics is the use of computer generated images. On the one hand they are used as testing instruments, on the other hand they are necessary tools for certain mathematical theories to be possible - as the outside reality is the object of observation of a physicist, computer generated images are the reality to be observed and perceived by the mathematician.

1. Introduction

The idea of mathematics as being pure and autonomous is very old, and was already formulated by Plato. He believed in an a priori ideal world, to which man can only gain knowledge through pure reasoning. This world is the perfection of the forms and ideas of the finite world we live in. The experienced world is just a shadow – a bad copy – of this ideal world. This opinion, that truth can only be reached through abstract reasoning without using our senses, has led many people to the conviction that mathematics is the only true science, because it is the only one of which we can be certain, since we do not have to use our deceitful senses. This idea is clearly reflected in the following words of Descartes [1].

> That is why it is not wrong to conclude that, when we say that physics, astronomy, medical sciences, and all the other sciences that depend on considering things that are composed (...) are doubtful and uncertain; but that arithmetic, geometry and the other sciences of this nature (...) without being bothered by the question whether they are part of nature, or not – contain something which is certain and indisputable. That is why, being asleep or not, two and three put together will always make the number five (...), and it seems not possible that truths so obvious, should be suspected of whatever falsity or uncertainty.

2. Images as Explanations

> Everybody else took an exam in algebra and complicated integrals, and I managed to take an exam in translation into geometry, and in thinking in terms of geometric shapes.
>
> (*Mandelbrot in an interview)*

The Platonic world consists of perfect forms. Forms and patterns grown or created in the world we live in, are merely shadows of these perfect forms. Imperfect though as they may be, history is full of examples of the usefulness of pictures to clear or explain these perfect ideas and forms, and it is our evenly imperfect visual perception that gives access to these visual explanations. Everybody probably remembers the extensive use of graphics in high school, for example to explain the concept of a derivative. But graphics are already quite important on the level of mathematical notation itself i.e. specific visual methods are used to make the structure of a proof or a calculation more transparent. For example, when writing down a calculation, the numbers are written down in a certain order with a certain structure. We will not write:

$$(((91/7) \text{ x } (4/13)) + (((48/3) - (76/19)) \text{ x } ((5/84) + (2/21))))^2 \quad (1)$$

but rather:

$$\left[\left(\frac{91}{7}\text{x}\frac{4}{13}\right)+\left[\left(\frac{48}{3}-\frac{76}{19}\right)\text{x}\left(\frac{5}{84}+\frac{2}{21}\right)\right]\right]^2 \quad (2)$$

This example is a simple one. However, when long calculations are involved, like the calculations performed by a long algorithm, this basic technique of structuring a calculation or a proof, can be a very useful and time-saving one. One only has to think about the use of flowcharts for designing an algorithm, or the use of colours and positioning of formulas to make a piece of code more transparent.

Visual information is also often used as a way to explain mathematical concepts. For example, try to consider a 2-dimensional, rectangular surface which results from rotating a line segment around a circle with half twist, and can be parameterized in the following way:

$$x(u, v) = a\{\cos(u) + v\cos(u/2)\cos(u)\} \quad (3)$$

$$y(u, v) = a\{\sin(u) + v\cos(u/2)\sin(u)\} \quad (4)$$

$$z(u, v) = a\,v\sin(u/2) \quad (5)$$

where a is the radius of the circle about which the line is rotated, u the angular position around the circle of the segment, and v the position along the horizontal line.(ref.) This strip has some remarkable topological properties: it is a one-sided non-orientable surface, with only one edge. Roughly speaking, this means that a figure such as the number "3" can be moved about on the surface so that it becomes mirror-reversed, when it is back at the starting point. It is thus also a one-sided (one-edged) surface since there can be made no distinction between front and reverse side when moving an object along the surface. This topological surface is known as the Möbius band, after the mathematician August Ferdinand Möbius (1790-1868) whom first described it in 1858.

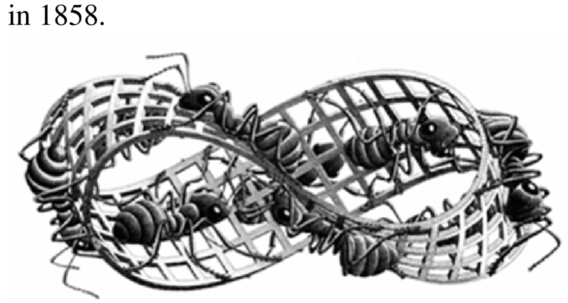

Fig. 1: Here M.C. Eschers Möbiusband II is shown.

