TREES WITH EQUAL 2-DOMINATION AND 2-INDEPENDENCE NUMBERS

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\textbf{Abstract}

Let \(G = (V, E)\) be a graph. A subset \(S\) of \(V\) is a 2-dominating set if every vertex of \(V - S\) is dominated at least 2 times, and \(S\) is a 2-independent set of \(G\) if every vertex of \(S\) has at most one neighbor in \(S\). The minimum cardinality of a 2-dominating set \(a\) of \(G\) is the 2-domination number \(\gamma_2(G)\) and the maximum cardinality of a 2-independent set of \(G\) is the 2-independence number \(\beta_2(G)\). Fink and Jacobson proved that \(\gamma_2(G) \leq \beta_2(G)\) for every graph \(G\). In this paper we provide a constructive characterization of trees with equal 2-domination and 2-independence numbers.

\textbf{Keywords:} 2-domination number, 2-independence number, trees.

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1. \textbf{Introduction}

Let \(G = (V(G), E(G))\) be a simple graph with vertex set \(V(G)\) and edge set \(E(G)\). The \textit{open neighborhood} \(N(v)\) of a vertex \(v\) consists of the vertices adjacent to \(v\), the \textit{closed neighborhood} of \(v\) is defined by \(N[v] = N(v) \cup \{v\}\) and \(d_G(v) = |N(v)|\) is the \textit{degree} of \(v\). A vertex of degree one is called a \textit{leaf} and its neighbor is called a \textit{support vertex}. If \(u\) is a support vertex, then \(L_u\) will denote the set of leaves attached at \(u\). We denote by \(K_{1,t}\) a \textit{star} of order \(t + 1\). A tree \(T\) is a \textit{double star} if it contains exactly two vertices that are not leaves. A double star with, respectively \(p\) and \(q\) leaves attached at each support vertex is denoted by \(S_{p,q}\). A

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graph is called a corona if it is constructed from a graph of $H$ by adding for each vertex $v \in V(H)$, a new vertex $v'$ and a pendant edge $vv'$.

In [4], Fink and Jacobson generalized the concepts of independent and dominating sets. Let $k$ be a positive integer, a subset $S$ of $V(G)$ is $k$-independent if the maximum degree of the subgraph induced by the vertices of $S$ is less or equal to $k - 1$. The subset $S$ is $k$-dominating if every vertex of $V(G) - S$ has at least $k$ neighbors in $S$. The $k$-domination number $\gamma_k(G)$ is the minimum cardinality of a $k$-dominating set and the $k$-independence number $\beta_k(G)$ is the maximum cardinality of a $k$-independent set. A minimum $k$-dominating set and a maximum $k$-independent set of a graph $G$ is called a $\gamma_k(G)$-set and $\beta_k(G)$-set, respectively.

It is well known that every graph $G$ satisfies $\gamma_1(G) \leq \beta_1(G)$. In [4], Fink and Jacobson proved that $\gamma_2(G) \leq \beta_2(G)$ and conjectured that for every graph $G$ and positive integer $k$, $\gamma_k(G) \leq \beta_k(G)$. The conjecture has been proved by Favaron [3] by showing that every graph $G$ admits a set that is both a $k$-independent and a $k$-dominating. It follows from this stronger result that if $G$ is a graph such that $\beta_k(G) = \gamma_k(G)$, then $G$ has a set that is both $\gamma_k(G)$-set and $\beta_k(G)$-set. This useful property will be used in the proof of the main result. Note that trees $T$ with $\gamma_1(T) = \beta_1(T)$ have been characterized in [1] by Borowiecki who proved that such trees must be either $K_1$ or coronas.

In this paper, we give a characterization of all trees $T$ with equal 2-domination and 2-independence numbers. We will call such trees $(\gamma_2, \beta_2)$-trees. Note that the difference $\beta_2(G) - \gamma_2(G)$ can be arbitrarily large even for trees. To see this consider a tree $T_j$ obtained from a path of order $2j + 1$ where the vertices are labelled from 1 to $2j + 1$ by attaching a path $P_2$ to each of the odd numbered vertices. Then $\beta_2(T_j) = 3j + 2$ and $\gamma_2(T_j) = 2j + 2$.

