On fuzzy h-ideals in h-regular Γ-hemiring and h-duo Γ-hemiring

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Abstract

In this paper, fuzzy h-bi-ideals, fuzzy h-quasi-ideals and fuzzy h-interior ideals of a Γ-hemiring are studied and some related properties are investigated. The notions of h-intra-hemiregularity and h-quasi-hemiregularity of a Γ-hemiring are introduced and studied along with h-hemiregularity and their characterizations in terms of fuzzy h-ideals are also obtained. The concept of fuzzy h-duo Γ-hemiring is introduced and some of its characterizations are obtained.

Keywords: Γ-hemiring, h-Duo Γ-hemiring, Fuzzy h-ideal, Fuzzy h-bi-ideal, Fuzzy h-quasi-ideal, Fuzzy h-interior ideal, Fuzzy h-duo, h-hemiregularity, h-quasi-hemiregularity, h-intra-hemiregularity.

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1 Introduction

Semiring is a well known universal algebra. This is a generalization of an associative ring (R, +, ·). If (R, +) becomes a semigroup instead of a group then (R, +, ·) reduces to a semiring. Semiring has been found very useful for solving problems in different areas of applied mathematics and information sciences, since the structure of a semiring provides an algebraic framework for modelling and studying the key factors in these applied areas. Ideals of semiring play a central role in the structure theory and useful for many purposes. However they do not in general coincide with the usual ring ideals and for this reason, their use is somewhat limited in trying to obtain analogues of ring theorems for semiring. To ammend this gap Henriksen[4] defined a more restricted class
of ideals, which are called k-ideals. A still more restricted class of ideals in hemirings are given by Iizuka[6], which are called h-ideals. LaTorre[9], investigated h-ideals and k-ideals in hemirings in an effort to obtain analogues of ring theorems for hemiring and to amend the gap between ring ideals and semiring ideals. The theory of Γ-semiring was introduced by Rao[12]. These concepts are extended by Dutta and Sardar[2]. The theory of fuzzy sets, proposed by Zadeh[15], has provided a useful mathematical tool for describing the behavior of the systems that are too complex or illdefined to admit precise mathematical analysis by classical methods and tools. The study of fuzzy algebraic structure has started by Rosenfeld[13]. Since then many researchers developed this ideas. Various relationship between the fuzzy sets and semirings have been considered by Dutta and Biswas[1]. The concept of fuzzy h-ideals in hemiring was studied by Jun et al[7]. Recently Zhan et al[16], Yin et al[14], Huang et al[5] introduced and studied the concepts of h-hemiregularity, h-intra-hemiregularity of a hemiring and gave some of their characterization using fuzzy h-ideals. Ma et al[10] extended some of these results in more general setting of hemiring i.e. Γ-hemiring. In this paper we study some of these properties by using o-composition instead of Γ-composition thereby encompassing some results of Ma et al[10]. We also study h-intra-hemiregularity and h-quasi-hemiregularity in Γ-hemirings.

2 Preliminaries

We recall the following preliminaries for subsequent use.

Definition 2.1. [3] A hemiring [respectively semiring] is a nonempty set S on which operations addition and multiplication have been defined such that the following conditions are satisfied:
(i) $(S,+)$ is a commutative monoid with identity 0.
(ii) $(S,.)$ is a semigroup [respectively monoid with identity $1_S$].
(iii) Multiplication distributes over addition from either side.
(iv) $0s=0=s0$ for all $s \in S$.
(v) $1_S \neq 0$

Definition 2.2. [11] Let $S$ and $\Gamma$ be two additive commutative semigroups with zero. Then $S$ is called a $\Gamma$-hemiring if there exists a mapping $S \times \Gamma \times S \rightarrow S$ ( $(a,\alpha,b) \mapsto a\alpha b$) satisfying the following conditions:
(i) $(a+b)\alpha c = a\alpha c + b\alpha c$,
(ii) $a\alpha (b+c) = a\alpha b + a\alpha c$,
(iii) $a(\alpha + \beta)b = a\alpha b + a\beta b$,
(iv) $a\alpha (b\beta c) = (a\alpha b)\beta c$,
(v) $0s\alpha a = 0s = a\alpha 0s$. 

(vi) \(a_0 \Gamma b = 0_S = b_0 \Gamma a\)
for all \(a, b, c \in S\) and for all \(\alpha, \beta \in \Gamma\).

For simplification we write 0 instead of 0\(_S\) and 0\(_\Gamma\).

**Example 2.3.** Let \(S\) be the set of all \(m \times n\) matrices over \(\mathbb{Z}_0\) (the set of all non-positive integers) and \(\Gamma\) be the set of all \(n \times m\) matrices over \(\mathbb{Z}_0\), then \(S\) forms a \(\Gamma\)-hemiring with usual addition and multiplication of matrices.

**Example 2.4.** Let \(S=\Gamma=\) a non-positive cone of a totally ordered ring. Then \(S\) is a \(\Gamma\)-hemiring where ternary operation is the multiplication of the ring. (It may be noted here that though non-negative cones of totally ordered rings form semirings, non-positive cones do not as the induced multiplication is not a binary composition in this case).

Throughout this paper, unless otherwise mentioned \(S\) denotes a \(\Gamma\)-hemiring.

A subset \(A\) of a \(\Gamma\)-hemiring \(S\) is called a left (resp. right) ideal of \(S\) if \(A\) is closed under addition and \(S \Gamma A \subseteq A\) (resp. \(A \Gamma S \subseteq A\)). A subset \(A\) of a hemiring \(S\) is called an ideal if it is both left and right ideal of \(S\).

A subset \(A\) of a \(\Gamma\)-hemiring \(S\) is called a quasi-ideal of \(S\) if \(A\) is closed under addition and \(S \Gamma A \cap A \Gamma S \subseteq A\).

A subset \(A\) of a \(\Gamma\)-hemiring \(S\) is called a bi-ideal (resp. interior ideal) if \(A\) is closed under addition and \(A \Gamma S \Gamma A \subseteq A\) (resp. \(A \Gamma S \subseteq A\)).

A left ideal \(A\) of \(S\) is called a left \(h\)-ideal if \(x, z \in S, a, b \in A\) and \(x + a + z = b + z\) implies \(x \in A\). A right \(h\)-ideal is defined analogously.

The \(h\)-closure \(\overline{A}\) of \(A\) in \(S\) is defined as \(\overline{A} = \{x \in S \mid x + a + z = b + z, \text{ for some } a, b \in A \text{ and } z \in S\}\).

Now if \(A\) is a left (right) ideal of \(S\), then \(\overline{A}\) is the smallest left (right) \(h\)-ideal containing \(A\).

A quasi-ideal (bi-ideal, interior-ideal) \(A\) of \(S\) is called an \(h\)-quasi-ideal (resp. \(h\)-bi-ideal, \(h\)-interior-ideal) of \(S\) if \(S \Gamma \overline{A} \cap \overline{A} \Gamma S \subseteq A\) and \(x + a + z = b + z\) implies \(x \in A\) for all \(x, z \in S\) and \(a, b \in A\).

**Lemma 2.5.** For a \(\Gamma\)-hemiring \(S\), we have
i) \(A \subseteq \overline{A}\) for all \(A \subseteq S\).
ii) \(A \subseteq B \subseteq S \Rightarrow \overline{A} \subseteq \overline{B}\).
iii) \(A \subseteq \overline{A}\) for all \(A \subseteq S\).
iv) \(\overline{A \Gamma B} = \overline{A} \Gamma \overline{B}\)
v) For any left (resp. right) \(h\)-ideal, \(h\)-bi-ideal, \(h\)-quasi-ideal \(A\) of \(S\), we have \(A = \overline{A}\).

Note that every \(h\)-ideal of \(S\) is an \(h\)-quasi-ideal and \(h\)-interior ideal; Every \(h\)-quasi-ideal of \(S\) is an \(h\)-bi-ideal of \(S\). However the converse of these properties do not hold in general.
Example 2.6. Let $S=\Gamma = M_2(Z^-)$, be the set of all $2 \times 2$ matrices over $Z^-$, the set of all non-positive integers. Let $Q=\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} : a \in Z^- \}$. Then $Q$ is a $h$-quasi-ideal of $S$, but not a (left,right) $h$-ideal of $S$.

Definition 2.7. [13] A fuzzy subset of a nonempty set $X$ is defined as a function $\mu : X \to [0,1]$.

Definition 2.8. [11] Let $\mu$ be a non empty fuzzy subset of a $\Gamma-$hemiring $S$ (i.e. $\mu(x) \neq 0$ for some $x \in S$). Then $\mu$ is called a fuzzy left ideal [fuzzy right ideal] of $S$ if

(i) $\mu(x + y) \geq \min\{\mu(x), \mu(y)\}$ and
(ii) $\mu(x \gamma y) \geq \mu(y)$ [respectively $\mu(x \gamma y) \geq \mu(x)$] for all $x, y \in S$, $\gamma \in \Gamma$.

A fuzzy ideal of a $\Gamma-$hemiring $S$ is a non empty fuzzy subset of $S$ which is a fuzzy left ideal as well as a fuzzy right ideal of $S$. Note that if $\mu$ is a fuzzy left or right ideal of a $\Gamma-$hemiring $S$, then $\mu(0) \geq \mu(x)$ for all $x \in S$.

Definition 2.9. [11] A fuzzy left ideal $\mu$ of a $\Gamma-$hemiring $S$ is called a fuzzy left $h$-ideal if for all $a, b, x, z \in S$, $x + a + z = b + z \Rightarrow \mu(x) \geq \min \{\mu(a), \mu(b)\}$.

A fuzzy right $h$-ideal is defined similarly.

