An Upper Bound to the Capacity of Discrete Time Gaussian Channel with Feedback, II

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Abstract

We give an upper bound to the finite block length capacity of discrete time nonstationary Gaussian channel with feedback. With the aid of minimization of a quadratic form, it is proved that the feedback capacity $C_{n,FB}(P)$ and the nonfeedback capacity $C_n(P)$ satisfies $C_n(P) \leq C_{n,FB}(P) \leq C_n(P^*)$, where $P^*$ is concretely given.

Key words: Gaussian Channel, Capacity, Feedback

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I. INTRODUCTION

The following model for the discrete time Gaussian channel with feedback is considered:

\[ Y_n = S_n + Z_n, \quad n = 1, 2, \ldots \]

where \( Z = \{Z_n; n = 1, 2, \ldots \} \) is a non-degenerate, zero mean Gaussian process representing the noise and \( S = \{S_n; n = 1, 2, \ldots \} \) and \( Y = \{Y_n; n = 1, 2, \ldots \} \) are stochastic processes representing input signals and output signals, respectively. The channel is with noiseless feedback, so \( S_n \) is a function of a message to be transmitted and the output signals \( Y_1, \ldots, Y_{n-1} \). For a code of rate \( R \) and length \( n \), with code words \( x^n(W, Y^{n-1}) \), \( W \in \{1, \ldots, 2^{nR}\} \), and a decoding function \( g_n : \mathbb{R}^n \to \{1, 2, \ldots, 2^{nR}\} \), the probability of error is

\[ P_e^{(n)} = Pr\{g_n(Y^n) \neq W; Y^n = x^n(W, Y^{n-1}) + Z^n\}, \]

where \( W \) is uniformly distributed over \( \{1, 2, \ldots, 2^{nR}\} \) and independent of \( Z^n \). The signal is subject to an expected power constraint

\[ \frac{1}{n} \sum_{i=1}^{n} E[S_i^2] \leq P, \]

and the feedback is causal, i.e., \( S_i \) is dependent of \( Z_1, \ldots, Z_{i-1} \) for \( i = 1, 2, \ldots, n \). Similarly, when there is no feedback, \( S_i \) is independent of \( Z^n \). We introduce a finite length block capacity in the following:

\[ C_{n,FB}(P) = \max \frac{1}{2n} \log \frac{(I + B)R_Z^{(n)}(I + B)^t + R_V^{(n)}}{|R_Z^{(n)}|}, \]

where the maximum is on \( B \) strictly lower triangular and \( R_V^{(n)} \) nonnegative definite, such that

\[ Tr[BR_Z^{(n)}B^t + R_V^{(n)}] \leq nP. \tag{1} \]

Similarly, let \( C_n(P) \) be the maximal value when \( B = 0 \), i.e., when there is no feedback. Under these conditions, Cover and Pombra proved the following.

**Theorem 1 (Cover and Pombra [1])** For every \( \epsilon > 0 \) there exist codes, with block length \( n \) and \( 2^{n(C_{n,FB}(P) - \epsilon)} \) codewords, \( n = 1, 2, \ldots \), such that \( P_e^{(n)} \to 0 \), as \( n \to \infty \). Conversely, for every \( \epsilon > 0 \) and any sequence of codes with \( 2^{n(C_{n,FB}(P) + \epsilon)} \) codewords and block length \( n \), \( P_e^{(n)} \) is bounded away from zero for all \( n \). The same theorem holds in the special case without feedback upon replacing \( C_{n,FB}(P) \) by \( C_n(P) \).

When block length \( n \) is fixed, we are interested in the relationship between \( C_{n,FB}(P) \) and \( C_n(P) \). \( C_n(P) \) was obtained in the following.

**Proposition 1 (Gallager [4])**

\[ C_n(P) = \frac{1}{2n} \sum_{i=1}^{k} \log \frac{nP + r_1 + \cdots + r_k}{kr_i}, \]

where \( 0 < r_1 \leq r_2 \leq \cdots \leq r_n \) are eigenvalues of \( R_Z^{(n)} \) and \( k(\leq n) \) is the largest integer satisfying \( nP + r_1 + \cdots + r_k > kr_k \).

Recently, Ihara and Yanagi [6] and Yanagi [11] obtained the necessary and sufficient conditions under which the finite block length capacity is increased by feedback. On the other hand, we are also interested in the upper bounds to \( C_{n,FB}(P) \). The following six propositions were already obtained.
Proposition 2 (Ebert [3], Pinsker [8], Cover and Pombra [1])

\[ C_n(P) \leq C_{n,FB}(P) \leq 2C_n(P). \]

Proposition 3 (Cover and Pombra [1])

\[ C_n(P) \leq C_{n,FB}(P) \leq C_n(P) + \frac{1}{2}. \]

Proposition 4 (Dembo [2]; Claim 1, Yanagi [11]; Lemma 2)

\[ C_n(P) \leq C_{n,FB}(P) \leq \frac{1}{2} \log(1 + \frac{P}{\lambda_{\min}(R_Z^{(n)})}), \]

where \( \lambda_{\min}(R_Z^{(n)}) \) is the smallest eigenvalue of \( R_Z^{(n)}. \)