2. $(\gamma_2, \beta_2)$-Trees

2.1. Observations

We give some useful observations.

**Observation 1.** Every 2-dominating set of a graph $G$ contains every leaf.

**Observation 2.** Let $T$ be a non-trivial tree and $w \in V(T)$. Then $\gamma_2(T) \leq \gamma_2(T - w) + 1$. 
Proof. If \( D \) is a \( \gamma_2(T - w) \)-set, then \( D \cup \{w\} \) is a 2-dominating set of \( T \) and hence \( \gamma_2(T) \leq \lvert D \rvert + 1 \).

Observation 3. Let \( T \) be a non-trivial tree and \( v \) a vertex of \( T \). Then \( \beta_2(T - v) \leq \beta_2(T) \leq \beta_2(T - v) + 1 \).

Proof. \( \beta_2(T - v) \leq \beta_2(T) \) follows from the fact that any 2-independent set of \( T - v \) is also a 2-independent set of \( T \). Now if \( D \) is a \( \beta_2(T) \)-set, then \( D - \{v\} \) is a 2-independent set of \( T - v \) and hence \( \beta_2(T - v) \geq \lvert D \rvert - 1 \).

Observation 4. Let \( T \) be a tree obtained from a nontrivial tree \( T' \) and a star \( K_{1,p} \) of center vertex \( v \) by adding an edge \( vw \) at any vertex \( w \) of \( T' \). Then,

\begin{enumerate}
\item \( \gamma_2(T') \leq \gamma_2(T) - p \), with equality if either \( p \geq 2 \) or \( w \) is a leaf of \( T' \).
\item If \( p \geq 2 \), then \( \beta_2(T) = \beta_2(T') + p \).
\end{enumerate}

Proof. (1) Let \( D \) be a \( \gamma_2(T) \)-set. Then by Observation 1, \( L_v \subset D \) and, without loss of generality, \( v \notin D \) (else substitute \( v \) by \( w \) in \( D \)). Then \( D \cap V(T') \) 2-dominates \( T' \) and so \( \gamma_2(T') \leq \lvert D \cap V(T') \rvert = \gamma_2(T) - p \). Now if \( p \geq 2 \), then every \( \gamma_2(T') \)-set can be extended to a 2-dominating set of \( T \) by adding the \( p \) leaves of the added star, and hence \( \gamma_2(T) \leq \gamma_2(T') + p \). Assume now that \( p = 1 \) and let \( v' \) be the unique leaf adjacent to \( v \). If \( w \) is a leaf in \( T' \), then \( w \) belongs to every \( \gamma_2(T') \)-set \( D' \) and \( D' \cup \{v'\} \) is a 2-dominating set of \( T' \), implying that \( \gamma_2(T) \leq \gamma_2(T') + 1 \). In both cases the equality is obtained.

(2) Let \( S' \) be any \( \beta_2(T') \)-set. Then clearly \( S' \cup L_v \) is a 2-independent set of \( T \), and so \( \beta_2(T) \geq \beta_2(T') + \lvert L_v \rvert \). Now among all \( \beta_2(T) \)-sets, let \( S \) be one containing the maximum number of leaves. If there exists a leaf \( v' \in L_v \) such that \( v' \notin S \), then \( v \in S \) (else \( S \cup \{v'\} \) is a 2-independent set larger than \( S \)) but then \( \{v'\} \cup S - \{v\} \) is a 2-independent set of \( T \) containing more leaves than \( S \), a contradiction. Hence \( L_v \subset S \) and so \( S - L_v \) is a 2-independent set of \( T' \). It follows that \( \beta_2(T') \geq \beta_2(T) - \lvert L_v \rvert \) and the equality holds.