By a fuzzy $h$-ideal $\mu$, we mean that $\mu$ is both fuzzy left and fuzzy right $h$-ideal.

Example 2.10. [11] Let $S$ be the additive commutative semigroup of all non positive integers and $\Gamma$ be the additive commutative semigroup of all non positive even integers. Then $S$ is a $\Gamma-$hemiring if $a \gamma b$ denotes the usual multiplication of integers $a, \gamma, b$ where $a, b \in S$ and $\gamma \in \Gamma$. Let $\mu$ be a fuzzy subset of $S$, defined as follows $\mu(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0.7 & \text{if } x \text{ is even} \\ 0.1 & \text{if } x \text{ is odd} \end{cases}$.

The fuzzy subset $\mu$ of $S$ is both fuzzy ideal and fuzzy $h$-ideal of $S$.

Definition 2.11. [10] Let $\mu$ and $\theta$ be two fuzzy sets of a $\Gamma-$hemiring $S$. Define $h$-product of $\mu$ and $\theta$ by

$\mu \Gamma_h \theta(x) = \sup \{ \min \{ \mu(a_1), \mu(a_2), \theta(b_1), \theta(b_2) \} | \sum_{i=1}^{n} a_i \gamma_i b_i + z = a \delta b + z \}$

for $x, z, a_1, a_2, b_1, b_2 \in S$ and $\gamma, \delta \in \Gamma$.

Definition 2.12. Let $\mu$ and $\theta$ be two fuzzy sets of a $\Gamma-$hemiring $S$. Define a generalized $h$-product of $\mu$ and $\theta$ by

$\mu \odot_h \theta(x) = \sup \{ \min \{ \mu(a_i), \mu(c_i), \theta(b_i), \theta(d_i) \} | \sum_{i=1}^{n} a_i \gamma_i b_i + z = \sum_{i=1}^{n} c_i \delta_i d_i + z \}$.
Lemma 2.13. Let $\mu_1, \mu_2$ be two fuzzy $h$-ideal of a $\Gamma$-hemiring $S$. Then

$$\mu_1 \mu_2 \subseteq \mu_1 \mu_2 \subseteq \mu_1 \cap \mu_2 \subseteq \mu_1, \mu_2.$$ 

Lemma 2.14. Let $S$ be a $\Gamma$-hemiring and $A, B \subseteq S$. Then we have

i) $A \subseteq B$ if and only if $\chi_A \subseteq \chi_B$.

ii) $\chi_A \cap \chi_B = \chi_{A \cap B}$

iii) $\chi_A \mu_2 = \chi_{A \mu_2}$

For more preliminaries of semirings (hemirings) and $\Gamma$-semirings we refer to [3] and [2] respectively.

3 Fuzzy $h$-ideals in $\Gamma$-hemiring

Definition 3.1. [10] A fuzzy subset $\mu$ of a $\Gamma$-hemiring $S$ is called fuzzy $h$-bi-ideal if for all $x, y, z, a, b \in S$ and $\alpha, \beta \in \Gamma$ we have

(i) $\mu(x + y) \geq \min\{\mu(x), \mu(y)\}$

(ii) $\mu(x \alpha y) \geq \min\{\mu(x), \mu(y)\}$

(iii) $\mu(x \alpha y \beta z) \geq \min\{\mu(x), \mu(z)\}$

(iv) $x + a + z = b + z \Rightarrow \mu(x) \geq \min\{\mu(a), \mu(b)\}$

Definition 3.2. A fuzzy subset $\mu$ of a $\Gamma$-hemiring $S$ is called fuzzy $h$-quasi-ideal if for all $x, y, z, a, b \in S$ we have

(i) $\mu(x + y) \geq \min\{\mu(x), \mu(y)\}$

(ii) $(\mu \alpha \chi_S) \cap (\chi_S \mu h) \subseteq \mu$

(iii) $x + a + z = b + z \Rightarrow \mu(x) \geq \min\{\mu(a), \mu(b)\}$

Definition 3.3. [10] A fuzzy subset $\mu$ of a $\Gamma$-hemiring $S$ is called fuzzy $h$-interior-ideal if for all $x, y, z, a, b \in S$ and $\alpha, \beta \in \Gamma$ we have

(i) $\mu(x + y) \geq \min\{\mu(x), \mu(y)\}$

(ii) $\mu(x \alpha y) \geq \min\{\mu(x), \mu(y)\}$

(iii) $\mu(x \alpha y \beta z) \geq \mu(y)$

(iv) $x + a + z = b + z \Rightarrow \mu(x) \geq \min\{\mu(a), \mu(b)\}$

Example 3.4. Let $S = \Gamma = Z^-$, the set of all non-positive integers. Then $S$ is a $\Gamma$-hemiring. Define $\mu : S \rightarrow [0, 1]$ by $\mu(x) = \begin{cases} t & \text{if } x \in 2S \\ 0 & \text{otherwise} \end{cases}$ Let $B = 2S$. Then for any $t \in [0, 1]$, $\mu_t = \{2S\}$. Since $\{2S\}$ is a $h$-bi-ideal (resp. $h$-quasi-ideal) of $Z^-$, $\mu_t$ is a $h$-bi-ideal of $Z^-$ for all $t$. Hence $\mu$ is a fuzzy $h$-bi-ideal (resp. $h$-quasi-ideal) of $Z^-$. 

= 0, if $x$ cannot be expressed as above where $x, z, a_i, b_i, c_i, d_i \in S$ and $\gamma_i, \delta_i \in \Gamma$, for $i = 1, \ldots, n$. 

= 0, if $x$ cannot be expressed as above where $x, z, a_i, b_i, c_i, d_i \in S$ and $\gamma_i, \delta_i \in \Gamma$, for $i = 1, \ldots, n$. 

= 0, if $x$ cannot be expressed as above where $x, z, a_i, b_i, c_i, d_i \in S$ and $\gamma_i, \delta_i \in \Gamma$, for $i = 1, \ldots, n$. 

= 0, if $x$ cannot be expressed as above where $x, z, a_i, b_i, c_i, d_i \in S$ and $\gamma_i, \delta_i \in \Gamma$, for $i = 1, \ldots, n$. 

= 0, if $x$ cannot be expressed as above where $x, z, a_i, b_i, c_i, d_i \in S$ and $\gamma_i, \delta_i \in \Gamma$, for $i = 1, \ldots, n$. 

For any fuzzy subset in a set $X$ and any $t \in [0,1]$, define $\mu_t = \{ x \in X \mid \mu(x) \geq t \}$, which is called a level subset of $\mu$. In [8], Kondo et al. introduced the Transfer Principle in fuzzy set theory, from which a fuzzy set can be characterized by its level subsets. For any algebraic system $U = (X,F)$, where $F$ is a family of operations defined on $X$, the Transfer Principle can be formulated as follows:

**Lemma 3.5.** A fuzzy subset defined on $U$ has the property $\mathcal{P}$ if and only if all non-empty level subset $\mu_t$ have the property $\mathcal{P}$.

As a direct consequence of the above Lemma, the following results can be obtained.

**Lemma 3.6.** [10] Let $S$ be a $\Gamma$-hemiring. Then the following conditions hold:

(i) $\mu$ is a fuzzy left (resp. right) $h$-ideal of $S$ if and only if all non-empty level subsets $\mu_t$ are left (resp. right) $h$-ideals of $S$.

(ii) $\mu$ is a fuzzy $h$-bi-ideal of $S$ if and only if all non-empty level subsets $\mu_t$ are $h$-bi-ideals of $S$.

(iii) $\mu$ is a fuzzy $h$-quasi-ideal of $S$ if and only if all non-empty level subsets $\mu_t$ are $h$-quasi-ideals of $S$.

**Lemma 3.7.** Let $S$ be a $\Gamma$-hemiring and $A \subseteq S$. Then the following conditions hold:

(i) $A$ is a left (resp. right) $h$-ideal of $S$ if and only if $\chi_A$ is a fuzzy left (resp. right) $h$-ideal of $S$.

(ii) $A$ is an $h$-bi-ideal of $S$ if and only if $\chi_A$ is a fuzzy $h$-bi-ideal of $S$.

(iii) $A$ is an $h$-quasi-ideal of $S$ if and only if $\chi_A$ is a fuzzy $h$-quasi-ideal of $S$.

**Lemma 3.8.** A fuzzy subset $\mu$ of a $\Gamma$-hemiring $S$ is a fuzzy left (resp. right) $h$-ideal of $S$ if and only if for all $x,y,z,a,b \in S$, we have

(i) $\mu(x + y) \geq \min\{\mu(x), \mu(y)\}$

(ii) $\chi_{S \circ_h \mu} \subseteq \mu$ (resp. $\mu \circ_h \chi_S \subseteq \mu$)

(iii) $x + a + z = b + z \Rightarrow \mu(x) \geq \min\{\mu(a), \mu(b)\}$.