Proposition 5 (Dembo [2]; Claim 3)

\[ C_n(P) \leq C_{n,FB}(P) \leq \log(\sqrt{P} + \sqrt{\frac{1}{n} \text{Tr}[R_Z^{(n)}]}) - \frac{1}{2n} \log|R_Z^{(n)}|, \]

where \( |R_Z^{(n)}| \) is the determinant of \( R_Z^{(n)}. \)

Proposition 6 (Dembo [2]; Proposition 3) Let \( \lambda_1, \ldots, \lambda_n \) be the eigenvalues of \( R_Z^{(n)} \), and for \( 0 < \mu < \infty \), let

\[ P(\mu) = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{4} (\sqrt{\mu + \lambda_i} - \sqrt{\lambda_i})^2. \]

Then

\[ C_n(P(\mu)) \leq C_{n,FB}(P(\mu)) \leq \frac{1}{n} \sum_{i=1}^{n} \log \frac{1}{2} (1 + \sqrt{1 + \frac{\mu}{\lambda_i}}). \]

Proposition 7 (Yanagi [10])

\[ C_n(P) \leq C_{n,FB}(P) \leq C_n(P_1), \]

where \( P_1 = \sup\{ \| (I + B)^{-1} \|^2 (P - \frac{1}{n} \text{Tr}[BR_Z^{(n)} B^t]); B \text{ are strictly lower triangle matrices} \}. \)

In this paper we obtain a tight upper bound to \( C_{n,FB}(P) \), which is represented by \( C_n(P^*) \) for some \( P^* (> P) \). In section II, we consider a real quadratic form and give several optimization theorems. In section III, we give an upper bound to \( C_{n,FB}(P) \). Finally in section IV, we give the relationship between all these bounds for \( n = 2 \) or \( 3 \) by comparing the new bound of Theorem 5 and other bounds. Furthermore we state the upper bounds for stationary and ergodic noise processes.

II. Minimization of a Quadratic Form
We consider the following real quadratic form:

$$Tr[Q^tR^{-1}Q - 2WQ] = \sum_{k=1}^{n} \sum_{i=k}^{n} \sum_{j=k}^{n} r_{ij}q_{ik}q_{jk} - 2 \sum_{k=1}^{n} \sum_{i=k}^{n} w_{ki}q_{ik},$$

(2)

where $R$ is a positive definite symmetric matrix and $R^{-1} = \{r_{ij}; 1 \leq i, j \leq n, q_{ij} = 0(i < j)\}$ is a lower triangle matrix, $Q = \{q_{ij}; 1 \leq i, j \leq n\}$ is an orthogonal matrix. We want to minimize (2) under any lower triangle matrices $Q$. Let $R^{-1}(k, \ldots, n)$ be the submatrix of $R^{-1}$ generated by $k, \ldots, n$ rows and columns and let $|R^{-1}(k, \ldots, n)|$ be the determinant of $R^{-1}(k, \ldots, n)$. And also let $R_{11}(k)$ be the submatrix of $R$ generated by $1, \ldots, k - 1$ rows and $1, \ldots, k - 1$ columns, $R_{12}(k)$ be the submatrix of $R$ generated by $1, \ldots, k - 1$ rows and $k, \ldots, n$ columns and $R_{21}(k) = R_{12}(k)^t$. Now we obtain the following.

**Theorem 2**  The minimal value of (2) is given by

$$-\sum_{k=1}^{n} |[R^{-1}(k, \ldots, n)]|^{-1} \begin{pmatrix} w_{kk} \\
  w_{kk+1} \\
  \vdots \\
  w_{kn} \end{pmatrix} \begin{pmatrix} w_{kk} \\
  w_{kk+1} \\
  \vdots \\
  w_{kn} \end{pmatrix},$$

(3)

where $(\cdot, \cdot)$ is the inner product in $\mathbb{R}^n$.

**Proof.**  Let $k \leq \ell \leq n$. We have

$$\sum_{i=k}^{n} \sum_{j=k}^{n} r_{ij}q_{ik}q_{jk} - 2 \sum_{i=k}^{n} w_{ki}q_{ik}$$

$$= \sum_{i \neq \ell} \sum_{j \neq \ell} r_{ij}q_{ik}q_{jk} + 2 \sum_{i \neq \ell} r_{i\ell}q_{ik}q_{\ell k} + r_{\ell k}^2q_{\ell k} - 2 \sum_{i \neq \ell} w_{ki}q_{ik} - 2 w_{k\ell}q_{\ell k}.$$  

(4)

We partially differentiate (4) at $q_{\ell k}$ and we get

$$\sum_{i=k}^{n} r_{ij}q_{ik} = w_{k\ell}.$$

Hence

$$\begin{pmatrix} r_{kk} & r_{kk+1} & \cdots & r_{kn} \\
  r_{k+1k} & r_{k+1k+1} & \cdots & r_{k+n} \\
  \vdots & \vdots & \ddots & \vdots \\
  r_{nk} & r_{nk+1} & \cdots & r_{nn} \end{pmatrix} \begin{pmatrix} q_{kk} \\
  q_{k+1k} \\
  \vdots \\
  q_{nk} \end{pmatrix} = \begin{pmatrix} w_{kk} \\
  w_{kk+1} \\
  \vdots \\
  w_{kn} \end{pmatrix}.$$  