Observation 5. Let \( T \) be a tree obtained from a nontrivial tree \( T' \) and a double star \( S_{1,p} \) with support vertices \( u \) and \( v \), where \( \lvert L_v \rvert = p \) by adding an edge \( vw \) at a vertex \( w \) of \( T' \). Then,

\begin{enumerate}
\item \( \beta_2(T) = \beta_2(T') + (p + 2) \).
\item \( \gamma_2(T) \leq \gamma_2(T') + (p + 2) \), with equality if \( \beta_2(T) = \gamma_2(T) \).
\end{enumerate}

Proof. (1) Let \( u' \) be the unique leaf neighbor of \( u \) and let \( S \) be a \( \beta_2(T) \)-set containing the maximum number of leaves. Then as seen in the proof of Observation 4, \( L_v \cup \{u'\} \subset S \). Also \( S \) contains either \( u \) or \( v \) for otherwise \( S \cup \{u\} \) is a 2-independent set of \( T \) larger than \( S \). Without loss of generality, \( u \in S \) and so \( S - (L_v \cup \{u, u'\}) \) is a 2-independent set of \( T' \). Hence \( \beta_2(T') \geq \beta_2(T) - (\lvert L_v \rvert + 2) \).
The equality is obtained from the fact that every \( \beta_2(T') \)-set can be extended to a 2-independent set of \( T \) by adding \( L_u \cup \{ u, u' \} \).

(2) Clearly if \( D' \) is a \( \gamma_2(T') \)-set, then \( D' \cup L_v \cup \{ u', v \} \) is a 2-dominating set of \( T \) and so \( \gamma_2(T) \leq \gamma_2(T') + (p + 2) \). Now assume that \( \beta_2(T) = \gamma_2(T) \) and suppose that \( \gamma_2(T) < \gamma_2(T') + (p + 2) \). Then by item (1) we have

\[
\beta_2(T') + (p + 2) = \beta_2(T) = \gamma_2(T) < \gamma_2(T') + (p + 2),
\]

implying that \( \beta_2(T') < \gamma_2(T') \), a contradiction. Therefore if \( \beta_2(T) = \gamma_2(T) \), then \( \gamma_2(T) = \gamma_2(T') + (p + 2) \).

\[\blacksquare\]

Observation 6. Let \( T \) be a tree obtained from a nontrivial tree \( T' \) and a path \( P_3 = xyz \) by adding an edge \( xw \) at a vertex \( w \) of \( T' \). Then

\[
1) \beta_2(T) = \beta_2(T') + 2.
2) \gamma_2(T) \leq \gamma_2(T') + 2, \text{ with equality if } \beta_2(T) = \gamma_2(T).
\]

Proof. (1) If \( D' \) is a \( \beta_2(T') \)-set, then \( D' \cup \{ y, z \} \) is a 2-independent set of \( T \) and so \( \beta_2(T) \geq \beta_2(T') + 2 \). Now let \( D \) be a \( \beta_2(T) \)-set. Clearly \( 1 \leq |D \cap \{ x, y, z \}| \leq 2 \). If \( |D \cap \{ x, y, z \}| = 1 \), then, without loss of generality, \( z \in D \) but \( D \cup \{ y \} \) is a larger 2-independent set of \( T \), a contradiction. Thus \( |D \cap \{ x, y, z \}| = 2 \). Also \( D \cap V(T') \) is a 2-independent set of \( T' \), implying that \( \beta_2(T') \geq \beta_2(T) - 2 \). Hence \( \beta_2(T) = \beta_2(T') + 2 \).

(2) If \( S' \) is a \( \gamma_2(T') \)-set, then \( S' \cup \{ z, x \} \) is a 2-dominating set of \( T \), and so \( \gamma_2(T) \leq \gamma_2(T') + 2 \). Assume now that \( T \) satisfies \( \beta_2(T) = \gamma_2(T) \). If \( \gamma_2(T) < \gamma_2(T') + 2 \), then by item (1) we have

\[
\beta_2(T') + 2 = \beta_2(T) = \gamma_2(T) < \gamma_2(T') + 2,
\]

implying that \( \beta_2(T') < \gamma_2(T') \), a contradiction. Therefore if \( \beta_2(T) = \gamma_2(T) \), then \( \gamma_2(T) = \gamma_2(T') + 2 \).

\[\blacksquare\]

2.2. Main result

For the purpose of characterizing \( (\gamma_2, \beta_2) \)-trees, we define the family \( \mathcal{O} \) of all trees \( T \) that can be obtained from a sequence \( T_1, T_2, \ldots, T_k \) \( (k \geq 1) \) of trees, where \( T_1 \) is a star \( K_{1,p} \) \( (p \geq 1) \), \( T_1 = T_k \), and, if \( k \geq 2 \), \( T_{i+1} \) is obtained recursively from \( T_i \) by one of the operations defined below.