**Proof.** Assume that $\mu$ is a fuzzy left $h$-ideal of $S$. Then it is sufficient to show that the condition (ii) is satisfied. Let $x \in S$. If $(\chi_{S \circ_h \mu})(x) = 0$, it is clear that $(\chi_{S \circ_h \mu})(x) \leq \mu(x)$. Otherwise, there exist elements $z,a_i, b_i, c_i, d_i \in S$ and $\gamma_i, \delta_i \in \Gamma$, for $i=1,...,n$ such that $x + \sum_{i=1}^{n} a_i \gamma_i b_i + z = \sum_{i=1}^{n} c_i \delta_i d_i + z$. Then we have

\[
(\chi_{S \circ_h \mu})(x) = \sup_i \{ \min \{ \min \{ \chi_S(a_i), \chi_S(c_i), \mu(b_i), \mu(d_i) \} \} \} \\
= x + \sum_{i=1}^{n} a_i \gamma_i b_i + z = \sum_{i=1}^{n} c_i \delta_i d_i + z
\]
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$$= \sup \{\min \{\mu(b_i), \mu(d_i)\}\}$$

$$= x + \sum_{i=1}^{n} a_i \gamma_i b_i + z = \sum_{i=1}^{n} c_i \delta_i d_i + z$$

$$\leq \sup \{\min \{\mu(a_i \gamma_i b_i), \mu(c_i \delta_i d_i)\}\}$$

$$= x + \sum_{i=1}^{n} a_i \gamma_i b_i + z = \sum_{i=1}^{n} c_i \delta_i d_i + z$$

$$\leq \sup \mu(x) = \mu(x).$$

This implies that $\chi_{S \cap \mu} \subseteq \mu.$ Conversely, assume that the given conditions hold. Then it is sufficient to show the 2nd condition of the definition of $h$-ideal. Let $x,y \in S$ and $\gamma \in \Gamma.$ Then we have

$$\mu(x \gamma y) \geq (\chi_{S \cap \mu})(x \gamma y) = \sup \{\min \{\mu(b_i), \mu(d_i)\}\} \geq \mu(y) \text{(since } x \gamma y + 0 = x \gamma y + 0).$$

Hence $\mu$ is a fuzzy left $h$-ideal of $S.$ The case for fuzzy right $h$-ideal can be proved similarly. \hfill \Box

**Lemma 3.9.** Let $\mu$ and $\nu$ be a fuzzy right $h$-ideal and a fuzzy left $h$-ideal of a $\Gamma$-hemiring $S$, respectively. Then $\mu \cap \nu$ is a fuzzy $h$-quasi-ideal of $S.$

**Proof.** Let $x,y$ be any element of $S.$ Then $(\mu \cap \nu)(x + y) = \min(\mu(x + y), \nu(x + y)) \geq \min(\min(\mu(x), \mu(y)), \min(\nu(x), \nu(y))) = \min(\min(\mu(x), \nu(x)), \min(\mu(y), \nu(y))) = \min((\mu \cap \nu)(x), (\mu \cap \nu)(y)).$ Now let $a,b,x,z \in S$ such that $x + a + z = b + z.$ Then

$$(\mu \cap \nu)(x) = \min(\mu(x), \nu(x)) \geq \min(\min(\mu(a), \mu(b)), \min(\nu(a), \nu(b))) = \min(\min(\mu(a), \nu(a)), \min(\mu(b), \nu(b))) = \min((\mu \cap \nu)(a), (\mu \cap \nu)(b)).$$

On the other hand, we have $(\mu \cap \nu)(x \gamma y) \cap (\chi_{S \cap \mu} \cap \mu \cap \nu) \subseteq (\mu \cap \mu \gamma S) \cap (\chi_{S \cap \mu} \cap \mu \cap \nu) \subseteq \mu \cap \nu.$

This completes the proof. \hfill \Box

**Lemma 3.10.** Any fuzzy $h$-quasi-ideal of $S$ is a fuzzy $h$-bi-ideal of $S.$

**Proof.** Let $\mu$ be any fuzzy $h$-quasi-ideal of $S.$ It is sufficient to show that $\mu(x \alpha y \beta z) \geq \min\{\mu(x), \mu(z)\}$ and $\mu(x \alpha y) \geq \min\{\mu(x), \mu(y)\}$ for all $x,y,z \in S$.
and \(\alpha, \beta \in \Gamma\).

In fact, by the assumption, we have
\[
\mu(x\alpha y\beta z) \geq ((\mu_0\chi S) \cap (\chi S_0\mu))(x\alpha y\beta z)
\]
\[
= \min\{(\mu_0\chi S)(x\alpha y\beta z), (\chi S_0\mu)(x\alpha y\beta z)\}
\]
\[
= \min\{\sup(\min(\mu(a_i), \mu(c_i))), \sup(\min(\mu(b_i), \mu(d_i)))\}
\]
\[
x + \sum_{i=1}^{n} a_i \gamma_i b_i + z = \sum_{i=1}^{n} c_i \delta_i d_i + z + \sum_{i=1}^{n} a_i \gamma_i b_i + z = \sum_{i=1}^{n} c_i \delta_i d_i + z
\]
\[
\geq \min\{\min(\mu(0), \mu(x)), \min(\mu(0), \mu(z))\} (\text{since } x\alpha y\beta z + 0\gamma 0 + 0 = x\alpha y\beta z + 0)
\]
\[
= \min\{\mu(x), \mu(z)\}.
\]

Similarly, we can show that \(\mu(x\alpha y) \geq \min\{\mu(x), \mu(y)\}\) for all \(x, y \in S\) and for all \(\alpha \in \Gamma\).

4 \ h-hemiregularity

In this section we study the concept of h-hemiregularity in \(\Gamma\)-hemiring by using h-ideal, h-bi-ideal, h-quasi-ideal.

**Definition 4.1.** [10] A \(\Gamma\)-hemiring \(S\) is said to be h-hemiregular if for each \(x \in S\), there exist \(a, b \in S\) and \(\alpha, \beta, \gamma, \delta \in \Gamma\) such that \(x + a\alpha x\beta a + z = b\gamma x\delta b + z\).

**Definition 4.2.** A \(\Gamma\)-hemiring \(S\) is said to be h-interior-regular if for each \(x \in S\), there exist \(a, a', b, b' \in S\) and \(\alpha, \beta, \gamma, \delta \in \Gamma\) such that \(x + a\alpha x\beta a' + z = b\gamma x\delta b' + z\).

**Lemma 4.3.** [10] A \(\Gamma\)-hemiring \(S\) is h-hemiregular if and only if for any right h-ideal \(R\) and any left h-ideal \(L\) of \(S\) we have \(R\Gamma L = R \cap L\).

**Theorem 4.4.** A \(\Gamma\)-hemiring \(S\) is h-hemiregular if and only if for any fuzzy right h-ideal \(\mu\) and any fuzzy left h-ideal \(\nu\) of \(S\) we have \(\mu_0 \nu = \mu \cap \nu\).

**Proof.** The proof of this theorem follows from simple generalization of the proof of Theorem 5.5 of [10].

**Proposition 4.5.** Let \(S\) be a \(\Gamma\)-hemiring. Then if \(S\) is h-interior-regular, \(I = \text{STIT}S\) for every h-interior-ideal \(I\) of \(S\).

**Proof.** Let \(S\) be a \(\Gamma\)-hemiring and \(x \in I\). Since \(S\) is h-interior-regular, \(x + a\alpha x\beta a' + z = b\gamma x\delta b' + z\) for some \(a, a', b, b', z \in S\) and \(\alpha, \beta, \gamma, \delta \in \Gamma\). Now \(a\alpha x\beta a' \in \text{STIT}S\). So, \(x \in \text{STIT}S\). On the other hand, since \(I\) is an h-interior-ideal of \(S\), we have \(\text{STIT}S \subseteq I\) and so \(\text{STIT}S \subseteq I = I\). Therefore \(I = \text{STIT}S\).

We need the following Lemma which was stated in [10] without proof.
Lemma 4.6. [11] Let \( S \) be a \( \Gamma \)-hemiring. Then the following conditions are equivalent.

(i) \( S \) is \( h \)-hemiregular.

(ii) \( B = B \Gamma ST\Gamma B \) for every \( h \)-bi-ideal \( B \) of \( S \).

(iii) \( Q = Q \Gamma STQ \) for every \( h \)-quasi-ideal \( Q \) of \( S \).

Proof. (i)\(\Rightarrow\)(ii) Assume that (i) holds. Let \( B \) be any \( h \)-bi-ideal of \( S \) and \( x \) be any element of \( B \). Then there exist \( a, b, z \in S \) and \( \alpha, \beta, \gamma, \delta \in \Gamma \) such that \( x + x\alpha a\beta x + z = x\gamma b\delta x + z \). Then it is easy to see that \( x\alpha a\beta x, x\gamma b\delta x \in B \Gamma ST\Gamma B \) and so \( x \in B \Gamma ST\Gamma B \). Hence \( B \subseteq B \Gamma ST\Gamma B \). On the other hand, since \( B \) is an \( h \)-bi-ideal of \( S \), we have \( B \Gamma ST\Gamma B \subseteq B \) and so \( B \Gamma ST\Gamma B \subseteq B = B \). Therefore \( B = B \Gamma ST\Gamma B \).

(ii)\(\Rightarrow\)(iii) This is straightforward.

(iii)\(\Rightarrow\)(i) Assume that (iii) holds. Let \( R \) and \( L \) be any right and left \( h \)-ideal of \( S \) respectively. Then we have \( (R \cap L)\Gamma S \cap \Gamma ST(R \cap L) \subseteq \Gamma R \cap L \subseteq R \cap L = R \cap L \) and thus \( R \cap L \) is a \( q \)-quasi-ideal of \( S \). Also, \( R \cap L = (R \cap L)\Gamma ST(R \cap L) \subseteq R \Gamma STL \subseteq R \cap L \subseteq R \cap L = R \cap L \). Therefore \( R \Gamma L = R \cap L \). Hence from Lemma [4, 13] \( S \) is \( h \)-hemiregular. \( \square \)

Now we obtain the following characterizations of \( h \)-hemiregular \( \Gamma \)-hemirings.

Theorem 4.7. Let \( S \) be a \( \Gamma \)-hemiring. Then the following conditions are equivalent.

(i) \( S \) is \( h \)-hemiregular.

(ii) \( \mu \subseteq \mu_0h\chi S_0h\mu \) for every fuzzy \( h \)-bi-ideal \( \mu \) of \( S \).

(iii) \( \mu \subseteq \mu_0h\chi S_0h\mu \) for every fuzzy \( h \)-quasi-ideal \( \mu \) of \( S \).

