Optimality and Duality for Minimax Fractional Semi-Infinite Programming

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Abstract—The purpose of this paper is to consider a class of nonsmooth minimax fractional semi-infinite programming problem. Based on the concept of $H$–tangent derivative, a new generalization of convexity, namely generalized uniform $(B_\alpha, \rho)$–invexity, is defined for this problem. For such semi-infinite programming problem, several sufficient optimality conditions are established and proved by utilizing the above defined new classes of functions. The results extend and improve the corresponding results in the literature. Subsequently, these optimality conditions are utilized as a basis for formulating dual problems. Weak, strong and reverse duality theorems are also derived for dual programs, using generalized invexity on the functions involved. Some previous duality results for differentiable minimax fractional programming problems turn out to be special cases for the results described in the paper.

Index Terms—$H$–tangent derivative, generalized convexity, minimax fractional semi-infinite programming, optimality conditions, duality

I. INTRODUCTION

In recent years, the concept of convexity and generalized convexity is well known in optimization theory and plays a central role in mathematical economics, management science, and optimization theory. Therefore, the research on convexity and generalized convexity is one of the most important aspects in mathematics programming. To relax convexity assumptions imposed on theorems on optimality conditions for generalized mathematical programming problems, various generalized convexity notions have been introduced. In particular, the concept of generalized $(F, \rho)$–convexity, introduced by Preda [1] is in turn an extension of the convexity and was used by several authors to obtain relevant results. In [2, 3], the concept of $V–\rho$–invexity and $(F, \alpha, \rho, d)$–convexity were introduced, respectively. Other classes of generalized type I functions have been discussed in [4, 5].

On the other hand, a large literature was developed around generalized convexity and its applications in mathematical programming. Many authors investigated the optimality conditions and duality results for min-max programming problems under the conditions of generalized convexity. In particular, Aparna Mehra [6] employed various optimality conditions and duality results under arcwise connectedness and generalized arcwise connectedness assumptions for a static minimax programming problem. Lin [7] and Wu [8] derived the sufficient optimality conditions for the generalized minimax fractional programming in the framework of $(F, \rho)$–convex functions and invex functions. In [9], the Karush-Kuhn-Tucker-type sufficient optimality conditions and duality theorems for a nondifferentiable minimax fractional programming problem under the assumptions of alpha-univex and related functions were derived. Hang-Chin Lai [10] established the necessary and sufficient optimality conditions of nondifferentiable minimax fractional programming problem with complex variables under generalized convexities. Lai and Liu [11] employed the elementary method and technique to prove the necessary and sufficient optimality conditions for nondifferentiable minimax fractional programming problem involving convexity. In [12], a unified higher-order dual for a nondifferentiable minimax programming problem was formulated involving generalized higher-order $(F, \alpha, \rho, d)$-Type I functions.

Semi-infinite programming have been a subject of wide interest since they play a key role in a particular physical or social science situation, i.e., control of robots, mechanical stress of materials, and air pollution abatement etc. Recently, Qingxiang Zhang [13] obtained the necessary and sufficient optimality conditions for the nondifferentiable nonlinear semi-infinite programming involving B-arcwise connected functions. In [14, 15, 16, 17], the optimality conditions under various constraints qualification for semi-infinite programming problems were established.

In this paper, motivated by the above work, we first define a kind of generalize convexity about the $H$-tangent derivative. Then, the sufficient optimality conditions are obtained for a class of min-max fractional semi-infinite programming problem involving the new generalized convexity. Further, we develop duality theory. Several duality results are established for the optimization problem.

II. DEFINITIONS AND PRELIMINARIES

Let $X \subseteq \mathbb{R}^n$ be a nonempty set, $x_0 \in X, d \in \mathbb{R}^n$ and $f : X \to \mathbb{R} \cup \{+\infty\}$ be a function, $T_{epf}^d (x_0, f(x_0))$ be
with respect to \((x_0, f(x_0))\).

We say that \(f^H(x^0,d)\) is \(H\)–tangent derivative of \(f\) at \(x^0\) along the direction \(d\), where

\[
f^H(x^0,d) = \inf_{[d,\eta] \in T^H_{x_0}(x_0, f(x_0))} \eta \quad \text{subject to} \quad f(x^0,\eta) = f(x_0) + \eta^T d + o(\eta, d).
\]

To define a class of new functions, we suppose that \(X\) is nonempty open subset of \(\mathbb{R}^n\), real valued function \(f : X \to \mathbb{R}\), \(h : X \times [0,1] \to \mathbb{R}\), \(\eta : X \times [0,1] \to \mathbb{R}\), \(\theta : X \times [0,1] \to \mathbb{R}^n\), where \(\eta\) and \(\theta\) are vector-valued functions.