Since $R^{-1}(k, \ldots, n)$ is regular, we have

$$\begin{pmatrix} q_{kk} \\
  q_{k+1k} \\
  \vdots \\
  q_{nk} \end{pmatrix} = [R^{-1}(k, \ldots, n)]^{-1} \begin{pmatrix} w_{kk} \\
  w_{kk+1} \\
  \vdots \\
  w_{kn} \end{pmatrix}.$$  

Then we obtain

$$q_{ik} = \frac{1}{|R^{-1}(k, \ldots, n)|} \sum_{j=k}^{n} r_{ij}(k, \ldots, n)w_{kj},$$
where \( \tilde{r}_{ij}(k, \ldots, n) \) is the cofactor of \((i, j)\) component in \( R^{-1}(k, \ldots, n) \). Then the minimal value can be represented by

\[
\sum_{i=k}^{n} \sum_{j=k}^{n} r_{ij} \tilde{r}_{i\ell}(k, \ldots, n) w_{k\ell} \sum_{m=k}^{n} \tilde{r}_{jm}(k, \ldots, n) w_{km} - \frac{2}{|R^{-1}(k, \ldots, n)|} \sum_{i=k}^{n} w_{ki} \sum_{m=k}^{n} \tilde{r}_{i\ell}(k, \ldots, n) w_{k\ell}.
\]

We remark that

\[
\sum_{i=k}^{n} \sum_{j=k}^{n} r_{ij} \tilde{r}_{i\ell}(k, \ldots, n) w_{k\ell} \sum_{m=k}^{n} \tilde{r}_{jm}(k, \ldots, n) w_{km} = \sum_{j=k}^{n} \sum_{\ell=k}^{n} \delta_{j\ell} |R^{-1}(k, \ldots, n)| w_{k\ell} \sum_{m=k}^{n} \tilde{r}_{jm}(k, \ldots, n) w_{km} = \sum_{j=k}^{n} |R^{-1}(k, \ldots, n)| w_{kj} \sum_{m=k}^{n} \tilde{r}_{jm}(k, \ldots, n) w_{km}.
\]

Hence the minimal value of (4) is given by

\[
-\frac{1}{|R^{-1}(k, \ldots, n)|} \sum_{i=k}^{n} w_{ki} \sum_{\ell=k}^{n} \tilde{r}_{i\ell}(k, \ldots, n) w_{k\ell} = -\langle [R^{-1}(k, \ldots, n)]^{-1} \left( \begin{array}{ccc} w_{kk} & w_{kk+1} & \vdots \\ w_{kk+1} & \ddots & \vdots \\ \vdots & \vdots & w_{kn} \end{array} \right) \right),
\]

(5)

Thus we obtain the result. \( \square \)

Now we want to estimate the maximal value of (3) under any orthogonal matrices \( W \). We obtain the following:

**Theorem 3** Under any orthogonal matrices \( W \), the upper bound of (3) is given by

\[
-Tr[R] + \sum_{k=2}^{n} \lambda_{\max} \left( \begin{array}{cc} R_{11}(k) & R_{12}(k) \\ R_{21}(k) & R_{22}(k) - R_{11}(k)^{-1} R_{12}(k) \end{array} \right),
\]

where \( \lambda_{\max}(A) \) is the maximal eigenvalue of \( A \).

**Proof.** Let \( R_{22}(k) \) be the submatrix of \( R \) generated by \( k, \ldots, n \) rows and \( k, \ldots, n \) columns and let \( w_k^* = \left( \begin{array}{c} w_{k1} \\ w_{k2} \\ \vdots \\ w_{kn} \end{array} \right) \). By (3) we have to maximize

\[
-\sum_{k=1}^{n} \langle [R^{-1}(k, \ldots, n)]^{-1} \left( \begin{array}{c} w_{kk} \\ w_{kk+1} \\ \vdots \\ w_{kn} \end{array} \right) \right),
\]

under any orthogonal matrices \( W \). Let \( 2 \leq k \leq n \). Since

\[
[R^{-1}(k, \ldots, n)]^{-1} = R_{22}(k) - R_{21}(k) R_{11}(k)^{-1} R_{12}(k),
\]

under any orthogonal matrices \( W \). Let \( 2 \leq k \leq n \). Since

\[
[R^{-1}(k, \ldots, n)]^{-1} = R_{22}(k) - R_{21}(k) R_{11}(k)^{-1} R_{12}(k),
\]

under any orthogonal matrices \( W \). Let \( 2 \leq k \leq n \). Since

\[
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\]

under any orthogonal matrices \( W \). Let \( 2 \leq k \leq n \). Since

\[
[R^{-1}(k, \ldots, n)]^{-1} = R_{22}(k) - R_{21}(k) R_{11}(k)^{-1} R_{12}(k),
\]