Proof. (i)\(\Rightarrow\)(ii) Assume that (i) holds. Let \( \mu \) be any fuzzy \( h \)-bi-ideal of \( S \) and \( x \) be any element of \( S \). Since \( S \) is \( h \)-hemiregular there exist \( a, b, z \in S \) and \( \alpha, \beta, \gamma, \delta \in \Gamma \) such that \( x + x\alpha a\beta x + z = x\gamma b\delta x + z \).

\[
(\mu_0h\chi S_0h\mu)(x) = \sup(\min\{(\mu_0h\chi S)(a_i), (\mu_0h\chi S)(c_i), (\mu_0h\chi S)(b_i), (\mu_0h\chi S)(d_i)\})
\]

\[
x + \sum_{i=1}^{n} a_i\gamma_i b_i + z = \sum_{i=1}^{n} c_i\delta_i d_i + z
\]

\[
\geq \min\{(\mu_0h\chi S)(x\alpha a), (\mu_0h\chi S)(x\gamma b), (\mu_0h\chi S)(x)\}
\]

\[
= \min\{\sup(\min\{(\mu(a_i), (\mu(c_i))\}), \sup(\min\{(\mu(a_i), (\mu(c_i))\}), (\mu(x))\}
\]

\[
x\alpha a + \sum_{i=1}^{n} a_i\gamma_i b_i + z = \sum_{i=1}^{n} c_i\delta_i d_i + z \quad x\gamma b + \sum_{i=1}^{n} a_i\gamma_i b_i + z = \sum_{i=1}^{n} c_i\delta_i d_i + z
\]

\[
\geq \min\{\mu(x), (\mu_0h\chi S_0h\mu)(x)\}(\text{since } x\alpha a + x\alpha a\beta x + z = x\gamma b\delta x + z\alpha a + z\gamma b + x\alpha a\beta x + z = x\gamma b\delta x + z\alpha a + z\gamma b).
\]

This implies that \( \mu \subseteq \mu_0h\chi S_0h\mu \).

(ii)\(\Rightarrow\)(iii) This is straightforward from Lemma [3, 10].

(iii)\(\Rightarrow\)(i) Assume that (iii) holds and \( Q \) be any \( h \)-quasi-ideal of \( S \). Then the
characteristic function $\chi_Q$ of $Q$ is a fuzzy $h$-quasi-ideal of $S$. Now $\chi_Q \subseteq \chi_Q^oh \chi_S^oh \chi_Q = \chi_{QTSQ}$. Therefore $Q \subseteq QTSQ$. On the other hand, since $Q$ is any $h$-quasi-ideal of $S$, we have $QTSQ \subseteq STQ \cap QTS \subseteq Q$ and so $QTSQ = Q$. Therefore $S$ is $h$-hemiregular by Lemma 4.6.

**Theorem 4.8.** Let $S$ be a $\Gamma$-hemiring. Then the following conditions are equivalent.

(i) $S$ is $h$-hemiregular.

(ii) $\mu \cap \nu \subseteq \mu ho\nu h\mu$ for every fuzzy $h$-bi-ideal $\mu$ and every fuzzy $h$-ideal $\nu$ of $S$.

(iii) $\mu \cap \nu \subseteq \mu ho\nu h\mu$ for every fuzzy $h$-quasi-ideal $\mu$ and every fuzzy $h$-ideal $\nu$ of $S$.

**Proof.** (i)$\Rightarrow$(ii) Assume that (i) holds. Let $\mu$ and $\nu$ be any fuzzy $h$-bi-ideal and fuzzy $h$-ideal of $S$, respectively, and $x$ be any element of $B$. Since $S$ is $h$-hemiregular, there exist $a, b, z \in S$ and $\alpha, \beta, \gamma, \delta \in \Gamma$ such that $x + xaoa$ $\beta x + z = x\gamma b\delta x + z$.

$$(\mu ho\nu h\mu)(x) = \sup(\min\{(\mu ho\nu)(a_i), (\mu ho\nu)(c_i), (\mu ho\nu)(d_i)\})$$

$$x + \sum_{i=1}^{n} a_i \gamma_i b_i + z = \sum_{i=1}^{n} c_i \delta_i d_i + z$$

$$\geq \min\{(\mu ho\nu)(xaoa), (\mu ho\nu)(x\gamma b), (\mu ho\nu)(x\gamma b, (\mu ho\nu)(x\gamma b))\} = \min\{\sup(\min\{(\mu(a_i), (\mu(c_i), (\nu(b_i), (\nu(d_i))\}), (\mu(x), (\nu(a_b), (\nu(b_d(x\gamma b)))\}, (\mu(x), (\nu(a_b), (\nu(b_d(x\gamma b))))\}, (\mu(x), (\nu(a_b), (\nu(b_d(x\gamma b))))\} \geq \min\{\mu(x), (\nu(a_b), (\nu(b_d(x\gamma b)))\}$$

$$(\mu ho\nu h\mu)(x) = \sup(\min\{(\mu ho\nu)(a_i), (\mu ho\nu)(c_i), (\mu ho\nu)(d_i)\})$$

(ii)$\Rightarrow$(iii) This is straight forward from Lemma 3.10.

(iii)$\Rightarrow$(i) Assume that (iii) holds. Let $\mu$ be any fuzzy $h$-quasi-ideal of $S$. Then since $\chi_s$ is a fuzzy $h$-ideal of $S$, we have $\mu = \mu \cap \chi_s \subseteq \mu ho\chi_s^oh\mu$. Therefore $S$ is $h$-hemiregular by Theorem 4.7.

**Theorem 4.9.** Let $S$ be a $\Gamma$-hemiring. Then the following conditions are equivalent.

(i) $S$ is $h$-hemiregular.

(ii) $\mu \cap \nu \subseteq \mu ho\nu h\mu$ for every fuzzy $h$-bi-ideal $\mu$ and every fuzzy left $h$-ideal $\nu$ of $S$.

(iii) $\mu \cap \nu \subseteq \mu ho\nu h\mu$ for every fuzzy $h$-quasi-ideal $\mu$ and every fuzzy left $h$-ideal $\nu$ of $S$.

(iv) $\mu \cap \nu \subseteq \mu ho\nu h\mu$ for every fuzzy right $h$-ideal $\mu$ and every fuzzy $h$-bi-ideal $\nu$ of $S$.

(v) $\mu \cap \nu \subseteq \mu ho\nu h\mu$ for every fuzzy right $h$-ideal $\mu$ and every fuzzy $h$-quasi-ideal $\mu$ and every fuzzy $h$-ideal $\nu$ of $S$.
\( \nu \) of \( S \).

(vi) \( \mu \cap \nu \cap \omega \subseteq \mu_0 \nu_0 \omega_0 \) for every fuzzy right \( h \)-ideal \( \mu \), for every fuzzy \( h \)-bi-ideal \( \nu \) and for every fuzzy left \( h \)-ideal \( \omega \) of \( S \).

(vii) \( \mu \cap \nu \cap \omega \subseteq \mu_0 \nu_0 \omega_0 \) for every fuzzy right \( h \)-ideal \( \mu \), for every fuzzy \( h \)-quasi-ideal \( \nu \) and for every fuzzy left \( h \)-ideal \( \omega \) of \( S \).

**Proof.** The proof follows by simple verification.

**Proposition 4.10.** If a \( \Gamma \)-hemiring \( S \) is \( h \)-hemiregular then any right \( h \)-ideal \( R \) and left \( h \)-ideal \( L \) are idempotent and \( R \Gamma L \) is an \( h \)-quasi-ideal of \( S \).

**Proof.** Assume that \( S \) is \( h \)-hemiregular and \( R, L \) be any right and left \( h \)-ideal of \( S \) respectively. Now \( R \Gamma R \subseteq \overline{R} = R \). Let \( x \in R \). Since \( S \) is \( h \)-hemiregular, there exist \( a, b \in S \) and \( \alpha, \beta, \gamma, \delta \in \Gamma \) such that \( x + x \alpha a \beta x + z = x \gamma b \delta x + z \). Since \( R \) is a right \( h \)-ideal of \( S \), \( x \alpha a, x \gamma b \in R \) and \( x \alpha a \beta x, x \gamma b \delta x \in R \Gamma R \). Hence \( x \in R \Gamma R \) that is \( R \subseteq R \Gamma R \). Thus \( R = \overline{R} \Gamma R \) and so \( R \) is idempotent. Similarly we can show that \( L \) is idempotent. Now since \( S \) is \( h \)-hemiregular we have \( R \cap L = \overline{R \Gamma L} \), and so \( R \Gamma L \) is an \( h \)-quasi-ideal of \( S \).

**Proposition 4.11.** If a \( \Gamma \)-hemiring \( S \) is \( h \)-hemiregular then any fuzzy right \( h \)-ideal \( \mu \) and fuzzy left \( h \)-ideal \( \nu \) are idempotent and \( \mu_0 \nu_0 \) is an \( h \)-quasi-ideal of \( S \).

**Proof.** Suppose \( S \) is \( h \)-hemiregular and \( \mu \) be any fuzzy right \( h \)-ideal of \( S \). Then \( \mu_0 \mu \subseteq \mu_0 \chi S \subseteq \mu \). Moreover, since \( S \) is \( h \)-hemiregular, \( \mu \subseteq \mu_0 \mu \) and so \( \mu = \mu_0 \mu \). Hence \( \mu \) is idempotent. Similarly, we can prove for fuzzy left \( h \)-ideals.

Now since \( S \) is \( h \)-hemiregular \( \mu_0 \nu_0 = \mu \cap \nu \), which implies \( \mu_0 \nu_0 \) is a fuzzy \( h \)-quasi-ideal of \( S \) by Lemma 3.9.