Graphics and our perception of them can also be used as visual proofs. The idea of proving a mathematical statement through visual methods goes back to the old Greeks, like for example the proof of the theorem of Pythagoras by Euclid. A recent development in the theory of the education of computer science proclaims the usefulness of visual methods in teaching how algorithms work, and how they process an input to an output. In their article *Teaching the analysis of Algorithms with Visual Proofs*, Goodrich and Tamassia remark that some key concepts in computer science are not fully comprehended by their students, because in proving or representing these concepts often sophisticated mathematical arguments are used [2]. To them this sophistication is often unnecessary, because these concepts and proofs can be represented visually:

In this era of real-time video games and *MTV*, students these days seem more visually oriented than ever. They learn most naturally by seeing a concept described with a picture, and they remember that concept by recalling the picture that goes with it. (...) The main idea of this educational paradigm is to justify important claims about data structures and algorithms by using pictures that visualize proofs so clearly that the pictures can qualify as proofs themselves. The advantage of using this appraoch (...) is that it augments or even replaces inductive arguments that many students find difficult.

One of the examples of Goodrich and Tamassia is a visual proof of the proposition that for any $i \geq 1$, the result of the following summation:

$$\sum_{i=1}^{n} i = 1 + 2 + 3 + ... + (n - 2) + (n - 1) + n \quad (6)$$

can be reduced to:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \quad (7)$$

Figure 2 shows this identity by means of the total area being the result of summing the rectangular areas with unit-width and heights 1, 2, 3...,n.

These areas cover a big triangle with area $A_t = n^2/2$ (A_t = (base n x height n)/2 plus n small triangles of area $A_i = 1/2$ (i = 1, 2,..., n). We then get $(n/2) + (n^2/2) = n(n + 1)/2$. □

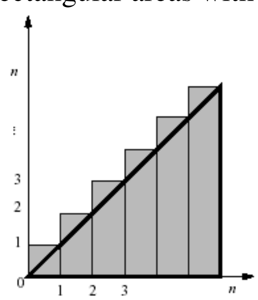

Fig. 2: Visual proof of the fact that Eq. 4 can be reduced to Eq. 5.

Many other examples of the usefulness of visual information could be given. Even the simple fact that the widespread communication of mathematical information would be impossible without mathematical notation serves, trivial though as it may seem at first sight, as an example in this context. But one could still object that even if visual representations can be very useful in gaining mathematical insight or understanding in terms, proofs or calculations, they are nevertheless not *necessary* conditions for gaining *new* mathematical knowledge.

3. Computer Generated Images as Mathematical Objects

"*This even beats shorthand*"

Translating mathematical into visual information can be very useful in gaining mathematical insights. However none of the examples discussed contains any new information. It is although this concept of information that leads in to a more fundamental use of visual information and perception in mathematics. To be more precise, this fundamental use follows from the fact that our pure reasoning can't handle the calculations, nor the information that becomes available through the calculations. The computer has given us the possibility of computing millions of data in a couple of seconds, and thus helps us to overcome the first limitation of this reasoning. Having at our disposal this instrument, a huge number of new mathematical data became available, data that were inaccessible and non-reachable before – even for the platonic minded. Having access to millions of new data although does not immediately imply access to new mathematical knowledge. You have to be able to interpret these data. In many cases this is done by performing tests, for example statistical tests. But very often these tests are combined with computer generated images. There are several reasons for the use of visualisations when large datasets are concerned. For example, a visualisation can be a very quick first test to verify whether the output of a mathematical function is mathematically random or not [3]. In Fig. 4 the first n numbers calculated by two different functions, are shown as a function of time. As can be seen, data from both functions fluctuate between 0 and 1 in an at first sight random way.

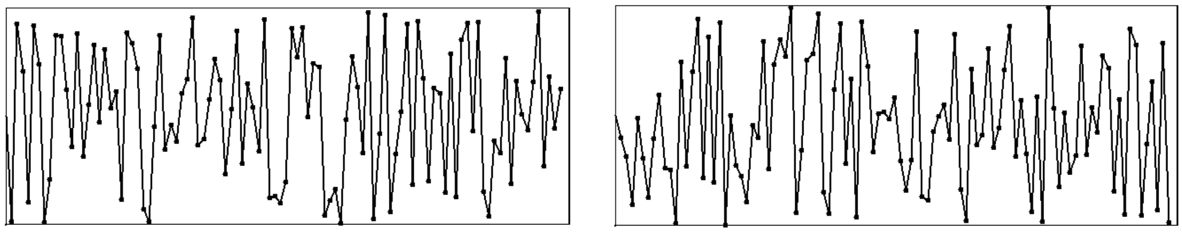

Fig. 3: Timeseries of the first 500 numbers calculated by two different functions.