under any orthogonal matrices \( W \). Let \( 2 \leq k \leq n \). Since
we get
\[
-\langle [R^{-1}(k, \ldots, n)]^{-1} \begin{pmatrix} w_{kk} \\
w_{kk+1} \\
\vdots \\
w_{kn} \\
\end{pmatrix}, \begin{pmatrix} w_{kk} \\
w_{kk+1} \\
\vdots \\
w_{kn} \\
\end{pmatrix} \rangle
\]
\[
= -\langle R_{22}(k) \begin{pmatrix} w_{kk} \\
w_{kk+1} \\
\vdots \\
w_{kn} \\
\end{pmatrix}, \begin{pmatrix} w_{kk} \\
w_{kk+1} \\
\vdots \\
w_{kn} \\
\end{pmatrix} \rangle + \langle R_{21}(k)R_{11}(k)^{-1}R_{12}(k) \begin{pmatrix} w_{kk} \\
w_{kk+1} \\
\vdots \\
w_{kn} \\
\end{pmatrix}, \begin{pmatrix} w_{kk} \\
w_{kk+1} \\
\vdots \\
w_{kn} \\
\end{pmatrix} \rangle
\]
\[
= -\langle \begin{pmatrix} R_{11}(k) \\
R_{21}(k) \\
R_{22}(k) \\
\end{pmatrix} w^*_k, w^*_k \rangle + \langle \begin{pmatrix} R_{11}(k) \\
R_{21}(k) \\
R_{22}(k)R_{11}(k)^{-1}R_{12}(k) \\
\end{pmatrix} w^*_k, w^*_k \rangle.
\]

Then we have
\[
-\sum_{k=1}^{n} \langle [R^{-1}(k, \ldots, n)]^{-1} \begin{pmatrix} w_{kk} \\
w_{kk+1} \\
\vdots \\
w_{kn} \\
\end{pmatrix}, \begin{pmatrix} w_{kk} \\
w_{kk+1} \\
\vdots \\
w_{kn} \\
\end{pmatrix} \rangle
\]
\[
= -\langle Rw^*_1, w^*_1 \rangle - \sum_{k=2}^{n} \langle Rw^*_k, w^*_k \rangle - \langle \begin{pmatrix} R_{11}(k) \\
R_{21}(k) \\
R_{22}(k)R_{11}(k)^{-1}R_{12}(k) \\
\end{pmatrix} w^*_k, w^*_k \rangle
\]
\[
= -\sum_{k=1}^{n} \langle Rw^*_k, w^*_k \rangle + \sum_{k=2}^{n} \langle \begin{pmatrix} R_{11}(k) \\
R_{21}(k) \\
R_{22}(k)R_{11}(k)^{-1}R_{12}(k) \\
\end{pmatrix} w^*_k, w^*_k \rangle
\]
\[
= -\text{Tr}[R] + \sum_{k=2}^{n} \langle \begin{pmatrix} R_{11}(k) \\
R_{21}(k) \\
R_{22}(k)R_{11}(k)^{-1}R_{12}(k) \\
\end{pmatrix} w^*_k, w^*_k \rangle.
\]

Since \( \lambda_{\max}(A) = \max \{ \langle Ax, x \rangle ; \| x \| = 1 \} \), we have
\[
\langle \begin{pmatrix} R_{11}(k) \\
R_{21}(k) \\
R_{22}(k)R_{11}(k)^{-1}R_{12}(k) \\
\end{pmatrix} w^*_k, w^*_k \rangle \leq \lambda_{\max}(\begin{pmatrix} R_{11}(k) \\
R_{21}(k) \\
R_{22}(k)R_{11}(k)^{-1}R_{12}(k) \\
\end{pmatrix}).
\]

Then we obtain the result. \( \square \)

We also consider the real quadratic form given by replacing \( W \) by \( I \) (= identity). That is
\[
\text{Tr}[Q^t R^{-1} Q - 2Q].
\]

(6)

Then we want to minimize (6) under any lower triangle matrices \( Q \). By Theorem 2 we obtain the following.

**Theorem 4** The minimum value of (6) is given by
\[
-|R_{11}(2)| - \frac{|R_{11}(3)|}{|R_{11}(2)|} - \frac{|R_{11}(4)|}{|R_{11}(3)|} - \cdots - \frac{|R|}{|R_{11}(n)|}.
\]

In order to prove Theorem 4 we need the following lemma.
Lemma 1. For $2 \leq k \leq n$

$$|R^{-1}(k, \ldots, n)| = \frac{|R_{11}(k)|}{|R|}$$

Proof. Let

$$R = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

where $A_{11}$ is $(k-1) \times (k-1)$ matrix, $A_{12}$ is $(k-1) \times (n-k+1)$ matrix, $A_{21}$ is $(n-k+1) \times (k-1)$ matrix and $A_{22}$ is $(n-k+1) \times (n-k+1)$ matrix. And let

$$R^{-1} = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

be the inverse matrix of $R$, where $B_{11}$ is $(k-1) \times (k-1)$ matrix, $B_{12}$ is $(k-1) \times (n-k+1)$ matrix, $B_{21}$ is $(n-k+1) \times (k-1)$ matrix and $B_{22}$ is $(n-k+1) \times (n-k+1)$ matrix. Since $R$ is regular, $A_{11}$ is also regular. Since