**Definition 2.1.** \(f\) is said to be generalized uniform \((B_\eta, \rho)\)–inex function at \(x^0 \in X\), if for any \(x \in X\), there exists \(b, \phi, \rho, \eta, \theta, \rho\), such that

\[
b(x,x^0)\phi \left( f(x) - f(x^0) \right) + \rho \mid \theta(x,x^0) \mid^2 < 0
\]

**Definition 2.2.** \(f\) is said to be strictly generalized uniform \((B_\eta, \rho)\)–inex function at \(x^0 \in X\), if for any \(x \in X\), \(x \neq x^0\), there exists \(b, \phi, \rho, \eta, \theta, \rho\), such that

\[
b(x,x^0)\phi \left( f(x) - f(x^0) \right) > \rho \mid \theta(x,x^0) \mid^2 < 0
\]

**Definition 2.3.** \(f\) is said to be weakly generalized uniform \((B_\eta, \rho)\)–inex function at \(x^0 \in X\), if for any \(x \in X\), \(x \neq x^0\), there exists \(b, \phi, \rho, \eta, \theta, \rho\), such that

\[
b(x,x^0)\phi \left( f(x) - f(x^0) \right) \leq 0
\]

**Definition 2.4.** \(f\) is said to be strictly generalized uniform \((B_\eta, \rho)\)–quasiinex function at \(x^0 \in X\), if for any \(x \in X\), \(x \neq x^0\), there exists \(b, \phi, \rho, \eta, \theta, \rho\), such that

\[
b(x,x^0)\phi \left( f(x) - f(x^0) \right) \leq 0
\]

**Definition 2.5.** \(f\) is said to be generalized uniform \((B_\eta, \rho)\)–quasiinex function at \(x^0 \in X\), if for any \(x \in X\), \(x \neq x^0\), there exists \(b, \phi, \rho, \eta, \theta, \rho\), such that

\[
b(x,x^0)\phi \left( f(x) - f(x^0) \right) < 0
\]

**Definition 2.6.** \(f\) is said to be weakly generalized uniform \((B_\eta, \rho)\)–quasiinex function at \(x^0 \in X\), if for any \(x \in X\), \(x \neq x^0\), there exists \(b, \phi, \rho, \eta, \theta, \rho\), such that

\[
b(x,x^0)\phi \left( f(x) - f(x^0) \right) < 0
\]

In this section, we consider the following minimax fractional semi-infinite programming problem:

\[
(SIFP) \quad \min_{x \in X} f(x, y)
\]

subject to \(g(x,u) \leq 0, \; u \in U, \; x \in X\), where \(X\) is a nonempty open subset of \(\mathbb{R}^n\), \(Y\) is compact subset of \(\mathbb{R}^m\), \(f(\cdot, \cdot) : X \times Y \to \mathbb{R}\), \(h(\cdot, \cdot) : X \times Y \to \mathbb{R}\), \(f(x, \cdot)\) and \(h(x, \cdot)\) are continuous on \(Y\) for every \(x \in X\), \(g : X \times U \to \mathbb{R}\) and \(U \subset \mathbb{R}^r\) is an infinite index set; \(f(x,y) \geq 0\) and \(h(x,y) > 0\) for each \((x,y) \in X \times Y\). We assume that \(f(x, \cdot), h(x, \cdot)\) and \(g(\cdot, u)\) are \(H\)–tangent derivable at \(x \in X\). We put \(X^0 = \{x \mid g(x,u) \leq 0, \; u \in U\}\) for the feasible set of problem (SIFP).

For each \(x \in X^0\), we define

\[
\Lambda = \{j \mid g(x,u) \leq 0, x \in X, u \in U\},
\]

\[
J(x^0) = \{j \mid g(x^0,u) = 0, x^0 \in X, u \in U\},
\]

\[
U^* = \{u \in U \mid g(x^0,u) \leq 0, x \in X, j \in \Delta\},
\]

\[
\Delta = \{\mu_j \mid \mu_j \geq 0, j \in \Delta\}
\]

Where \(U^*\) is a countable subset of \(U\), in the set \(\Lambda\), every \(\mu_j \geq 0\), for all \(j \in \Delta\), and only finitely many are strictly positive.

\[
\overline{Y}(x) = \{y \in Y \mid f(x, y) \leq \sup_{x \in X^0} f(x, z)\}
\]

\[
Q = \{(s, \lambda, \gamma) \mid s + \lambda \leq 0, s \geq 0, \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_s) \in R^s, \lambda_i \geq 0, i = 1, \ldots, s\}
\]

In view of the continuity of \(f(\cdot, \cdot)\) and \(h(x, \cdot)\) on \(Y\) and compactness of \(Y\), it is clear that \(\overline{Y}(x)\) is nonempty compact subset of \(Y\) for each \(x \in X\), and for any \(\gamma \in \overline{Y}(x^0)\), we let \(\gamma^* = \frac{f(x^0, \gamma)}{h(x^0, \gamma)}\), which is always a constant.