## 5 \ h \text{-intra-hemiregularity}

In this section we introduce the concept of \( h \)-intra-hemiregularity in \( \Gamma \)-hemiring and characterize some of its properties using \( h \)-ideal, \( h \)-bi-ideal and \( h \)-quasi-ideal.

**Definition 5.1.** A \( \Gamma \)-hemiring \( S \) is said to be \( h \)-intra-hemiregular if for each \( x \in S \), there exist \( z, a_i, a_i', b_i, b_i' \in S \), and \( \alpha_i, \beta_i, \gamma_i, \delta_i, \eta \in \Gamma \), \( i \in \mathbb{N} \), the set of natural numbers, such that \( x + \sum_{i=1}^{n} a_i \alpha_i x \eta x \beta_i + z = \sum_{i=1}^{n} b_i \gamma_i x \eta x \delta_i + z \).

**Example 5.2.** Let \( S=\Gamma = \{0, a, b\} \) with addition(+) and multiplication(\). as follows:

\[
\begin{array}{ccc}
+ & 0 & a & b \\
0 & 0 & a & b \\
a & a & a & a \\
b & b & a & b \\
\end{array}
\] and
\[
\begin{array}{ccc}
. & 0 & a & b \\
0 & 0 & 0 & 0 \\
a & 0 & a & a \\
b & 0 & a & b \\
\end{array}
\]
Then $S$ is a $\Gamma$-hemiring where the ternary composition is defined as $x \circ y = x.\alpha.y$. Here $S$ is both $h$-hemiregular and $h$-intra-hemiregular.

**Lemma 5.3.** Let $S$ be a $\Gamma$-hemiring. Then the following conditions are equivalent

(i) $S$ is $h$-intra-hemiregular.

(ii) $L \cap R \subseteq \overline{LR}$ for every left $h$-ideal $L$ and every right $h$-ideal $R$ of $S$.

**Proof.** (i)$\Rightarrow$(ii). Assume that (i) holds. Let $L$ and $R$ be any left and right $h$-ideal of $S$ respectively. Since $S$ is $h$-intra-hemiregular, we have $L \cap R \subseteq \sigma(L \cap R) \Gamma(L \cap R) = \sigma \Gamma(L \cap R) S \subseteq \sigma \Gamma(S \subseteq \Gamma(S)$.

(ii)$\Rightarrow$(i). Assume that (ii) holds. Let $x \in S$. Then $\sigma x + N x$ and $x \Gamma S + x N$ where $N=\{0,1,2,\ldots\}$ are the principal left $h$-ideal and principal right $h$-ideal of $S$ generated by $x$, respectively. Then $x = 0 \gamma x + 1 x = x 0 + x 1 \in \sigma x + N x \cap x \Gamma S + x N \subseteq (\sigma x + N x) \Gamma x \Gamma S + x N$.

Thus we have $x + \sum_{i=1}^{n} a_i \alpha_i x \eta x \beta_i a_i' + z = \sum_{i=1}^{n} b_i \gamma_i x \eta x \delta b_i' + z$, for some $z, a_i, a_i', b_i, b_i' \in S$, and $\alpha_i, \beta_i, \gamma_i, \delta_i, \eta \in \Gamma$, $i \in \mathbb{N}$, the set of natural numbers. This implies $S$ is $h$-intra-hemiregular.

**Theorem 5.4.** Let $S$ be a $\Gamma$-hemiring. Then $S$ is $h$-intra-hemiregular if and only if $\mu \cap \nu \subseteq \mu_0 \nu$ for every fuzzy left $h$-ideal $\mu$ and every fuzzy right $h$-ideal $\nu$ of $S$.

**Proof.** Suppose $S$ is $h$-intra-hemiregular. Let $\mu$ and $\nu$ be any fuzzy left $h$-ideal and fuzzy right $h$-ideal of $S$ respectively. Now let $x \in S$. Then by hypothesis there exist $z, a_i, a_i', b_i, b_i' \in S$, and $\alpha_i, \beta_i, \gamma_i, \delta_i, \eta \in \Gamma$, $i \in \mathbb{N}$, the set of natural numbers, such that $x + \sum_{i=1}^{n} a_i \alpha_i x \eta x \beta_i a_i' + z = \sum_{i=1}^{n} b_i \gamma_i x \eta x \delta b_i' + z$. Therefore

$(\mu_0 \nu)(x) = \sup \{ \min \{ \min \{ \mu(a_i), \mu(c_i) \}, \theta(b_i) \} \}$

$= \min_{i \in \mathbb{N}} \{ \min \{ \mu(a_i), \mu(b_i, \nu(x)) \} \} \geq \min_{i \in \mathbb{N}} \{ \mu(x), \nu(x) \} = (\mu \cap \nu)(x)$.

Conversely, Suppose the condition holds. Let $L$ and $R$ be any left $h$-ideal and right $h$-ideal of $S$, respectively. Then the characteristic function $\chi_L$ and $\chi_R$ are fuzzy left $h$-ideal and fuzzy right $h$-ideal of $S$, respectively. So, from Lemma 2.14 we obtain $\chi_{L \cap R} = \chi_L \cap \chi_R \subseteq \chi_L \cap \chi_R = \chi_{LR}$. From which it follows that $L \cap R \subseteq LR$, whence $S$ is $h$-intra-hemiregular by Lemma 5.3.

**Theorem 5.5.** Let $S$ be a $\Gamma$-hemiring and $x \in S$. Then $S$ is $h$-intra-hemiregular if and only if $\mu(x) = \nu(x \gamma x)$, for all fuzzy $h$-ideal(resp. fuzzy $h$-interior ideal) $\mu$ of $S$ and for all $\gamma \in \Gamma$. 

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Proof. Suppose \( S \) is h-intra-hemiregular \( \Gamma \)-hemiring and \( \mu \) be any fuzzy h-ideal of \( S \). Let \( x \in S \). Then by hypothesis there exist \( z, a_i, a'_i, b_i, b'_i \in S \), and \( \alpha_i, \beta_i, \gamma_i, \delta_i, \eta \in \Gamma \), \( i \in \mathbb{N} \), the set of natural numbers, such that \( x + \sum_{i=1}^{n} a_i \alpha_i x \eta x \beta_i a'_i + z = \sum_{i=1}^{n} b_i \gamma_i x \eta x \delta_i b'_i + z \).

Then \( \mu(x) \geq \min \{ \min \{ \mu(a_i \alpha_i x \eta x \beta_i a'_i), \mu(b_i \gamma_i x \eta x \delta_i b'_i) \} \} = \mu(x \eta x) \geq \mu(x) \).

Therefore \( \mu(x) = \mu(x \gamma x) \).

Conversely, suppose the condition holds. Now \( \overline{M} x \eta x + ST x \eta x + x \eta x \Gamma S + ST x \eta x \Gamma S \), where \( M=\{0,1,2,...\} \) is the principal h-ideal of \( S \) generated by \( x \eta x \).

Since \( x \gamma = \sum_{i=1}^{n} a_i \alpha_i x \eta x \beta_i a'_i + z \), for some \( z, a_i, a'_i, b_i, b'_i \in S \), and \( \alpha_i, \beta_i, \gamma_i, \delta_i, \eta \in \Gamma \), \( i \in \mathbb{N} \), the set of natural numbers. Therefore \( S \) is h-intra-hemiregular.

Lemma 5.6. Let \( S \) be a \( \Gamma \)-hemiring. Then the following conditions are equivalent.

(i) \( S \) is both h-hemiregular and h-intra-hemiregular.

(ii) \( B = B \Gamma B \) for every h-bi-ideal \( B \) of \( S \).

(iii) \( Q = Q \Gamma Q \) for every h-quasi-ideal \( B \) of \( S \).

Proof. (i) \( \Rightarrow \) (ii) Assume that \( S \) is both h-hemiregular and h-intra-hemiregular. Let \( x \in B \). Then \( x \) can be expressed as \( x = \sum_{i=1}^{n} a_i \alpha_i x \eta x \beta_i a'_i b_i \beta_i x + z \), where \( z, a_i, b_i, c_i, d_i \in S \) and \( \alpha_i, \beta_i, \alpha'_i, \beta'_i, \gamma_i, \delta_i, \gamma'_i, \delta'_i, \eta \in \Gamma \). Since \( B \) is a bi-ideal \( x = \sum_{i=1}^{n} a_i \alpha_i x \eta x \beta_i b_i \beta_i x, \)

\[ \sum_{i=1}^{n} x \gamma_i c_i \gamma'_i x \eta x \delta'_i d_i \delta_i x \in B \Gamma B, \]

which implies \( x \in B \Gamma B \) that is \( B \subseteq B \Gamma B \). Since \( B \) is a h-bi-ideal we have \( B \Gamma B \subseteq B \). Therefore \( B = B \Gamma B \).

(ii) \( \Rightarrow \) (iii) follows easily.

(iii) \( \Rightarrow \) (i) Suppose \( L \) and \( R \) be left and right h-ideal of \( S \) respectively. Then \( L \cap R \) is a h-quasi-ideal of \( S \). So, \( L \cap R = (L \cap R) \Gamma (L \cap R) \subseteq R \Gamma L \subseteq R \cap L = R \cap R \) and \( L \cap R = (L \cap R) \Gamma (L \cap R) \subseteq L \Gamma R \). Therefore \( S \) is both h-hemiregular and h-intra-hemiregular. \( \square \)
Theorem 5.7. Let $S$ be a $\Gamma$-hemiring. Then the following conditions are equivalent.
(i) $S$ is both $h$-hemiregular and $h$-intra-hemiregular.
(ii) $\mu = \mu_0 \mu$ for every $h$-bi-ideal $\mu$ of $S$.
(iii) $\mu = \mu_0 \mu$ for every $h$-quasi-ideal $\mu$ of $S$.