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} I & -A_{11}^{-1}A_{12} \\ 0 & I \end{pmatrix} = \begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} - A_{21}A_{11}^{-1}A_{12} \end{pmatrix},$$

we have

$$|R| = |A_{11}| |A_{22} - A_{21}A_{11}^{-1}A_{12}|.$$ (7)

On the other hand, since

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} = I,$$

we have the following relations:

$$A_{11}B_{12} + A_{12}B_{22} = 0$$ (8)
$$A_{21}B_{12} + A_{22}B_{22} = I.$$ (9)

By (8), we get

$$B_{12} = -A_{11}^{-1}A_{12}B_{22}. \quad \text{(10)}$$

It follows from (9) and (10) that

$$-A_{21}A_{11}^{-1}A_{12}B_{22} + A_{22}B_{22} = I$$

and then

$$(A_{22} - A_{21}A_{11}^{-1}A_{12})B_{22} = I.$$ (11)

Then

$$|A_{22} - A_{21}A_{11}^{-1}A_{12}| |B_{22}| = 1.$$ (12)

By (7) we get

$$\frac{|R|}{|A_{11}|} |B_{22}| = 1.$$ (13)

Then we have

$$|B_{22}| = \frac{|A_{11}|}{|R|}.$$ (14)

That is

$$|R^{-1}(k, \ldots, n)| = \frac{|R_{11}(k)|}{|R|}.$$ (15)

$\square$
Proof of Theorem 4. We put \( w_{k\ell} = \delta_{k\ell}, \ell = k, \ldots, n \) in (5). Then the minimal value of (6) is given by

\[
- \sum_{k=1}^{n} \frac{\tilde{r}_{kk}(k, \ldots, n)}{|R^{-1}(k, \ldots, n)|},
\]

where \( \tilde{r}_{kk}(k, \ldots, n) \) is the cofactor of \((k, k)\) component in \(R^{-1}(k, \ldots, n)\). By the relation of \( \tilde{r}_{kk}(k, \ldots, n) = |R^{-1}(k+1, \ldots, n)| \) and by Lemma 1, we have as the minimal value of (6)

\[
- \sum_{k=1}^{n} \frac{|R^{-1}(k+1, \ldots, n)|}{|R^{-1}(k, \ldots, n)|} = -|R_{11}(2)| - \frac{|R_{11}(3)|}{|R_{11}(2)|} - \cdots - \frac{|R|}{|R_{11}(n)|}.
\]

Then we complete the proof. \( \square \)

III. Tight Upper Bound to \( C_{n,FB}(P) \)

Let \( R_Z^{11}(k) \) be the submatrix of \( R_Z^{(n)} \) generated by \( 1, \ldots, k - 1 \) rows and \( 1, \ldots, k - 1 \) columns, \( R_Z^{12}(k) \) be the submatrix of \( R_Z^{(n)} \) generated by \( 1, \ldots, k - 1 \) rows and \( k, \ldots, n \) columns and \( R_Z^{21}(k) = R_Z^{12}(k)^t \).

Lemma 2 For any matrices \( A, C \),

\[
Tr[AC^t] \leq Tr[AA^t]^{1/2} Tr[CC^t]^{1/2}.
\]

Proof. See [2]. \( \square \)

Lemma 3 Under the condition (1), an upper bound of

\[
\frac{1}{n} \left( Tr[R_{Z'}^{(n)}] + BR_Z^{(n)} B^t + B R_Z^{(n)} + R_Z^{(n)} B^t \right)
\]

is given by

\[
P + 2 \sqrt{n \left( Tr[R_Z^{(n)}] - |R_{Z'}^{(1)}(2)| - \frac{|R_{Z'}^{(1)}(3)|}{|R_{Z'}^{(2)}(2)|} - \cdots - \frac{|R_{Z'}^{(n)}|}{|R_{Z'}^{(n)}|} \right)}.
\]

Proof. To avoid the complexity we denote \( R_Z^{(n)} \) by \( R_Z \). We apply Lemma 2 for \( A = BR_Z^{1/2}, C = R_Z^{1/2} - Q^t R_Z^{-1/2} \), where \( Q \) is a lower triangular. And we remark that \( Tr[BQ] = 0 \). Then we obtain

\[
Tr[BR_Z] = Tr[BR_Z] - Tr[BQ]
= Tr[BR_Z^{1/2}(R_Z^{1/2} - R_Z^{-1/2} Q^t)]
= Tr[BR_Z^{1/2}(R_Z^{1/2} - Q^t R_Z^{-1/2})]
\leq Tr[BR_Z] B^{t 1/2} Tr[(R_Z^{1/2} - Q^t R_Z^{-1/2})(R_Z^{1/2} - Q^t R_Z^{-1/2})^t]^{1/2}
= Tr[BR_Z] B^{t 1/2} Tr[R_Z - Q - Q^t + Q^t R_Z^{-1/2} Q]^{1/2}.
\]