- **Operation \( O_1 \) :** Add a star \( K_{1,p} \), \( p \geq 2 \), centered at a vertex \( u \) and join \( u \) by an edge to a vertex of \( T_i \).

- **Operation \( O_2 \) :** Add a double star \( S_{1,p} \) with support vertices \( u \) and \( v \), where \( |L_u| = p \) and join \( v \) by an edge to a vertex \( w \) of \( T_i \) with the condition that if \( \gamma_2(T_i - w) = \gamma_2(T_i) - 1 \), then no neighbor of \( w \) in \( T_i \) belongs to a \( \gamma_2(T_i - w) \)-set.
• **Operation $O_3$**: Add a path $P_2 = w'u$ and join $u$ by an edge to a leaf $v$ of $T_i$ that belongs to every $\beta_2(T_i)$-set and satisfies in addition $\beta_2(T_i-v)+1 = \beta_2(T_i)$.

• **Operation $O_4$**: Add a path $P_3 = w'uv$ and join $v$ by an edge to a vertex $w$ that belongs to a $\gamma_2(T_i)$-set and satisfies further $\gamma_2(T_i-w) \leq \gamma_2(T_i)$, with the condition that if $\gamma_2(T_i-w) = \gamma_2(T_i)-1$, then no neighbor of $w$ in $T_i$ belongs to a $\gamma_2(T_i-w)$-set.

We state the following lemma.

**Lemma 7.** If $T \in O$ then, $\gamma_2(T) = \beta_2(T)$.

**Proof.** Let $T$ be a tree of $O$. Then $T$ is obtained from a sequence $T_1, T_2, \ldots, T_k$ ($k \geq 1$) of trees, where $T_1$ is a star $K_{1,p}$ ($p \geq 1$), $T = T_k$, and, if $k \geq 2$, $T_{k+1}$ is obtained recursively from $T_k$ by one of the four operations defined above. We use an induction on the number of operations performed to construct $T$. Clearly the property is true if $k = 1$. This establishes the basis case.

Assume now that $k \geq 2$ and that the result holds for all trees $T \in O$ that can be constructed from a sequence of length at most $k-1$, and let $T' = T_{k-1}$. By the inductive hypothesis, $T'$ is a $(\gamma_2, \beta_2)$-tree. Let $T$ be a tree obtained from $T'$ by using one of the operations $O_1, O_2, O_3$ and $O_4$. We examine each of the following cases. Note that we will use in the proof the same notation as used for the construction.

**Case 1.** $T$ is obtained from $T'$ by using operation $O_1$. By Observation 4, $\gamma_2(T) = \gamma_2(T') + p$ and $\beta_2(T) = \beta_2(T') + p$. Since $T'$ is a $(\gamma_2, \beta_2)$-tree it follows that $\gamma_2(T) = \beta_2(T)$.