Proof. (i)$\Rightarrow$(ii) Assume that (i) holds. Let $x \in S$ and $\mu$ be any fuzzy $h$-bi-ideal of $S$. Since $S$ is both $h$-hemiregular and $h$-intra-hemiregular there exist $z, a_i, b_i, c_i, d_i \in S$ and $\alpha_i, \beta_i, \alpha_i', \beta_i', \gamma_i, \delta_i, \gamma_i', \delta_i' \in \Gamma$, $i \in N$ such that

$$x + \sum_{i=1}^{n} x\alpha_i a_i \alpha_i' x\eta x \beta_i b_i \beta_i x + z = \sum_{i=1}^{n} x\gamma_i c_i \gamma_i' x\eta x \delta_i d_i \delta_i x + z.$$

Therefore

$$\mu_0 \mu(x) \leq \min\{\mu(a_i), \mu(c_i), \mu(b_i), \mu(d_i)\} \Rightarrow \mu(x).$$

Now $\mu_0 \mu \subseteq \mu_0 \chi_S \subseteq \mu$. Hence $\mu_0 \mu = \mu$ for every fuzzy $h$-bi-ideal $\mu$ of $S$.

(ii)$\Rightarrow$(i) This is straightforward from the Lemma 3.10

(iii)$\Rightarrow$(i) Assume that (iii) holds. Let $Q$ be any $h$-quasi-ideal of $S$. Then the characteristic function $\chi_Q$ of $Q$ is a fuzzy $h$-quasi ideal of $S$. Now $\chi_Q = \chi Q_0 \chi_Q = \chi_Q$, which implies $Q \subseteq Q_0$. Hence $S$ is both $h$-hemiregular and $h$-intra-hemiregular by Lemma 5.6.

Theorem 5.8. Let $S$ be a $\Gamma$-hemiring. Then the following conditions are equivalent.
(i) $S$ is both $h$-hemiregular and $h$-intra-hemiregular.
(ii) $\mu \cap \nu \subseteq \mu_0 \nu$ for all fuzzy $h$-bi-ideals $\mu$ and $\nu$ of $S$.
(iii) $\mu \cap \nu \subseteq \mu_0 \nu$ for every fuzzy $h$-bi-ideals $\mu$ and every fuzzy $h$-quasi-ideal $\nu$ of $S$.
(iv) $\mu \cap \nu \subseteq \mu_0 \nu$ for every fuzzy $h$-quasi-ideals $\mu$ and every fuzzy $h$-bi-ideal $\nu$ of $S$.
(v) $\mu \cap \nu \subseteq \mu_0 \nu$ for all fuzzy $h$-quasi-ideals $\mu$ and $\nu$ of $S$.

Proof. (i)$\Rightarrow$(ii) Assume that (i) holds. Let $x \in S$ and $\mu$ and $\nu$ be any fuzzy $h$-bi-ideals of $S$. Since $S$ is both $h$-hemiregular and $h$-intra-hemiregular there exist there exist $z, a_i, b_i, c_i, d_i \in S$ and $\alpha_i, \beta_i, \alpha_i', \beta_i', \gamma_i, \delta_i, \gamma_i', \delta_i' \in \Gamma$, $i \in N$ such that

$$x + \sum_{i=1}^{n} x\alpha_i a_i \alpha_i' x\eta x \beta_i b_i \beta_i x + z = \sum_{i=1}^{n} x\gamma_i c_i \gamma_i' x\eta x \delta_i d_i \delta_i x + z.$$
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\[(\mu_{oh}\nu)(x) = \sup \{\min_i \{\mu(a_i), \mu(c_i), \nu(b_i), \nu(d_i)\}\} \]
\[= \sup_i \{\min \{\mu(a_i), \mu(c_i), \nu(b_i), \nu(d_i)\}\} \]
\[= \sup_i \{\min \{\mu(x_{i1}a_i\alpha_i', x), \mu(x_{i2}c_i\gamma_i', x), \nu(x_{i3}b_i\beta_i, x), \nu(x_{i4}d_i\delta_i, x)\}\} \]
\[\geq \min \{\mu(x_{i1}a_i\alpha_i', x), \mu(x_{i2}c_i\gamma_i', x), \nu(x_{i3}b_i\beta_i, x), \nu(x_{i4}d_i\delta_i, x)\} \]
\[\geq \min_i \{\mu(x_i), \nu(x_i)\} = (\mu \cap \nu)(x), \text{ which implies } \mu \cap \nu \subseteq \mu_{oh}\nu.\]

(ii)⇒(iii)⇒(v) and (ii)⇒(iv)⇒(v) are obvious from Lemma 3.10.

Now for (v)⇒(i), suppose Q be any h-quasi-ideal of S. Then the characteristic function \(\chi_Q\) of Q is a fuzzy h-quasi ideal of S. Now \(\chi_Q = \chi_Q \cap \chi_Q = \chi_{Q \cap Q}\), which implies \(Q \subseteq Q \cap Q\). Since the converse inclusion always holds; Therefore \(Q = Q \cap Q\). Hence S is both h-hemiregular and h-intra-hemiregular, by Lemma 5.6.

\[\square\]

6 h-quasi-hemiregularity

**Definition 6.1.** A Γ-hemiring S is called (left, right) h-quasi hemiregular if every (left, right) h-ideal of S is idempotent.

Then it is easily seen that, a Γ-hemiring S is left(right) h-quasi-hemiregular if and only if \(a \in \text{STa} \Gamma \text{STa}(\text{resp. } a \in \text{a} \Gamma \text{STa} S)\).

**Theorem 6.2.** A Γ-hemiring S is left(right) h-quasi-hemiregular if and only if every fuzzy left(right) h-ideal is idempotent.

**Proof.** Let S be a left h-quasi-hemiregular Γ-hemiring and \(\mu\) be any fuzzy left h-ideal of S. Let \(a \in S\). Since S is left h-quasi-hemiregular \(a \in (\text{STa}) \Gamma (\text{STa})\).

So, there exist \(z, x_i, y_i \in S\) and \(\alpha_i, \beta_i, \gamma_i \in \Gamma\), \(i = 1, 2\) such that \(a + (x_{i1}a_1\alpha_i a_2a_1) + (x_{i2}c_2 a_2 a_2) + z = (x_{i3}a_2 a_2) + (y_{i2}a_2 a_2) + z\). Therefore

\[(\mu_{oh}\mu)(a) \]
\[= \sup_i \{\min \{\mu(a), \mu(c), \mu(b), \mu(d)\}\} \]
\[= \sup_i \{\min \{\mu(x_{i1}a_1\alpha_i a_2a_1), \mu(x_{i2}c_2 a_2 a_2), \mu(y_{i2}a_2 a_2)\}\} \]
\[\geq \min \{\mu(x_{i1}a_1\alpha_i a_2a_1), \mu(x_{i2}c_2 a_2 a_2), \mu(y_{i2}a_2 a_2)\} \]
\[\geq \min \{\mu(a), \mu(c), \mu(b), \mu(d)\} = \mu(a).\]

Hence \(\mu \subseteq \mu_{oh}\mu\). Since \(\mu\) is a fuzzy left h-ideal of S, \(\mu_{oh}\mu \subseteq \mu\). Therefore \(\mu = \mu_{oh}\mu\).

Conversely, assume that every fuzzy left h-ideal is idempotent. Let \(a \in S\). Then \(a\Gamma a\) is a left h-ideal of S. So, \(\chi_{\leq a\Gamma a\leq a}\) is a fuzzy left h-ideal of S. Now,

\[\chi_{\leq a\Gamma a\leq a}(a) = (\chi_{\leq a\Gamma a\leq a}) \cap (\chi_{\leq a\Gamma a\leq a}) = \chi_{\leq a\Gamma a\leq a}(a) = 1.\]

So, \(a \in a\Gamma a\Gamma a\) \(= \text{STa} \Gamma \text{STa} S\). Thus S is left h-quasi-hemiregular.

\[\square\]
Theorem 6.3. Let \( S \) be a \( \Gamma \)-hemiring. Then the following conditions are equivalent:

(i) \( S \) is \( h \)-quasi-hemiregular

(ii) \( \mu = (\mu_0\chi_s)^2 \cap (\chi_s\mu_0)^2 \) for every fuzzy \( h \)-quasi-ideal \( \mu \) of \( S \).

Proof. (i) \( \Rightarrow \) (ii)
Assume that (i) holds. Let \( \mu \) be a fuzzy \( h \)-quasi-ideal of \( S \). Since \( S \) is \( h \)-quasi-hemiregular, the fuzzy right \( h \)-ideal \( (\mu_0\chi_s) \) and the fuzzy left \( h \)-ideal \( (\chi_s\mu_0) \) are idempotent. Therefore \( (\mu_0\chi_s)^2 \cap (\chi_s\mu_0)^2 = (\mu_0\chi_s) \cap (\chi_s\mu_0) \subseteq \mu \).