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It is clear that
\[ \text{Tr}[BR_Z B^t] \leq \text{Tr}[R_V + BR_Z B^t] \leq nP. \]

By Theorem 4, we obtain
\[
\min_Q \text{Tr}[R_Z - Q - Q^t + Q^t R_Z^{-1} Q]
= \text{Tr}[R_Z] + \min_Q \text{Tr}[Q^t R_Z^{-1} Q - 2Q]
= \text{Tr}[R_Z] - |R_Z^1(2)| - \frac{|R_Z^1(3)|}{|R_Z^1(2)|} - \cdots - \frac{|R_Z|}{|R_Z^1(n)|}.
\]

Then
\[
\frac{1}{n} \text{Tr}[R_V + BR_Z B^t + BR_Z + R_Z B^t]
= \frac{1}{n} \text{Tr}[R_V + BR_Z B^t] + \frac{2}{n} \text{Tr}[BR_Z]
\leq P + 2\sqrt{P \frac{n}{n}} \sqrt{\text{Tr}[R_Z] - |R_Z^1(2)| - \frac{|R_Z^1(3)|}{|R_Z^1(2)|} - \cdots - \frac{|R_Z|}{|R_Z^1(n)|}}.
\]

Now we can state the following proposition which is an extension of Proposition 5.

**Proposition 8**

\[ C_n(P) \leq C_{n,FB}(P) \leq \frac{1}{2} \log(P_2 + \frac{1}{n} \text{Tr}[R_z^{(n)}]) - \frac{1}{2n} \log |R_z^{(n)}|, \]

where
\[ P_2 = P + 2\sqrt{P \frac{n}{n}} \sqrt{\text{Tr}[R_z^{(n)}] - |R_z^1(2)| - \frac{|R_z^1(3)|}{|R_z^1(2)|} - \cdots - \frac{|R_z|}{|R_z^1(n)|}}. \]

**Proof.** To avoid the complexity we denote \( R_V, \ldots \) by \( R_V, \ldots \), respectively.

\[
\frac{1}{2n} \log |R_V + BR_Z B^t + BR_Z + R_Z B^t + R_Z| - \frac{1}{2n} \log |R_Z|
\leq \frac{1}{2} \log \frac{1}{n} \text{Tr}[R_V + BR_Z B^t + BR_Z + R_Z B^t + R_Z] - \frac{1}{2n} \log |R_Z|
\quad (\text{Because } \frac{1}{2n} \log |A| \leq \frac{1}{2} \log \frac{1}{n} \text{Tr}[A])
\]
\[
= \frac{1}{2} \log \left( \frac{1}{n} \text{Tr}[R_V + BR_Z B^t + BR_Z + R_Z B^t] + \frac{1}{n} \text{Tr}[R_Z] \right) - \frac{1}{2n} \log |R_Z|
\leq \frac{1}{2} \log(P_2 + \frac{1}{n} \text{Tr}[R_Z]) - \frac{1}{2n} \log |R_Z|
\quad (\text{by Lemma 3})
\]

Then
\[ C_n(P) \leq C_{n,FB}(P) \leq \frac{1}{2} \log(P_2 + \frac{1}{n} \text{Tr}[R_Z]) - \frac{1}{2n} \log |R_Z|. \]
We let $R_U^{(n)} = R_V^{(n)} + BR_Z^{(n)} B^t + BR_Z^{(n)} + R_Z^{(n)} B^t$. Then $R_U^{(n)}$ is symmetric, but not necessarily positive definite. We denote $\langle R_U^{(n)} \rangle = ((R_U^{(n)})^t R_U^{(n)})^{1/2}$. We can represent

$$R_U^{(n)} = \frac{(R_U^{(n)}) + R_U^{(n)}}{2} - \frac{(R_U^{(n)}) - R_U^{(n)}}{2}.$$ 

Since $(R_U^{(n)}) + R_U^{(n)}$ and $(R_U^{(n)}) - R_U^{(n)}$ are positive definite, we have

$$R_Z^{(n)} + R_U^{(n)} \leq R_U^{(n)} + \frac{(R_U^{(n)}) + R_U^{(n)}}{2}.$$

Then

$$|R_Z^{(n)} + R_U^{(n)}| \leq |R_Z^{(n)} + \frac{(R_U^{(n)}) + R_U^{(n)}}{2}|.$$

We want to obtain an upper bound of $\frac{1}{n} Tr[\frac{(R_U^{(n)}) + R_U^{(n)}}{2}]$ under the condition (1). It follows from Lemma 3 that an upper bound of $\frac{1}{n} Tr[R_U^{(n)}]$ is given by $P_2$. Then we have to obtain an upper bound of $\frac{1}{n} Tr[\langle R_U^{(n)} \rangle]$ under the condition (1).