**Definition 3.1.** For the problem (SIFP), a point \(x^0 \in X^0\) is said to be an optimal solution, if for any \(x \in X^0\) such that

\[
\sup_{x \in X^0} f(x, y) = \sup_{x \in X^0} f(x, y)
\]

**Definition 3.2.** It is said that \(x^*\) satisfies the Kuhn-Tucker constraint qualification for (SIFP), if there exists

\[
s^* > 0, \lambda^*_i \geq 0, 1 \leq i \leq s, \mu_j^* \in \Lambda, j \in \Delta, \overline{y}_i \in \overline{Y}(x^*), 1 \leq i \leq s^*
\]

and \(\lambda^*_i \in R\), such that
\[\sum_{j=1}^{s'} \lambda'_j (f - q'h)_{ij}(x', y'; \eta(x, x')) + \sum_{j=1}^{s'} \mu'_j g_{ij}(x', u'; \eta(x, x')) \geq 0, \forall u' \in U'\]
\[f(x', y') - q'h(x', y') = 0, i = 1, \ldots, s'\]
\[\mu'_j g_{ij}(x', u') = 0, j \in \Delta\]
\[\sum_{j=1}^{s'} \lambda'_j + \sum_{j=1}^{s'} \mu'_j = 0, \sum_{j=1}^{s'} \lambda'_j = 1\]

**Theorem 3.1.** Let \(x' \in X^0\) and for any \(x \in X^0\), we assume that there exists \((s', \bar{\lambda}', \bar{\gamma}) \in Q, q' \in R_+, \mu'_j \in \Lambda, j \in \Delta\)
and \(b_0, \phi_0, h_1, \phi_1, \eta, \theta, \rho^* \in R^*, \tau^* \in R^{(\Delta)}\), such that

(i) For any \(y'_j \in \bar{Y}(x'), i = 1, \ldots, s', (f - q'h)(x', y')\) is

genralized uniform \((B_j, \rho^*_j)\) - in the x' with respect to \(b_0\) \& \(\phi_1\)

(ii) For any \(u' \in U', j \in J(x'), g(\cdot, u')\) is

genralized uniform \((B_j, \tau^*_j)\) - in the x' with respect to \(b_0\) \& \(\phi_1\)

(iii) \[\sum_{j=1}^{s'} \lambda'_j (f - q'h)_{ij}(x', y'; \eta(x, x')) + \sum_{j=1}^{s'} \mu'_j g_{ij}(x', u'; \eta(x, x')) \geq 0, \forall u' \in U', j \in \Delta;\]

(iv) \[\sum_{j=1}^{s'} \mu'_j g_{ij}(x', u'; \eta(x, x')) = 0, \forall u' \in U', j \in \Delta;\]

(v) \[f(x', y') - q'h(x', y') = 0, i = 1, \ldots, s';\]

(vi) \[a < 0 \Rightarrow \phi_1(a) < 0 \quad \text{and} \quad \phi_1(0) = a, a = 0 \Rightarrow \phi_1(a) = 0, b_0(x, x') > 0, b_0(x, x') \geq 0;\]

(vii) \[\sum_{j=1}^{s'} \lambda'_j \rho^*_j + \sum_{j=1}^{s'} \mu'_j \tau^*_j \geq 0.\]

Then \(x'\) is an optimal solution of (SITP).

**Proof:** Suppose that \(x'\) is not an optimal solution of (SITP). Then there exists \(\bar{x} \in X^0\), such that

\[\sup_{y \in \bar{x}} \frac{f(\bar{x}, y)}{h(x', y)} > \sup_{y \in \bar{x}} \frac{f(x', y)}{h(x', y)}\]

Also

\[\sup_{y \in \bar{x}} \frac{f(x', y)}{h(x', y)} = \sup_{y \in \bar{x}} \frac{f(x', y)}{h(x', y)} = q', \forall y'_j \in \bar{Y}(x'), i = 1, \ldots, s'.\]

Further

\[\frac{f(\bar{x}, y)}{h(x', y)} \leq \sup_{y \in \bar{x}} \frac{f(\bar{x}, y)}{h(x', y)}\]

Thus, we have

\[\frac{f(\bar{x}, y)}{h(x', y)} \leq q^*, i = 1, \ldots, s^* .\]

That is

\[f(\bar{x}, y) - q'h(\bar{x}, y) < 0, i = 1, \ldots, s^*\]

By (v), we obtain

\[f(\bar{x}, y) - q'h(\bar{x}, y) < 0 = f(x', y') - q'h(x', y'), i = 1, \ldots, s'\]

From (vi), we get

\[b_0(\bar{x}, x') \phi_1[(f(\bar{x}, y') - q'h(x', y'))] < 0\]

Then from (i), we have

\[(f - q'h)(x', y'; \eta(x, x')) + \rho^*_j \|\theta(\bar{x}, x')\| < 0\]