To prove the reverse inclusion, let \( aS \in S \). Since \( S \) is left \( h \)-quasi-hemiregular, \( a \in (Sta) \implies (Sta) \). So there exist \( z, x, y \in S \) and \( \alpha, \beta, \gamma \in \Gamma \), \( i = 1,2 \) such that \( a + (x_1\alpha_1a)\beta_1(y_1\gamma_1a) + z = (x_2\alpha_2a)\beta_2(y_2\gamma_2a) + z \ldots (1) \). Therefore
\[
(\chi_s\mu_0)^2(a) = ((\chi_s\mu_0)\mu_0(\chi_s\mu_0))(a) = \sup\{\min\{\min\{\chi_s\mu_0(\mu_0(\chi_s\mu_0)), \chi_s\mu_0(\chi_s\mu_0)\}(a_i), (\chi_s\mu_0)(\chi_s\mu_0)(b_i), (\chi_s\mu_0)(\chi_s\mu_0)(c_i)\}\}
\]

\[
a + \sum_{i=1}^{n} a_i\gamma_i b_i + z = \sum_{i=1}^{n} c_i\delta_i d_i + z
\]

\[
\geq \min\{(\chi_s\mu_0)(x_1\alpha_1a), (\chi_s\mu_0)(x_2\alpha_2a), (\chi_s\mu_0)(y_1\gamma_1a), (\chi_s\mu_0)(y_2\gamma_2a)\} \ldots (2)
\]

Now from (1) we have, \( x_1\alpha_1z + x_1\alpha_1z = (x_1\alpha_1x_2\alpha_2a)\beta_2(y_2\gamma_2a) + z \)

So from (2) we have
\[
(\chi_s\mu_0)^2 \geq \min\{\chi_s(x_1\alpha_1x_1\alpha_1a\beta_1y_1), \chi_s(x_2\alpha_2x_1\alpha_1a\beta_1y_1), \chi_s(x_2\alpha_2x_2\alpha_2a\beta_2y_2), \chi_s(y_1\gamma_1x_1\alpha_1a\beta_1y_1), \chi_s(y_2\gamma_2x_1\alpha_1a\beta_1y_1), \chi_s(y_2\gamma_2x_2\alpha_2a\beta_2y_2), \chi_s(y_2\gamma_2x_2\alpha_2a\beta_2y_2), \mu(a), \mu(a), \mu(a), \mu(a), \mu(a), \mu(a)\} = \mu(a).
\]

Therefore \( (\chi_s\mu_0)^2 \geq \mu \). Similarly we obtain \( (\mu_0\chi_s)^2 \geq \mu \). So \( (\mu_0\chi_s)^2 \cap (\chi_s\mu_0)^2 \geq \mu \). Hence \( (\mu_0\chi_s)^2 \cap (\chi_s\mu_0)^2 = \mu \).

(ii) \( \Rightarrow \) (i)
Conversely, assume that (ii) holds. Let \( \mu \) be any fuzzy right \( h \)-ideal of \( S \). Then since \( \mu \) is a fuzzy \( h \)-quasi-ideal of \( S \), we have \( \mu = (\mu_0\chi_s)^2 \cap (\chi_s\mu_0)^2 \subseteq (\mu_0\chi_s)^2 \subseteq \mu_0\chi_0 \subseteq \mu_0\chi_0 \subseteq \mu \). Therefore it follows from Theorem 5.2 that \( S \) is right \( h \)-quasi-hemiregular. Similarly, it can be proved that \( S \) is left \( h \)-quasi-hemiregular. Thus (ii) \( \Rightarrow \) (i). \( \square \)

Theorem 6.4. Let \( S \) be a \( \Gamma \)-hemiring. Then the following conditions are equivalent:

(i) \( S \) is both \( h \)-intra-hemiregular and left \( h \)-quasi-hemiregular

(ii) \( \mu \cap \nu \cap \omega \subseteq \mu_0\nu_0\omega \), for every fuzzy \( h \)-bi-ideal \( \omega \), every fuzzy left \( h \)-ideal \( \mu \) and every right \( h \)-ideal \( \nu \) of \( S \).

(ii) \( \mu \cap \nu \cap \omega \subseteq \mu_0\nu_0\omega \), for every fuzzy \( h \)-quasi-ideal \( \omega \), every fuzzy left \( h \)-ideal \( \mu \) and every right \( h \)-ideal \( \nu \) of \( S \).

Proof. (i) \( \Rightarrow \) (ii)
Suppose (i) holds. Let \( \mu, \nu, \omega \) be fuzzy left \( h \)-ideal, fuzzy right \( h \)-ideal and...
fuzzy h-bi-ideal respectively. Since S is h-intra-hemiregular and left h-quasi-hemiregular, any element \( a \in S \) can be expressed as \( a + (x_1 \alpha_1 a) \beta_1 (a \eta a) \delta_1 (y_1 \gamma_1 a) + z = (x_2 \alpha_2 a) \beta_2 (a \eta a) \delta_2 (y_2 \gamma_2 a) + z \). (1) for \( z, x_i, y_i \in S \) and \( \eta, \alpha_i, \beta_i, \gamma_i, \delta_i \in \Gamma, i = 1, 2 \).

Therefore

\[
(\mu \circ \nu)(a) = \sup \{ \min \{ \min \{ \mu(a_i), \mu(c_i), (\nu \circ \omega)(b_i), (\nu \circ \omega)(d_i) \} \} \}
\]

\[
= \sum_{i=1}^{n} a_i \gamma_i b_i + z = \sum_{i=1}^{n} c_i \delta_i d_i + z
\]

\[
= \min \{ \mu(x_1 \alpha_1 a \beta_1 a), \mu(x_2 \alpha_2 a \beta_2 a), (\nu \circ \omega)(a \delta_1 y_1 \gamma_1 a)(\nu \circ \omega)(a \delta_2 y_2 \gamma_2 a) \}
\]

\[
\geq \min \{ \mu(a), \mu(a), (\nu \circ \omega)(a \delta_1 y_1 \gamma_1 a)(\nu \circ \omega)(a \delta_2 y_2 \gamma_2 a) \} \quad \text{(2)}
\]

Now, from (1) we have

\[
a \delta_1 y_1 \gamma_1 a + a \delta_1 y_1 \gamma_1 a x_1 \alpha_1 a \beta_1 a \eta a \delta_1 y_1 \gamma_1 a + a \delta_1 y_1 \gamma_1 z = a \delta_1 y_1 \gamma_1 x_2 \alpha_2 a \beta_2 a \eta a \delta_1 y_2 \gamma_2 a + a \delta_1 y_1 \gamma_1 z + a \delta_2 y_2 \gamma_2 a + a \delta_2 y_2 \gamma_2 a x_1 \alpha_1 a \beta_1 a \eta a \delta_1 y_1 \gamma_1 a + a \delta_2 y_2 \gamma_2 z = a \delta_2 y_2 \gamma_2 x_2 \alpha_2 a \beta_2 a \eta a \delta_2 y_2 \gamma_2 a + a \delta_2 y_2 \gamma_2 z.
\]

So, from (2), we have

\[
(\mu \circ \nu)(a) \geq \min \{ \mu(a), \mu(a), \nu(a \delta_1 y_1 \gamma_1 x_1 \alpha_1 a \beta_1 a), \nu(a \delta_1 y_1 \gamma_1 x_2 \alpha_2 a \beta_2 a), \omega(a \delta_1 y_1 \gamma_1 a), \omega(a \delta_2 y_2 \gamma_2 a), \nu(a \delta_2 y_2 \gamma_2 x_2 \alpha_1 a \beta_1 a), \nu(a \delta_2 y_2 \gamma_2 x_2 \alpha_2 a \beta_2 a) \} \geq \min \{ \mu(a), \nu(a), \omega(a) \}
\]

\[
= \{ \mu \cap \nu \cap \omega \}(a).
\]

(iii) \Rightarrow (i)

Let \( \mu \) be any fuzzy left h-ideal of S. Then \( \mu = \mu \cap \chi_S \cap \mu \subseteq \mu \circ \omega \cap \mu \subseteq \mu \circ \omega \cap \chi_S \subseteq \mu \). Therefore from Theorem 6.2, S is left h-quasi-hemiregular.

Now suppose \( \nu \) be a fuzzy right h-ideal of S. Then \( \mu \cap \nu = \mu \cap \nu \cap \chi_S \subseteq \mu \circ \omega \cap \nu \cap \chi_S \subseteq \mu \circ \omega \cap \nu \subseteq \mu h \cap \nu \cap \chi_S \subseteq \mu h \cap \nu \subseteq \mu h \cap \nu \). Hence from Theorem 5.4, we deduce that S is h-intra-hemiregular.

7 Fuzzy h-Duo \( \Gamma \)-hemiring

A \( \Gamma \)-hemiring S is called h-duo if every one-sided h-ideal of it is a h-ideal of S. A \( \Gamma \)-hemiring S is called fuzzy h-duo if every one-sided fuzzy h-ideal of S is a fuzzy h-ideal of S. A \( \Gamma \)-hemiring S is called h-hemiregular fuzzy h-duo if it is both h-hemiregular and fuzzy h-duo.

For simplicity in what follows we call h-duo \( \Gamma \)-hemiring a duo \( \Gamma \)-hemiring.

Lemma 7.1. Let S be a \( \Gamma \)-hemiring. Then the following conditions are equivalent.

(i) S is h-hemiregular duo hemiring.

(ii) \( L \cap R = \overline{L \cap R} \) for every left h-ideal \( L \) and every right h-ideal \( R \).

Proof. (i) \Rightarrow (ii) Assume that (i) holds. Let \( L \) and \( R \) be any left h-ideal and any right h-ideal of S respectively. Then since S is duo hemiring both
L and R are h-ideals of S. Moreover since S is h-hemiregular, it follows from Lemma 4.3 that \( L \cap R = \overline{LR} \).

Conversely, assume that (ii) holds. Let L and R be any left h-ideal and any right h-ideal of S respectively. Since S itself is an h-ideal of S, we have \( L = L \cap S = \overline{LS} \supseteq L \cap S \) and \( R = S \cap R = \overline{SR} \supseteq S \cap R \). Hence both L and R are h-ideals of S and so S is duo. Now for every right h-ideal R and left h-ideal L we have \( R \cap L = \overline{RL} \) whence S is h-hemiregular.