**Lemma 4** Under the condition (1), an upper bound of

$$\frac{1}{n} Tr[\langle R_U^{(n)} \rangle]$$

is given by

$$P + 2 \sqrt{\frac{P}{n}} \sum_{k=2}^{n} \lambda_{\text{max}} \left( R_Z^{11}(k) \begin{pmatrix} R_Z^{12}(k) & R_Z^{21}(k) & (R_Z^{22}(k))^{-1} R_Z^{12}(k) \end{pmatrix} \right).$$

**Proof.** To avoid the complexity we denote $R_Z^{(n)}, \ldots$ by $R_Z, \ldots$, respectively. By Schatten [9], p4, there exists an orthogonal matrix $W$ such that

$$\langle R_U \rangle = W^t R_U = W^t R_V + W^t B R_Z B^t + W^t B R_Z + W^t R_Z B^t.$$ 

It is clear that

$$Tr[W^t (R_V + B R_Z B^t)] \leq ||W|| |Tr[R_V + B R_Z B^t]| \leq nP.$$ 

We apply Lemma 2 for $A = B R_Z^{1/2}, C = W R_Z^{1/2} - Q^t R_Z^{-1/2}$, where $Q$ is a lower triangular. Since $Tr[BQ] = 0$, we obtain

$$\begin{align*}
Tr[W^t B R_Z] &= Tr[B R_Z W^t] \\
&= Tr[B R_Z W^t] - Tr[BQ] \\
&= Tr[B R_Z^{1/2} (R_Z^{1/2} W^t - R_Z^{-1/2} Q)] \\
&= Tr[B R_Z^{1/2} (W R_Z^{1/2} - Q^t R_Z^{-1/2})^t] \\
&\leq Tr[BR_Z B^t]^{1/2} Tr[(W R_Z^{1/2} - Q^t R_Z^{-1/2})(W R_Z^{1/2} - Q^t R_Z^{-1/2})^t]^{1/2} \\
&= Tr[BR_Z B^t]^{1/2} Tr[WR_Z W^t - WQ - Q^t W^t + Q^t R_Z^{-1/2} Q]^{1/2}.
\end{align*}$$

Now

$$\min_Q Tr[WR_Z W^t - WQ - Q^t W^t + Q^t R_Z^{-1/2} Q] = Tr[R_Z] + \min_Q Tr[Q^t R_Z^{-1} Q - 2WQ].$$
It follows from Theorem 2 that

\[
\min_Q \text{Tr}[Q^t R_Z^{-1} Q - 2WQ] = -\sum_{k=1}^n \langle [R_Z^{-1}(k,\ldots,n)]^{-1} \begin{pmatrix} w_{kk} \\ w_{kk+1} \\ \vdots \\ w_{kn} \end{pmatrix}, \begin{pmatrix} w_{kk} \\ w_{kk+1} \\ \vdots \\ w_{kn} \end{pmatrix} \rangle. \tag{11}
\]

By Theorem 3, an upper bound of (11) under any orthogonal matrices \(W\) is given by

\[-\text{Tr}[R_Z] + \sum_{k=2}^n \lambda_{\text{max}}\left( \begin{array}{ccc} R_{Z1}^1(k) & R_{Z2}^1(k) & R_{Z1}^{12}(k) \\ R_{Z2}^1(k) & R_{Z2}^{12}(k) & R_{Z2}^{12}(k) \\ R_{Z1}^{12}(k)^{-1}R_{Z2}^{12}(k) & R_{Z2}^{12}(k) & R_{Z2}^{12}(k) \end{array} \right) \].

Then under the condition (1), an upper bound of \(\frac{1}{n}\text{Tr}[W^t BR_Z]\) is given by

\[
\sqrt{\frac{P}{n}} \sqrt{\sum_{k=2}^n \lambda_{\text{max}}\left( \begin{array}{ccc} R_{Z1}^1(k) & R_{Z2}^1(k) & R_{Z1}^{12}(k) \\ R_{Z2}^1(k) & R_{Z2}^{12}(k) & R_{Z2}^{12}(k) \\ R_{Z1}^{12}(k)^{-1}R_{Z2}^{12}(k) & R_{Z2}^{12}(k) & R_{Z2}^{12}(k) \end{array} \right) }. \tag{12}
\]

Similarly under the condition (1), an upper bound of

\[
\frac{1}{n}\text{Tr}[W^t R_Z B^t]
\]

is also given by (12). Hence we obtain the result. \(\square\)

Now we can state the main theorem.

**Theorem 5**

\[ C_n(P) \leq C_{n,FB}(P) \leq C_n(P^*), \]

where

\[
P^* = P + \sqrt{\frac{P}{n}} \left\{ \sqrt{\text{Tr}[R_Z^{(n)}]} \right\} + \sqrt{\text{Tr}[R_Z^{(n)}] - |R_Z^{11}(2)| - |R_Z^{11}(3)| - \cdots - |R_Z^{(n)}|}. \]

It is easy to obtain the Corollary.