Since \(\lambda'_j \geq 0\) and \(\sum_{j=1}^{s'} \lambda'_j = 1\), we have

\[\sum_{j=1}^{s'} \lambda'_j (f - q'h)_{ij}(x', y'; \eta(x, x')) + \sum_{j=1}^{s'} \mu'_j \tau^*_j \|\theta(\bar{x}, x')\| < 0\]

Now from (iii) and (vii), we get

\[\sum_{j=1}^{s'} \mu'_j g_{ij}(x', u'; \eta(x, x')) + \sum_{j=1}^{s'} \mu'_j \tau^*_j \|\theta(\bar{x}, x')\| > 0\]

By (iv), we know that as \(j \in \Delta \setminus J(x')\), \(\mu'_j = 0\), always holds for any \(u' \in U'\).

Hence, as \(j \in J(x')\), we also have

\[\sum_{j=1}^{s'} \mu'_j g_{ij}(x', u'; \eta(x, x')) + \sum_{j=1}^{s'} \mu'_j \tau^*_j \|\theta(\bar{x}, x')\| > 0\]

But as \(j \in J(x')\), we know

\[g(\bar{x}, u') \leq 0 = g(x', u'), u' \in U'\]

From (vi), we get

\[b_0(\bar{x}, x') \phi_1[(g(\bar{x}, u') - g(x', u'))] \leq 0, \forall u' \in U'\]

By (ii), we have

\[g_{ij}(x', u'; \eta(x, x')) + \tau^*_j \|\theta(\bar{x}, x')\| \leq 0, \forall u' \in U', j \in J(x')\]

Since \(\mu'_j \in \Lambda, j \in J(x')\), we get

\[\sum_{j=1}^{s'} \mu'_j g_{ij}(x', u'; \eta(x, x')) + \sum_{j=1}^{s'} \mu'_j \tau^*_j \|\theta(\bar{x}, x')\| \leq 0, \forall u' \in U'\]

Finally, we have a contradiction. Thus the theorem is proved and \(x'\) is an optimal solution of (SITP).

**Theorem 3.2.** Let \(x' \in X^0\) and for any \(x \in X^0\), we assume that there exists \((s', \bar{\lambda}', \bar{\gamma}) \in Q, q' \in R_+, \mu'_j \in \Lambda, j \in \Delta\), \(b_0, \phi_0, b_1, \phi_1, \eta, \theta, \rho^* \in R^*, \tau^* \in R^{(\Delta)}\), such that
(i) For any \( y_i \in \overline{Y}(x'), i = 1, \ldots, s' \), \((f - q'h)(y_i)\) is generalized uniform \((B_{x'}, \rho')\) - pseudoinvex at \( x' \) with respect to \( b_i \) and \( \phi_i \).

(ii) For any \( u' \in U^* \), \( j \in J(x'), g(\cdot, u') \) is generalized uniform \((B_{x'}, r')\) - quasiinvex at \( x' \) with respect to \( b_i \) and \( \phi_i \).

(iii) \( \sum_{i=1}^s \lambda_i^*(f - q'h)^i(y_i, \overline{y}; \eta(x', x')) + \sum_{j=1}^m \mu_j g_j^*(x', u'; \eta(x, x')) \geq 0, \forall u' \in U^*, \ i \in \Delta; \)

(iv) \( \sum_{j=1}^m \mu_j g_j^*(x', u') = 0, \forall u' \in U^*, \ i \in \Delta; \)

(v) \( f(x', \overline{y}) - q'h(x', \overline{y}) = 0, i = 1, \ldots, s'; \)

(vi) \( a < 0 \Rightarrow \phi_i(a) > 0, a \leq 0 \Rightarrow \phi_i(a) \leq 0, \)

\( b_i(x', x*) > 0, \ i \in \Delta; \)

(vii) \( \sum_{i=1}^s \lambda_i^* r_i^* + \sum_{j=1}^m \mu_j r_j^* \geq 0. \)

Then \( x' \) is an optimal solution of (SIFP).

Proof: Suppose that \( x' \) is not an optimal solution of (SIFP). Then there exists \( \tilde{x} \in X^0 \), such that

\[
\sup_{y \in Y} \frac{f(\tilde{x}, y)}{h(\tilde{x}, y)} < \sup_{y \in Y} \frac{f(x', y)}{h(x', y)}
\]

Further

\[
\frac{f(\tilde{x}, \overline{y})}{h(\tilde{x}, \overline{y})} \leq \sup_{y \in Y} \frac{f(\tilde{x}, y)}{h(\tilde{x}, y)}
\]

Thus, we have

\[
\frac{f(\tilde{x}, \overline{y})}{h(\tilde{x}, \overline{y})} \leq q^*, i = 1, \ldots, s'.
\]

Which is equivalent to

\[
f(\tilde{x}, \overline{y}) - q'h(\tilde{x}, \overline{y}) < 0, i = 1, \ldots, s'.
\]