**Case 2.** $T$ is obtained from $T'$ by using operation $O_2$. By Observation 5, $\beta_2(T) = \beta_2(T') + (p + 2)$ and $\gamma_2(T) \leq \gamma_2(T') + (p + 2)$. Now assume that $\gamma_2(T) < \gamma_2(T') + (p + 2)$ and let $D$ be a $\gamma_2(T)$-set. Then, without loss of generality, $D$ contains $L_u \cup \{v\}$ and the unique leaf neighbor of $u$. If $w \in D$, then $D \cap V(T')$ is a 2-dominating set of $T'$ with cardinality $\gamma_2(T') - (p + 2) < \gamma_2(T')$, which is impossible. Hence $w \notin D$ and so $D' = D \cap V(T')$ is a 2-dominating set of $T' - w$. Note that since $w \notin D$ and $v \in D$, $D'$ contains a neighbor of $w$ in $T'$. Hence $\gamma_2(T' - w) \leq |D'| = \gamma_2(T) - (p + 2) < \gamma_2(T')$. It follows from Observation 2 that $\gamma_2(T' - w) = \gamma_2(T') - 1$ and $D'$ is a $\gamma_2(T' - w)$-set containing a neighbor of $w$, a contradiction with the construction. Therefore $\gamma_2(T) = \gamma_2(T') + (p + 2)$. Now using the fact that $\gamma_2(T') = \beta_2(T')$ we obtain $\gamma_2(T) = \beta_2(T)$, that is $T$ is a $(\gamma_2, \beta_2)$-tree.

**Case 3.** $T$ is obtained from $T'$ by using operation $O_3$. By Observation 4, $\gamma_2(T') = \gamma_2(T) - 1$. Also $\beta_2(T) \geq \beta_2(T') + 1$ since every $\beta_2(T')$-set can be extended to a 2-independent set of $T$ by adding $u'$. Now assume that $\beta_2(T) > \beta_2(T') + 1$ and let $S$ be a $\beta_2(T)$-set. Since $\beta_2(T') \geq |S \cap V(T')|$, it follows that
u, u' \in S. Hence v \notin S and S \cap V(T') is a 2-independent set of T' - v. Thus 
\beta_2(T' - v) \geq |S \cap V(T')| = \beta_2(T) - 2. Also from the construction v satisfies 
\beta_2(T' - v) + 1 = \beta_2(T'). Therefore

\[ \beta_2(T') - 1 = \beta_2(T' - v) \geq \beta_2(T) - 2 > (\beta_2(T') + 1) - 2, \]

a contradiction. Consequently \( \beta_2(T) = \beta_2(T') + 1 \). Since \( \gamma_2(T') = \beta_2(T') \) we obtain \( \gamma_2(T) = \beta_2(T) \).

Case 4. T is obtained from \( T' \) by using operation \( O_4 \). By Observation 6, 
\( \beta_2(T) = \beta_2(T') + 2 \) and \( \gamma_2(T) \leq \gamma_2(T') + 2 \). Assume that \( \gamma_2(T) < \gamma_2(T') + 2 \) and let \( D \) be a \( \gamma_2(T) \)-set. Clearly \( u' \in D \) and \( |D \cap \{u', v, u\}| = 2 \). If \( u \in D \), then \( v \notin D \) and so \( w \in D \). Hence \( D \cap V(T') \) is a 2-dominating set of \( T' \) having cardinality \( |D| - 2 < \gamma_2(T') \), a contradiction. Therefore \( u \notin D \) and so \( v \in D \). If \( w \in D \), then using the same argument than used above leads to a contradiction. 
Thus \( u \notin D \) and hence \( D \cap V(T') \) is a 2-dominating set of \( T' - w \). It follows that 
\[ \gamma_2(T' - w) \leq |D| - 2 < \gamma_2(T') \] and by Observation 2 we obtain 
\( \gamma_2(T' - w) = \gamma_2(T') - 1 \). Therefore \( D \cap V(T') \) is a \( \gamma_2(T' - w) \)-set. Note that \( w \) is 2-dominated in 
\( T \) by \( v \) and some vertex, say \( w' \in V(T') \). But then \( w' \) belongs to a \( \gamma_2(T' - w) \)-set, 
a contradiction with the construction. Consequently, \( \gamma_2(T) = \gamma_2(T') + 2 \) implying 
that \( \gamma_2(T) = \beta_2(T) \), that is, \( T \) is a \( (\gamma_2, \beta_2) \)-tree. \( \blacksquare \)

We now are ready to state our main result.

**Theorem 8.** Let \( T \) be a tree of order \( n \). Then \( \gamma_2(T) = \beta_2(T) \) if and only if 
\( T = K_1 \) or \( T \in \mathcal{O} \).