The question now is whether these datasets are random or not. This can be tested by using the chaos game method to generate the Sierpinski triangle – a famous fractal, which is defined by the following transformations:

$f_1\ (x, y) = (0.5x, 0.5y)$

$f_2\ (x, y) = (0.5x + 0.5, 0.5y)$

$f_3\ (x, y) = (0.5x, 0.5y + 0.5)$

The chaos game is played as follows. First choose an initial point (x_1, y_1), say (10, 10). We then generate a random number z_1 between 0 and 1. Depending on whether z_1 is in one of the three numeric intervals [0, 1/3), [1/3, 2/3) or [2/3, 1], we perform the first, second or third mapping respectively on the initial point, and then plot the newly calculated point (x_2, y_2). In the same way, the following transformation is determined by a newly generated random number z_2, and is then performed on (x_2, y_2), so that a new point (x_3, y_3) is calculated and plotted ...When iterating this procedure for several times, a visual representation of the Sierpinski triangle is generated. This method however only works when random numbers are used. When non-random numbers are used, certain combinations of the transformations will be excluded so that certain parts of the triangle cannot be visited by the point that is transformed, and will thus not be visualised. This method can now be used to test whether the datasets shown in Fig. 3 are random. The numbers which determine which transformation has to be performed, being now the first 10000 numbers calculated by the functions of which the

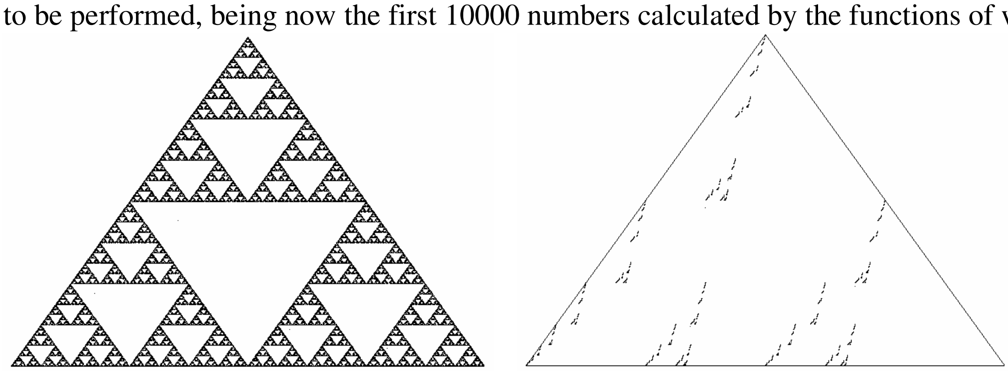

Fig. 4: Results of applying the chaos game method on the first 10000 numbers calculated by the functions from the timeseries of Fig. 3.

first 500 numbers were plotted in Fig. 3. If one of these datasets is random, it must result in a picture of the Sierpinski triangle. As can be seen in the left figure of Fig. 4, the first dataset must be random, since a Sierpinski triangle is indeed the result of substituting the random numbers for generating it, by the numbers of the first dataset. The second dataset on the other hand, does clearly not result in a Sierpinski triangle, so it cannot be random.

Several other examples can be given of the use of computer generated images as a method for analysing datasets, as for example the structuring of higher dimensional datasets by trees or the simple plotting of the results of a test in a graphic, when the number of results is too large or too complex to immediately observe the relation between these results. So one can indeed conclude that computer generated images are working instruments of analysis when large or complex datasets are involved. However, sometimes these instruments become fundamental for the research done i.e. certain branches of mathematics would have not been possible without using these computer images, since certain complex structural relations between the data generated cannot be observed without them. As a first example the research on cellular automata should be mentioned here. This research gained new interest when Conway developed his game of live. It works as follows: given a grid of square cells comparable to a chess board but infinitely extending in every direction. These cells can be in two states: dead (white) or alive (black). The game is started after the player has chosen some cells to be alive. A 'birth' then, occurs in a cell when exactly three of his neighbours are alive. Living cells which have none or only one neighbour alive, die, and cells with more than four neighbours will also die. Playing this game, a rich variety of complex behaviours can be observed. Trivial though this game may seem, looking at the behaviour of the cells and the patterns they can form, gives the impression of self-organizing life. These properties of game of life could not have been known without the possibility of observing the evolution of the game. This is due to the fact that the evolution of the game, given an arbitrary initial condition, is unpredictable. It can be that it stabilizes after some time, but it is also possible that the patterns formed keep on changing and growing without bound. Given an arbitrary initial configuration, the only way to know what the behaviour will be, is running the game and observing it. Following Conway, Stephen Wolfram further developed the theory of cellular automata. One of the observations he made, was that certain CA seem to behave random [5]. An example is shown in Fig. 5, where the right part of the automaton indeed looks like a random pattern. This observation led Wolfram to the implementation of this automaton as a random number generator in the programming language Mathematica, he and his team developed.