By (v), we get

\[
f(\tilde{x}, \overline{y}) - q'h(\tilde{x}, \overline{y}) = 0 = f(x', \overline{y}) - q'h(x', \overline{y}), \ i = 1, \ldots, s'.
\]

From (vi), we get

\[
\sum_{i=1}^s \lambda_i^* (f - q'h)^i(y_i, \overline{y}; \eta(x', x')) + \sum_{j=1}^m \mu_j g_j^*(x', u'; \eta(x, x')) \leq 0, \forall u' \in U^*, \ i \in \Delta; \)

Then by (i), we have

\[
\sum_{i=1}^s \lambda_i^* (f - q'h)^i(y_i, \overline{y}; \eta(x', x')) + \sum_{j=1}^m \mu_j g_j^*(x', u'; \eta(x, x')) \leq 0, \forall u' \in U^*, \ i \in \Delta; \)

Also, as \( j \in J(x') \), we have

\[
g(\tilde{x}, u') \leq g(x', u'), \forall u' \in U^*.
\]

Then by (vi), we get

\[
\sum_{j=1}^m \mu_j g_j^*(x', u'; \eta(x, x')) = 0, \forall u' \in U^*, \ i \in \Delta.
\]

Also by (v), as \( a \leq 0 \Rightarrow \phi_i(a) \leq 0, \)

\( b_i(x', x*) > 0, \ i \in \Delta; \)

we have

\[
\sum_{j=1}^m \mu_j g_j^*(x', u'; \eta(x, x')) \leq 0, \forall u' \in U^*, \ i \in \Delta.
\]

Now, adding (3) and (4), then from (vii), we have

\[
\sum_{i=1}^s \lambda_i^* (f - q'h)^i(y_i, \overline{y}; \eta(x', x')) + \sum_{j=1}^m \mu_j g_j^*(x', u'; \eta(x, x')) \leq 0, \forall u' \in U^*.
\]

Finally, we have a contradiction. Hence \( x' \) is an optimal solution of (SIFP).

**Theorem 3.3.** Let \( x' \in X^0 \) and for any \( x \in X^0 \), we assume that there exists \((s', \lambda', \gamma) \in Q^*, q \in R^* \), \( \mu \in \Lambda^* = \Lambda \times \Lambda \), \( j \in \Delta \), and \( b_i, \phi_i, b_i, \phi_i, \theta, \tau^* \in R^\prime \), \( r^* \in R^{\Delta^*} \), such that

(i) For any \( \overline{y} \in \overline{Y}(x'), i = 1, \ldots, s' \), \((f - q'h)(\overline{y})\) is strictly generalized uniform \((B_{x'}, \rho')\) - invex at \( x' \) with respect to \( b_i \) and \( \phi_i \)

(ii) For any \( u' \in U^*, j \in J(x'), g(\cdot, u') \) is strictly generalized uniform \((B_{x'}, r')\) - invex at \( x' \) with respect to \( b_i \) and \( \phi_i \)

(iii) \( \sum_{i=1}^s \lambda_i^* (f - q'h)^i(y_i, \overline{y}; \eta(x', x')) + \sum_{j=1}^m \mu_j g_j^*(x', u'; \eta(x, x')) \leq 0, \forall u' \in U^*, \ i \in \Delta; \)

(iv) \( \sum_{j=1}^m \mu_j g_j^*(x', u') = 0, \forall u' \in U^*, \ i \in \Delta; \)

(v) \( f(x', \overline{y}) - q'h(x', \overline{y}) = 0, i = 1, \ldots, s'; \)

(vi) \( a < 0 \Rightarrow \phi_i(a) < 0, a \leq 0 \Rightarrow \phi_i(a) \leq 0, \)

\( b_i(x', x*) > 0, b_i(x', x*) \geq 0; \)
Then \( x^* \) is an optimal solution of (SIFP).

**Theorem 3.4.** Let \( x^* \in X^0 \) and for any \( x \in X^0 \), we assume that there exists \((s^*, \lambda^*, \vec{y}) \in Q, q^* \in R, \mu^*_j \in \Lambda, j \in \Delta \) and \( b_0, \phi_0, b, \phi, \eta, \theta, \rho \in R \) such that:

(i) For any \( \vec{y}, i = 1, \ldots, s \), \((f - q^*h)(\cdot, \vec{y})\) is strictly generalized uniform \((B_{\mu^*}, \rho^*)\) – quasiinvex at \( x^* \) with respect to \( b \) and \( \phi \);

(ii) For any \( u^* \in U^*, j \in J(x^*), g(\cdot, u^*) \) is generalized uniform \((B_{\mu^*}, \rho^*)\) – quasiinvex at \( x^* \) with respect to \( b \) and \( \phi \);