**Proof.** If \( T = K_1 \), then \( \gamma_2(T) = \beta_2(T) \). If \( T \in \mathcal{O} \), then by Lemma 7, \( \gamma_2(T) = \beta_2(T) \). Let us prove the necessity. Obviously, \( \gamma_2(K_1) = \beta_2(K_1) \), so assume \( n \geq 2 \). We use an induction on the order \( n \) of \( T \). If \( n = 2 \), then \( T = K_{1,1} \) that belongs to \( \mathcal{O} \). Assume that every \( (\gamma_2, \beta_2) \)-tree \( T' \) of order \( 2 \leq n' < n \) is in \( \mathcal{O} \). Let \( T \) be 
\( (\gamma_2, \beta_2) \)-tree of order \( n \). If \( T \) is a star, then \( T \in \mathcal{O} \). If \( T \) is a double star, then 
\( T \) is obtained from \( T_1 \) by using Operation \( O_1 \) if \( n \geq 5 \), and \( T \) is obtained from 
\( T_3 = K_{1,1} \) by using Operation \( O_3 \) if \( n = 4 \). Therefore both stars and double stars 
are in \( \mathcal{O} \). Thus we may assume that \( T \) has diameter at least four.

We now root \( T \) at a leaf \( r \) of a longest path. Among all vertices at distance 
\( \text{diam}(T) - 1 \) from \( r \) on a longest path starting at \( r \), let \( u \) be one of maximum 
degree. Since \( \text{diam}(T) \geq 4 \), let \( v, w \) be the parents of \( u \) and \( v \), respectively. Also 
let \( D \) be a set that is both \( \beta_2(T) \)-set and \( \gamma_2(T) \)-set. Recall that such a set exists 
as mentioned in the introduction (see [3]). Denote by \( T_x \) the subtree induced by 
a vertex \( x \) and its descendants in the rooted tree \( T \). We examine the following cases.

**Case 1.** \( \deg_T(u) \geq 3 \), that is \( u \) is adjacent to at least two leaves. Let 
\( T' = T - T_u \). By Observation 4, \( \gamma_2(T) = \gamma_2(T') + |L_u| \) and \( \beta_2(T) = \beta_2(T') + |L_u| \).
Hence $\gamma_2(T') = \beta_2(T')$. By induction on $T'$, $T' \in \mathcal{O}$ and so $T \in \mathcal{O}$ because it is obtained from $T'$ by using operation $\mathcal{O}_1$.

**Case 2**. $\deg_T(u) = 2$. Let $u'$ be the unique leaf neighbor of $u$. By our choice of $u$, every child of $v$ has degree at most two. First we claim that every child of $v$ besides $u$ (if any) is a leaf. Suppose to the contrary that a child $b$ of $v$ is a support vertex with $L_b = \{b'\}$. Then $u', b' \in D$. If $v \in D$, then $u, b \notin D$ (since $D$ is a $\beta_2(T)$-set) but $\{u, b\} \cup D - \{v\}$ would be a 2-independent set of $T$ larger than $D$, a contradiction. Hence $v \notin D$ and so $u, b \in D$ but $\{v\} \cup D - \{u, b\}$ would be a 2-dominating set of $T$ smaller than $D$, a contradiction too. Thus every child of $v$ besides $u$ is a leaf. We consider two subcases.

**Subcase 2.1**. $\deg_T(v) \geq 3$. Hence $v$ is a support vertex and $T_v$ is a double star $S_{1,|L_v|}$. Let $T' = T - T_v$. Clearly $T'$ is nontrivial. By Observation 5, $\gamma_2(T) = \gamma_2(T') + |L_v| + 2$ and $\beta_2(T) = \beta_2(T') + |L_v| + 2$. It follows that $\gamma_2(T') = \beta_2(T')$ and by induction on $T'$, $T' \in \mathcal{O}$. Assume now that $T' - w$ admits a $\gamma_2(T' - w)$-set $D''$ such that $|D''| = \gamma_2(T' - 1)$ and $D''$ contains at least one vertex adjacent to $w$ in $T'$. Then $D'' \cup L_v \cup \{u', v\}$ is a 2-dominating set of $T'$, and so

$$
\gamma_2(T) \leq |D'' \cup L_v \cup \{u', v\}| = \gamma_2(T' - w) + |L_v| + 2 = \gamma_2(T') - 1 + |L_v| + 2 < \gamma_2(T') + |L_v| + 2,
$$

a contradiction. Hence such a case cannot occur and so $T$ can be obtained from $T'$ by using operation $\mathcal{O}_2$. Therefore $T \in \mathcal{O}$.