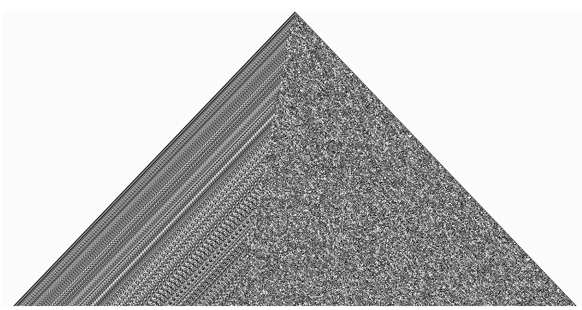

Fig. 5: A one-dimensional cellular automaton, which seems to behave random.

Several other interesting aspects of cellular automata could be mentioned here. These aspects could not have been known without the possibility of calculating, observing and looking at the behaviour of several automata. This is so because humans are incapable of remembering the output of hundreds or thousands of calculations and then observing the patterns that are formed by the output of these calculations.

As a further example of the fundamental use of observation in mathematics, consider the following transformation:

$$z \rightarrow z^2 + c \quad (8)$$

where c and z are complex numbers. This transformation is an iterative function, which means that the output after each iteration, becomes the input z for the following iteration. Now let us put the starting value z equal to the constant parameter c and ask the following question: what will be the long term behaviour for arbitrary values for c. Some fundamental facts can be deduced when trying out different values for c. It will for example become clear that for some values of c the sequence of subsequently generated outputs will escape to infinity, but for other values of *c* this sequence remains bounded within a certain area around zero. The complex numbers are thus subdivided into two sets under this iteration: a set which contains those complex numbers for which the sequence of outputs generated escapes to infinity – the escape set – and a set which contains those complex numbers for which the sequence of outputs remains bounded – the prisoner set. These observations lead to another question: what is the boundary between these two sets? This question is not yet answered completely, but the research already done would not have been possible, without the computer generated images of the prisoner set, which is known as the Mandelbrot set. A zoom in of this set can be seen in Fig. 6. This set is just one of the many fractals that were made since Mandelbrot first discovered it [6].

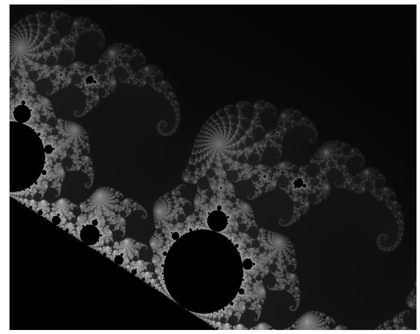

Fig. 6: **Zoom in of the Mandelbrot set.**

Fractal geometry is a branch of mathematics that is nowadays not thinkable without the computer generated images associated with it. One could say that it only became possible through these images, because it was only through these images that the structure of the output generated became known. Years before Mandelbrot first used the term fractal, the transformations that were behind it were *already* partly known. However these were identified as obscure monsters of mathematics, and were ignored for years, until Mandelbrot rediscovered these and generated his set. One of the most attracting aspects of fractal geometry is the fact that it seems to be able to capture those aspects of nature that belong to the domain of the complex – Mandelbrot uses the term *roughness* here, opposed to *smoothness*. It is possible to generate fractals that are not only as erratic as many natural forms, but are also identified as natural forms. It is for example possible to generate images of trees, clouds, mountains, islands, crystals....But it are not only the forms itself that reflect this *roughness*. Fractal geometry is very closely related to another mathematical theory namely chaos theory, which tries to formalise another erratic character of nature, namely its unpredictability.

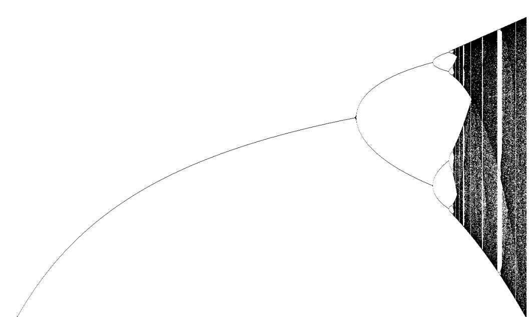

Fig. 7: The Feigenbaum diagram.