(iii) \( \sum_{j=1}^s \lambda^*_j (f - q^*h)^b_j (x^*, \vec{y}; \eta(x, x^*)) + \sum_{j \in \Lambda} \mu^*_j g^b_j (x^*, u^*; \eta(x, x^*)) \geq 0, \forall u^* \in U^*, j \in \Delta; \)

(iv) \( \sum_{j \in \Lambda} \mu^*_j g_j (x^*, u^*) = 0, \forall u^* \in U^*, j \in \Delta; \)

(v) \( f(x^*, \vec{y}) - q^*h(x^*, \vec{y}) = 0, i = 1, \ldots, s; \)

(vi) \( a < 0 \Rightarrow \phi(a) < 0, a \leq 0 \Rightarrow \phi(a) \leq 0, b(x, x^*) \geq 0, b_0(x, x^*) \geq 0; \)

(vii) \( \sum_{j=1}^s \lambda^*_j \rho^*_j + \sum_{j \in \Lambda} \mu^*_j \tau^*_j \geq 0. \)

Then \( x^* \) is an optimal solution of (SIFP).

### IV. DUALITY THEOREMS

In this section, we formulate a dual problem to the minmax problem (SIVP).

**Theorem 4.1.** (Weak duality) Let \( x \) be a feasible solution of (SIFP) and \((z, \mu, q, s, \lambda, \vec{y})\) be a feasible solution of (SIFD). For \( \lambda_i \geq 0, i = 1, 2, \ldots, s \) with \( \sum_{j=1}^s \lambda_j = 1, \mu_j \in \Lambda, j \in \Lambda \) and \( q \in R^s \), assume there exists \( b_0, \phi_0, b, \phi, \eta, \theta, \rho, \rho' \in R \) and \( r \in R^{s^*} \), such that:

(i) For all \( j, i = 1, \ldots, s \), \((f - q^*h)(\cdot, \vec{y})\) is generalized uniform \((B_{\mu^*}, \rho^*)\) – invex with respect to \( b_0 \) and \( \phi \) at \( z \);

(ii) For all \( \vec{y}, i = 1, \ldots, s \), \((f - q^*h)(\cdot, \vec{y})\) is generalized uniform \((B_{\mu^*}, \rho^*)\) – invex with respect to \( b \) and \( \phi \) at \( z \);

(iii) \( a < 0 \Rightarrow \phi(a) < 0, a \leq 0 \Rightarrow \phi(a) \leq 0, b(x, x^*) \geq 0, b_0(x, x^*) \geq 0 \);

(iv) \( \sum_{j=1}^s \lambda^*_j \rho^*_j + \sum_{j \in \Lambda} \mu^*_j \tau^*_j \geq 0. \)

Then \( x^* \) is an optimal solution of (SIFP).
According to the constraint condition (5), for all 
\( u^i \in U^* \), we let 
\( \mu_j = 0 \), as \( j \in \Delta \setminus J(z) \).

Thus, we have

\[
\sum_{j \in \Delta} \mu_j g^u_n(\nu, u^i; \eta(\nu, z)) + \sum_{j \in \Delta} \mu_j \lambda_j \theta(\nu, z) \leq 0
\]

We have a contradiction. Therefore, we conclude that

\[
\sup_{y \in Y} \frac{f(x, y)}{h(x, y)} \geq q.
\]

Hence, the proof of the theorem is complete.

Theorem 4.2. (Weak duality) Let \( x \) be a feasible solution of (SIFP) and \( (z, \mu, q, s, \lambda, \vec{y}) \) be a feasible solution of (SIFD). For \( \lambda_j \geq 0, i=1, 2, \ldots, s \) with

\[
\sum_{i=1}^s \lambda_i = 1, \mu_j \in \Lambda, \; j \in \Delta \text{ and } q \in R_+,
\]

assume there exists \( b_0, \phi_0, \bar{b}_i, \phi_i, \theta, \rho, \rho_0 \in R^* \) and \( \tau \in R^{(s)} \), such that

(i) For all \( \nu, i=1, \ldots, s, (f - qh)(\nu, y) \) is generalized uniform \( (B_{ii}, \rho_i) \) - pseudoinvx with respect to \( b_0 \) and \( \phi_i \) at \( z \);

(ii) For all \( u^i \in U^*, j \in J(z), g_i(u^i) \) is generalized uniform \( (B_{ij}, \tau_j) \) - quasiinvex with respect to \( b_i \) and \( \phi_i \) at \( z \);

(iii) \( a < 0 \Rightarrow \phi_i(a) < 0; a \leq 0 \Rightarrow \phi_i(a) \leq 0, b_i(x, z) > 0, b_i(x, z) \geq 0 \)

(iv) \( \sum_{i=1}^s \lambda_i \rho_i + \sum_{j \in J} \mu_j \tau_j \geq 0 \).