**Subcase 2.2**. $\deg_T(v) = 2$. Clearly $u' \in D$. Three possibilities can occur $(u \notin D$ and $v, w \in D), (u, w \notin D$ and $v \in D)$ and $(u, w \in D$ and $v \notin D)$. Observe that if the first situation occurs, then $\{u\} \cup D - \{v\}$ is both $\beta_2(T)$-set and $\gamma_2(T)$-set too. Hence we have to consider only the last two situations.

Assume that $u, w \notin D$ and $v \in D$ and let $T' = T - \{u, u'\}$. By Observation 4, $\gamma_2(T') = \gamma_2(T) - 1$. Also it is clear that $\beta_2(T) \geq \beta_2(T') + 1$. If $\beta_2(T') > \beta_2(T') + 1$, then $\gamma_2(T') + 1 = \gamma_2(T) = \beta_2(T) > \beta_2(T') + 1$, implying that $\gamma_2(T') > \beta_2(T')$, a contradiction. Hence $\beta_2(T) = \beta_2(T') + 1$ and so $\gamma_2(T') = \beta_2(T')$. By induction on $T'$, $T' \in \mathcal{O}$. Note that $v$ belongs to every $\beta_2(T')$-set, for otherwise if $S'$ is a $\beta_2(T')$-set such that $v \notin S'$, then $S' \cup \{u, u'\}$ would be a 2-independent set of $T$ larger than $D$, a contradiction. On the other hand, by Observation 3, $\beta_2(T' - v) \leq \beta_2(T' - v) + 1$. Clearly if $\beta_2(T' - v) = \beta_2(T')$, then every $\beta_2(T' - v)$-set is also $\beta_2(T')$-set but does not contain $v$, a contradiction with the fact that $v$ belongs to every $\beta_2(T')$-set. Therefore $v$ satisfies $\beta_2(T') = \beta_2(T' - v) + 1$. It follows that $T \in \mathcal{O}$ because it is obtained from $T'$ by using Operation $\mathcal{O}_3$.

Finally assume that $u, w \in D$ and $v \notin D$. Let $T' = T - \{v, u, u'\}$. Then by Observation 6, $\beta_2(T) = \beta_2(T') + 2$ and $\gamma_2(T) = \gamma_2(T') + 2$. Note that $D \cap V(T')$ is a $\gamma_2(T')$-set that contains $w$. Also by Observation 2, $\gamma_2(T' - w) \geq \gamma_2(T') - 1$. Therefore...
Assume that $\gamma_2(T'-w) > \gamma_2(T')$. Then using the fact that $\beta_2(T) \geq \beta_2(T'-w)+2$, it follows that

$$\beta_2(T) \geq \beta_2(T'-w) + 2 \geq \gamma_2(T'-w) + 2 > \gamma_2(T') + 2 = \gamma_2(T),$$

and so $\beta_2(T) > \gamma_2(T)$, a contradiction. Therefore $\gamma_2(T') \geq \gamma_2(T'-w) \geq \gamma_2(T') - 1$. Now we note that if $\gamma_2(T'-w) = \gamma_2(T') - 1$, then no neighbor of $w$ in $T'$ belongs to a $\gamma_2(T'-w)$-set, for otherwise such a set can be extended to 2-dominating set of $T$ by adding $u', v$ which leads to $\beta_2(T) > \gamma_2(T)$. Under these conditions it is clear that $T$ is obtained from $T'$ by using Operation $O_4$ and since $T' \in \mathcal{O}$ it follows immediately that $T \in \mathcal{O}$.

References


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