It is typical for natural phenomena that they can suddenly switch from one state into the next, from order into chaos. One of the most important discoveries of chaos theory – if not the most important – was that there is a fractal structure behind this route from order into chaos and this route is one with infinite detail and complexity. This fractal is shown in Fig. 7. It was named after his inventor Mitchell Feigenbaum. One essential aspect of this diagram is its period-doubling character. When looking from left to right, we see that out of one major branch two new branches bifurcate, which each splits up again into two new branches,...until a certain point is reached where an abrupt change is observed. From this point on, the systems' dynamics indeed becomes 'infinitely' complex.

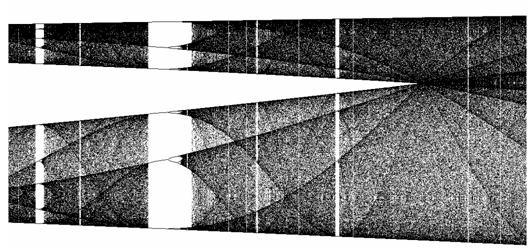

Fig. 8: Zoom in of the Feigenbaum diagram.

Zooming in on this part of the structure reveals a variety of structures, including infinitely many smaller copies of the diagram itself, which can be seen in Fig. 8. Without going further into the details of this diagram it must be remarked that this diagram is structured according to a number, the Feigenbaum number $\delta = 4.6692$...When looking at the bifurcations in the diagram, it is clear that the branches become shorter and shorter. It was shown by Feigenbaum that this length decreasing process is governed by this constant. This number always turns up when chaotic systems can be analyzed in a Feigenbaum diagram, and is thus called a universal number. One can conclude then that this computer image indeed is fundamental to chaos theory. Without this structure and the possibility to generate it by the computer, chaos theory would not have been what it is today. As Peitgen et.al. notice:

> It will most likely be an image which will remain as a landmark of the scientific progress of this century. The image is a computer generated image and is necessarily so. That is to say that the details of it could have never been obtained without the aid of a computer. Consequently, the beautiful mathematical properties attached to the structure would definitely be still in the dark and would probably have remained there if the computer had not been developed. The success of modern chaos theory would be unimaginable without the computer.

4. Conclusion

The ideal of pure mathematics as being free from the effects of our senses and the real world, is very old, and has dominated mathematics of the 20th century. However when comparing this ideal with the practice of mathematics, it seems that this ideal must and always will stay an ideal. First of all mathematics is practiced through a language and the written representation of this language. These are fundamental because otherwise communication of mathematical knowledge would be impossible. Secondly, it is often the case that mathematical concepts or proofs or made more transparent and understandable, by using visual representations of these concepts or proofs. The ideal of pure and autonomous mathematics is however not only unrealistic when comparing it to practice, it is also self-undermining. At the moment it became possible to calculate millions of data, computer generated images and observing them, became indispensable tools for mathematics, not using them leads to the impossibility of certain mathematical statements. It is a remarkable fact that this pure mathematics shows itself inadequate when complexity is involved. Trying to capture the interrelations and properties of a complex dataset containing millions of numbers without using any external input besides these numbers, is the same as trying to calculate these millions of numbers in a couple of seconds – ideal is it may be, realistic it is not. One can then conclude that it is in considering the realistic distinction between the *ideal* of pure mathematical reasoning and the fact that this reasoning is to be performed by a finite human being, with finite memory and finite calculation powers, that Plato seems to be wrong.

References

[1] Descartes, *Méditations Metaphysiques*, Paris: Le livre de poches, 1990, p. 39.

[2] Goodrich, Michael; Tamassia, Roberto, *Teaching the Analysis of Algorithms with Visual Proof* in: Proceedings *of the 29th SIGCSE Technical Symposium on Computer Science Education*, 1998.

[3] H.Peitgen, H.Jurgens and D. Saupe, “Chaos and Fractals. New Frontiers of Science.” Berlin: Springer Verlag, 1992

[4] Martin Gardner, *The fantastic combinations of John Conway's new solitaire game "life"*in: *Scientific American,* 223, 1970, 120-123.

[5] Stephen Wolfram, *Random Sequence Generation by Cellular Automata* in: *Advances in Applied Mathematics*, 1986, 123-169.

[6] Benoît Mandelbrot, *Les Objets Fractals: Forme, hasard et* dimension, Paris:Flammarion, 1995.