Then

\[
\sup_{y \in Y} \frac{f(x, y)}{h(x, y)} \geq q.
\]

Proof: Suppose contrary that

\[
\sup_{y \in Y} \frac{f(x, y)}{h(x, y)} < q.
\]

Then, we have

\[
f(x, y) - qh(x, y) < 0, \forall y \in Y.
\]

Using the constraint condition (5), it follows that for all \( \nu, i=1, \ldots, s \), we get

\[
f(\nu, \nu) - qh(\nu, \nu) < 0 \leq f(\nu, \nu) - qh(\nu, \nu)
\]

By (iii), we get

\[
b_i(\nu) \phi_i((f(x, y) - qh(x, y)) - (f(\nu, \nu) - qh(\nu, \nu))) < 0
\]

Then (i) yields

\[
(f - qh)(\nu, \nu; \eta(\nu, z)) + \rho_i \theta(\nu, z) < 0
\]

Since \( \lambda_i \geq 0 \) and \( \sum_{i=1}^s \lambda_i = 1 \), it follows that

\[
\sum_{i=1}^s \lambda_i (f - qh)(\nu, \nu; \eta(\nu, z)) + \sum_{j \in J} \mu_j \tau_j \theta(\nu, z) < 0
\]

Hence by the constraint condition (5) and (iv), we get

\[
\sum_{j \in \Delta} \lambda_j g^u_n(\nu, u^i; \eta(\nu, z)) + \sum_{j \in \Delta} \mu_j \tau_j \theta(\nu, z) \geq 0
\]

For \( j \in J(z) \), using the constraint condition (5), then we get

\[
g(\nu, u^i) \leq 0 \leq g(\nu, u^i), \forall u^i \in U^*
\]

From (iii), we get

\[
b_i(\nu) \phi_i((g(\nu, u^i) - g(\nu, u^i)) \leq 0, \forall u^i \in U^*, j \in J(z)
\]

By (ii), we have

\[
g^u_n(\nu, u^i; \eta(\nu, z)) + \tau_j \theta(\nu, z) \leq 0
\]

Since \( \lambda_j \in \Lambda, i \in \Delta \text{ and } q \in R_+ \), we get

\[
\sum_{i=1}^s \lambda_i \rho_i + \sum_{j \in J} \mu_j \tau_j \theta(\nu, z) \leq 0
\]

Adding (6) and (7), then from (iv), we obtain

\[
\sum_{i=1}^s \lambda_i (f - qh)(\nu, \nu; \eta(\nu, z)) + \sum_{j \in \Delta} \mu_j \tau_j \theta(\nu, z) \leq 0
\]

We have a contradiction. Hence, the proof of the theorem is complete.

Similarly, we can derive the following theorems.

Theorem 4.3. (Weak duality) Let \( x \) be a feasible solution of (SIFP) and \( (z, \mu, q, s, \lambda, \vec{y}) \) be a feasible solution of (SIFD). For \( \lambda_j \geq 0, i=1, 2, \ldots, s \) with

\[
\sum_{i=1}^s \lambda_i = 1, \mu_j \in \Lambda, \; j \in \Delta \text{ and } q \in R_+,
\]

assume there exists \( b_0, \phi_0, b_i, \phi_i, \theta, \rho, \rho_0 \in R^* \) and \( \tau \in R^{(s)} \), such that

(i) For all \( \nu, i=1, \ldots, s, (f - qh)(\nu, \nu) \) is generalized uniform \( (B_{ii}, \rho_i) \) - quasiinvex with respect to \( b_0 \) and \( \phi_i \) at \( z \);

(ii) For all \( u^i \in U^*, j \in J(z), g_i(u^i) \) is strictly generalized uniform \( (B_{ij}, \tau_j) \) - pseudoinvx with respect to \( b_i \) and \( \phi_i \) at \( z \);

(iii) \( a < 0 \Rightarrow \phi_i(a) < 0; a \leq 0 \Rightarrow \phi_i(a) \leq 0, b_i(x, z) > 0, b_i(x, z) \geq 0 \)

(iv) \( \sum_{i=1}^s \lambda_i \rho_i + \sum_{j \in J} \mu_j \tau_j \geq 0 \).

Then

\[
\sum_{i=1}^s \lambda_i (f - qh)(\nu, \nu; \eta(\nu, z)) + \sum_{j \in \Delta} \mu_j \tau_j \theta(\nu, z) \leq 0
\]
Theorem 4.4. (Strong duality) Let \( x^* \) be an optimal solution of problem (SIFP). Assume that \( x^* \) satisfies \( K-T \) constraint qualification for (SIFP). Then there exists \( (s^*, \lambda^*, \bar{y}) \in Q \) and \( (x^*, \mu^*, q^*) \in D(s^*, \lambda^*, \bar{y}) \), such that \( (x^*, \mu^*, q^*, s^*, \lambda^*, \bar{y}) \) is a feasible solution of (SIFD). If the hypothesis of theorem 4.1 is also satisfied, then \( (x^*, \mu^*, q^*, s^*, \lambda^*, \bar{y}) \) is an optimal of (SIFD), furthermore, the two problems (SIFP) and (SIFD) have the same optimal value.