**Corollary 1**

\[ C_n(P) \leq C_{n,FB}(P) \leq C_n(P_3), \]

where

\[
P_3 = P + \sqrt{\frac{P}{n}} \left\{ \sqrt{\text{Tr}[R_Z^{(n)}]} \right\} + \sqrt{\text{Tr}[R_Z^{(n)}] - |R_Z^{11}(2)| - |R_Z^{11}(3)| - \cdots - |R_Z^{(n)}|}. \]
Proof. To avoid the complexity we denote $R^{(n)}_Z$ by $R_Z$. We apply Lemma 2 for $A = BR_Z^{1/2}$, $C = WR_Z^{1/2}$. Then

$$\begin{align*}
\text{Tr}[W^tBR_Z] &= \text{Tr}[BR_ZW^t] \\
&\leq \text{Tr}[BR_ZB^{1/2}]\text{Tr}[WR_ZW^t]^{1/2} \\
&\leq \sqrt{nPTr[R_Z^{1/2}]}.
\end{align*}$$

As the same method of Lemma 4 we obtain the result. \qed

IV. Examples and Remarks

We want to compare the relationship between upper bounds given by Proposition 2, Proposition 3, Proposition 4, Proposition 5, Proposition 6, Proposition 7 and Theorem 5. As the upper bound in Proposition 8 is tighter than the upper bound in Proposition 5, we may compare the relationship between six types of upper bounds without Proposition 5. Since we can’t compare in general, we have two examples; one is for $n = 2$, another is for $n = 3$.

Example 1 Let $R_Z^{(2)} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$. Then $\lambda_1 = 1$, $\lambda_2 = 3$ and

$$P(\mu) = \frac{1}{4}(\mu + 4 - \sqrt{\mu + 1} - \sqrt{3(\mu + 3)}).$$

The upper bound in Proposition 4 is given by

$$\frac{1}{2} \log(1 + \frac{P}{\lambda_1}).$$

The upper bound in Proposition 6 is given by

$$\frac{1}{2} \log \frac{1}{4}(1 + \sqrt{1 + \frac{\mu}{\lambda_1}})(1 + \sqrt{1 + \frac{\mu}{\lambda_2}}).$$

We get Table 1.

Example 2 Let $R_Z^{(3)} = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$. Then $\lambda_1 = 2 - \sqrt{2}$, $\lambda_2 = 2$, $\lambda_3 = 2 + \sqrt{2}$ and

$$P(\mu) = \frac{1}{12}((\sqrt{\mu + \lambda_1} - \sqrt{\lambda_1})^2 + (\sqrt{\mu + \lambda_2} - \sqrt{\lambda_2})^2 + (\sqrt{\mu + \lambda_3} - \sqrt{\lambda_3})^2).$$

The upper bound in Proposition 4 is given by

$$\frac{1}{2} \log(1 + \frac{P}{\lambda_1}).$$

The upper bound in Proposition 6 is given by

$$\frac{1}{3} \log \frac{1}{8}(1 + \sqrt{1 + \frac{\mu}{\lambda_1}})(1 + \sqrt{1 + \frac{\mu}{\lambda_2}})(1 + \sqrt{1 + \frac{\mu}{\lambda_3}}).$$

We get Table 2. Since it is not easy to calculate $P_1$, we omit $P_1$ and $C_3(P_1)$ in Table 2.
From two examples, if $P$ is sufficiently small, then we hope that Proposition 7 gives the strongest upper bound in those obtained before, and if $P$ is sufficiently large, then we hope that Theorem 5 gives the strongest upper bound in those obtained before.

**Remark 1**
When the noise process is stationary, \{\(R_n\)\} is a sequence of leading minors of an infinite Toeplitz matrix \(R_Z\). Assume that the first row of \(R_Z\) is in \(\ell_1\) (i.e., \(\sum_{n=1}^{\infty} |(R_Z)_{1n}| < \infty\)), and define the noise spectrum

\[
R(w) = (R_Z)_{11} + 2 \sum_{n=1}^{\infty} (R_Z)_{1n} \cos[w(n - 1)].
\]

Then the asymptotic distribution of \(\lambda_1(R^{(n)}_Z), \ldots, \lambda_n(R^{(n)}_Z)\) is according to \(R(w)\), in the sense that for any bounded continuous function \(f(x)\):

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} f(\lambda_i(R^{(n)}_Z)) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(R(w)) dw.
\]

One of the upper bounds in [7] is

\[
C_{FB} = \lim_{n \to \infty} C_{n, FB}(P) \leq
\]

\[
\frac{1}{2} \log (P + \sqrt{P}) \sqrt{\frac{1}{2\pi} \int_{-\pi}^{\pi} R(w) dw - \exp \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} \log R(w) dw \right)} + \frac{1}{2\pi} \int_{-\pi}^{\pi} R(w) dw - \frac{1}{2\pi} \int_{-\pi}^{\pi} \log \sqrt{R(w)} dw.
\]

Unfortunately we can’t interpret the upper bound in Theorem 5 easily. Because the maximal eigenvalues

\[
\lambda_{\text{max}}(R^{(n)}_Z(k)) = \begin{pmatrix}
R^{11}_Z(k) & R^{12}_Z(k) \\
R^{21}_Z(k) & R^{22}_Z(k)
\end{pmatrix}
\]

are not able to be represented by the eigenvalues of \(R^{(n)}_Z\). So we are ready to treat the stationary case in the next paper.

**References**


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