**Lemma 7.2.** Let S be a \( \Gamma \)-hemiring. Then S is h-hemiregular duo hemiring if and only if \( A \cap Q = \overline{AQ} \cap A \) for every h-ideal A and every quasi-ideal Q of S.

**Proof.** Suppose S be a \( \Gamma \)-hemiring. Assume that S is h-hemiregular duo hemiring. Let A and Q be any left h-ideal and any h-quasi-ideal of S, respectively. Then \( \overline{AQ} \cap A \subseteq \overline{A} = A \) and \( \overline{AQ} \cap A \subseteq \overline{A} \cap Q \subseteq A \cap Q \). Hence \( \overline{AQ} \cap A \subseteq A \cap Q \). Now let \( x \in A \cap Q \). Since S is h-hemiregular, there exist \( a, b \in S \) and \( \alpha, \beta, \gamma, \delta \in \Gamma \) such that \( x + x\alpha a\beta x + z = x\gamma b\delta x + z \). Then we have \( x\alpha a\beta x + x\alpha a\beta x + z = x\gamma b\delta x + x\gamma b\delta x + z \) and \( x + x\alpha a\beta x + z + x\alpha a\beta x + z = x\gamma b\delta x + x\gamma b\delta x + z \). So, \( x + x\gamma b\delta x + x\gamma b\delta x + z \) where \( x \in A \cap Q \). Since A is an h-ideal of S, we have \( x\alpha a, x\gamma b, a\beta x, b\delta x \in A \) and in consequence \( x\gamma b\delta x + x\alpha a\beta x + x\gamma b\delta x + z \). Hence \( A \cap Q \cap A = A \cap Q \). Conversely, suppose the condition holds. Let L and R be any left and right h-ideal of S, respectively. Then both L and R are h-ideals of S. Since S itself is an h-ideal of S, we have \( S \cap L = L \) and \( S \cap R = R \). Now, \( L \cap S = S \cap L \subseteq S \cap S \subseteq S \cap S = S \cap S = S \). Hence both L and R are h-ideals of S, i.e. S is duo. Moreover by the assumption, we have \( R \cap L = \overline{RL} \subseteq \overline{RL} \subseteq R \cap L = R \cap L \). Therefore S is h-hemiregular.

**Lemma 7.3.** Let S be h-hemiregular \( \Gamma \)-hemiring. Then S is duo if and only if it is fuzzy duo.

**Proof.** Assume that S is duo. Let \( \mu \) be any fuzzy left h-ideal of S. Then since the set \( STx \) is a left h-ideal of S, it is an h-ideal by the assumption. Since S is h-hemiregular, we have \( x\gamma y \in x\gamma y + a\alpha x + z = b\beta x + z \). Since \( \mu \) is a fuzzy left h-ideal of S, we have \( \mu(x\gamma y) \geq \min\{\mu(a\alpha x), \mu(b\beta x)\} \geq \mu(x) \) and so \( \mu \) is a fuzzy right h-ideal of S. It can be seen in a similar way that any fuzzy right h-ideal of S is a fuzzy h-ideal of S. Thus S is fuzzy duo.

Conversely, assume that S is fuzzy duo. Let L be any left h-ideal of S, then the
characteristic function \( \chi_L \) of \( L \) is a fuzzy left h-ideal of \( S \). By the assumption \( \chi_L \) is a fuzzy h-ideal of \( S \) and so \( L \) is a h-ideal of \( S \). Thus \( S \) is duo. This completes the proof.

**Lemma 7.4.** Let \( S \) be a h-hemiregular fuzzy duo hemiring. Then every fuzzy h-bi-ideal of \( S \) is a h-ideal of \( S \).

**Proof.** Let \( \mu \) be any fuzzy h-bi-ideal of \( S \) and \( x,y \in S \). Then since the set \( STx \) is a left h-ideal of \( S \), it is an h-ideal of \( S \). Now since \( S \) is h-hemiregular, \( x\gamma y \in \frac{x\Gamma STx \Gamma y}{x \Gamma STx ST} \subseteq \frac{x \Gamma STx ST}{x \Gamma STx ST} \subseteq x \Gamma STx = x \Gamma STx \)
this implies \( x\gamma y + x\alpha\beta x + z = x\eta\delta x + z \) for \( a,b,z \in S \) and \( \alpha, \beta, \eta, \delta \in \Gamma \). Since \( \mu \) is fuzzy h-bi-ideal of \( S \), we have \( \mu(x\gamma y) \geq \min\{\mu(x\alpha\beta x), \mu(x\eta\delta x)\} \geq \mu(x) \)
and so \( \mu \) is a fuzzy right h-ideal of \( S \). In a similar way it can be easily seen that \( \mu \) is a fuzzy left h-ideal of \( S \) and so \( \mu \) is a fuzzy h-ideal of \( S \).

By a routine verification we can get the following result.

**Theorem 7.5.** Let \( S \) be a \( \Gamma \)-hemiring. Then the following conditions are equivalent.

1. \( S \) is h-hemiregular duo hemiring.
2. \( S \) is h-hemiregular fuzzy duo hemiring.
3. \( \mu \cap \nu = \mu o_h \nu \) for every fuzzy h-bi-ideal \( \mu \) and \( \nu \) of \( S \).
4. \( \mu \cap \nu = \mu o_h \nu \) for every fuzzy h-bi-ideal \( \mu \) and every fuzzy h-quasi-ideal \( \nu \) of \( S \).
5. \( \mu \cap \nu = \nu o_h \mu \) for every fuzzy h-bi-ideal \( \mu \) and every fuzzy right h-ideal \( \nu \) of \( S \).
6. \( \mu \cap \nu = \nu o_h \mu \) for every fuzzy h-quasi-ideal \( \mu \) and every fuzzy h-bi-ideal \( \nu \) of \( S \).
7. \( \mu \cap \nu = \nu o_h \mu \) for every fuzzy left h-ideal \( \mu \) and every fuzzy h-bi-ideal \( \nu \) of \( S \).
8. \( \mu \cap \nu = \nu o_h \mu \) for every fuzzy left h-ideal \( \mu \) and every fuzzy h-quasi-ideal \( \nu \) of \( S \).

**Theorem 7.6.** Let \( S \) be a \( \Gamma \)-hemiring. Then the following conditions are equivalent.

1. \( S \) is h-hemiregular duo \( \Gamma \)-hemiring.
2. \( \mu \cap \nu = \mu o_h \nu o_h \mu \) for every fuzzy h-ideal \( \mu \) and every fuzzy h-bi-ideal \( \nu \) of \( S \).
3. \( \mu \cap \nu = \mu o_h \nu o_h \mu \) for every fuzzy h-ideal \( \mu \) and every fuzzy h-quasi-ideal \( \nu \) of \( S \).
Proof. (i)⇒(ii) Assume that (i) holds. Let μ be any fuzzy h-ideal and ν be any fuzzy h-bi-ideal of S. Then we have 
\[ μ_o \mu_ν \subseteq (μ_o h_ν) \subseteq μ_o h_ν \subseteq μ. \]
Also, \( μ_o ν_μ h \subseteq h_s \subseteq ν \). Hence \( μ_o ν_μ h \subseteq μ \cap ν \).

Now let \( x \) be any element of \( S \). Since \( S \) is \( h \)-hemiregular, there exist \( a,b \in S \) and \( α,β,γ,δ \in Γ \) such that 
\[ x + xαaβx + z = xγbδx + z. \]
Then we have 
\[ xαaβx + xαaβαaβx + zαaβx = xγbδxαaβx + zαaβx, xγbδx + xαaβxγbδx + zγbδx = xγbδxγbδx + zγbδx \text{ and } x + xαaβx + z + xαaβxαaβx + zαaβx + xαaβxγbδx + zγbδx + xγbδx + z. \]
So, 
\[ x + xγbδxαaβx + xαaβxγbδx + z' = xαaβxαaβx + xγbδxγbδx + z' \text{ where } z' = z + zαaβx + zγbδx. \]
Hence we have 
\[ (μ_o ν_θ_h_ν)(x) = \sup\{\min\{(μ_o ν)(a), (μ_o ν)(c), μ(b), μ(d)\} \}
\[ ≥ \min\{(μ_o ν)(x), μ(aβxαaβx + bδxγbδx), μ(bδxαaβx + aβxγbδx)\}
\[ = \min\{\sup\{\min\{μ(a), μ(c), ν(b), ν(d)\}, μ(bδxαaβx + aβxγbδx), μ(aβxαaβx + bδxγbδx)\} \}
\[ ≥ \min\{μ(x), ν(x)\} = (μ ∩ ν)(x) \text{ and so } (μ ∩ ν) \subseteq μ_o ν_μ h. \]

Hence \( (μ ∩ ν) = μ_o ν_μ h. \).

(ii)⇒(iii) This is straightforward from Lemma 3.10.

(iii)⇒(i) Assume that (iii) holds. Let \( A \) and \( Q \) be any h-ideal and h-quasi-ideal of \( S \), respectively. Then the characteristic function \( γ_α, γ_Q \) of \( A \) and \( Q \) respectively is a fuzzy h-ideal and fuzzy h-quasi-ideal of \( S \). Now by the assumption we have \( γ_α \cap γ_Q = γ_α \cap Q = A h_ν γ_Q \cap γ_α = γ_α A h_ν γ_Q \) and so \( A \cap Q = A h_ν γ_Q \). Hence from Lemma 7.2 we have \( S \) is \( h \)-hemiregular duo \( Γ \)-hemiring.

8 Conclusion

Various relationships between fuzzy h-ideals of a \( Γ \)-hemiring and those of its operator hemirings\[ \] can be established which will be effective in extending the study of \( Γ \)-hemirings in terms of fuzzy subsets.

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