Proof: Since \( x^* \) is an optimal solution of problem (SIFP), and \( x^* \) satisfies \( K-T \) constraint qualification for (SIFP), then there exists \( (s^*, \lambda^*, \bar{y}) \in Q \), \( q^* \in R \), and \( \mu^*_j \in \Lambda \), \( j \in \Delta \), such that the relations (3.1)- (3.4) hold. Therefore, \( (x^*, \mu^*, q^*, s^*, \lambda^*, \bar{y}) \) is a feasible solution of (SIFD), and we have

\[
q^* = \frac{f(x^*, \bar{y})}{h(x^*, \bar{y})}
\]

The optimality of this feasible solution for (SIFD) follows from theorem 4.1. It is clear that the two problems have the same optimal values.

Remark 4.1. the result of strong duality under the hypothesis of theorem 4.2 (or 4.3) follows with the same lines as the argument given in theorem 4.1.

Theorem 4.5. (Strict reverse duality) Let \( x^* \) and \( (\bar{z}, \mu, q, s, \lambda, \bar{y}) \) be an optimal solution of problem (SIFP) and (SIFD), respectively. Assume that \( x^* \) satisfies \( K-T \) constraint qualification for (SIFP). And for \( \sum_{i=1}^{n} \bar{z}_j = 1 \),

\[
\bar{\mu}_j \in \Lambda \), \( j \in \Delta \) and \( \bar{q} \in R \), there exists \( b_i, \phi_i, \rho_i, \phi_i, \eta_i, \theta_i \), \( \bar{\rho}_i \in R^+, \bar{\tau}_i \in R^{\Delta} \), the following conditions are fulfilled:

(i) For all \( \bar{y}, f(\bar{y})-\bar{q}h(\bar{y}) \) is strictly generalized uniform \((B_{\mu}, \bar{\tau})\) - invex with respect to \( b_i \) and \( \phi_i \) at \( \bar{z} \);

(ii) For all \( u^j \in U^j, j \in J(\bar{z}), g(\bar{\rho}, u^j) \) is strictly generalized uniform \((B_{\mu}, \bar{\tau})\) - invex with respect to \( b_i \) and \( \phi_i \) at \( \bar{z} \);

(iii) \( a \leq 0 \Rightarrow \phi_i(a) \leq 0 \); \( a \leq 0 \Rightarrow \phi_i(a) \leq 0 \),

\[
b_i(x^*, \bar{z}) > 0, h_i(x^*, \bar{z}) \geq 0
\]

(iv) \( \sum_{j=1}^{n} \bar{z}_j \rho_j + \sum_{j=1}^{n} \phi_j \bar{\tau}_j \geq 0 \).

Then, \( x^* = \bar{z} \); that is, \( \bar{z} \) is also an optimal solution of (SIFP) and \( \bar{q} = \sup_{y \in D} f(z, y) / h(z, y) \).

Proof: Suppose on the contrary that \( x^* \neq \bar{z} \). From Theorem 4.4, we know that there exists \( (s^*, \lambda^*, \bar{y}) \in Q \) and \( (x^*, \mu^*, q^*) \in D(s^*, \lambda^*, \bar{y}) \), such that \( (x^*, \mu^*, q^*, s^*, \lambda^*, \bar{y}) \) is an optimal solution of (SIFD) with the optimal value.

\[
\bar{q} = \sup_{y \in D} \frac{f(x^*, \bar{y})}{h(x^*, \bar{y})}
\]

Now using the conditions (i)-(iv) and like the proof of theorem 4.1 (by replacing \( x \) by \( x^* \) and \( (\bar{z}, \mu, q, s, \lambda, \bar{y}) \) by \( (x^*, \mu^*, q^*, s^*, \lambda^*, \bar{y}) \), we arrive at the strict inequality.

\[
\sup_{y \in D} \frac{f(x^*, \bar{y})}{h(x^*, \bar{y})} > \bar{q}
\]

This contradicts the fact

\[
\sup_{y \in D} \frac{f(x^*, \bar{y})}{h(x^*, \bar{y})} = \bar{q}
\]

Therefore, we conclude that \( x^* = \bar{z} \). Hence, the proof of the theorem is complete.

V. CONCLUSION

Throughout this paper, we have defined a new generalized convex function, extending many well-known classes of generalized convex functions. Furthermore, we have achieved some sufficient optimality conditions for a class of multiobjective semi-infinite programming problem. Finally, we have formulated the multiobjective dual problem and proved the results concerning weak and strong duality between the primal (SIVP) and the dual (SIVD), there should be further opportunities for exploiting this structure of the semi-infinite programming problem.

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REFERENCES


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