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ORTHONORMAL BASES OF EXPONENTIALS FOR THE n-CUBE

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1. Introduction. A compact set Ω in $\mathbb{R}^n$ of positive Lebesgue measure is a spectral set if there is some set of exponentials

$$\mathcal{B}_\Lambda := \left\{e^{2\pi i\langle\lambda,x\rangle} : \lambda \in \Lambda\right\}, \tag{1.1}$$

which when restricted to Ω gives an orthogonal basis for $L^2(\Omega)$, with respect to the inner product

$$\langle f, g\rangle_\Omega := \int_\Omega \overline{f(x)}g(x)\,dx. \tag{1.2}$$

Any set Λ that gives such an orthogonal basis is called a spectrum for Ω. Only very special sets Ω in $\mathbb{R}^n$ are spectral sets. However, when a spectrum exists, it can be viewed as a generalization of Fourier series, because for the n-cube $\Omega = [0,1]^n$ the spectrum $\Lambda = \mathbb{Z}^n$ gives the standard Fourier basis of $L^2([0,1]^n)$.

The main object of this paper is to relate the spectra of sets Ω to tilings in Fourier space. We develop such a relation for a large class of sets and apply it to geometrically characterize all spectra for the n-cube $\Omega = [0,1]^n$.

THEOREM 1.1. *The following conditions on a set Λ in $\mathbb{R}^n$ are equivalent.*

(i) *The set $\mathcal{B}_\Lambda = \{e^{2\pi i\langle\lambda,x\rangle} : \lambda \in \Lambda\}$ when restricted to $[0,1]^n$ is an orthonormal basis of $L^2([0,1]^n)$.*

(ii) *The collection of sets $\{\lambda + [0,1]^n : \lambda \in \Lambda\}$ is a tiling of $\mathbb{R}^n$ by translates of unit cubes.*

This result was conjectured by Jorgensen and Pedersen [6], who proved it in dimensions $n \le 3$. We note that in high dimensions there are many "exotic" cube tilings. There are aperiodic cube tilings in all dimensions $n \ge 3$, while in dimensions $n \ge 10$ there are cube tilings in which no two cubes share a common $(n-1)$-face; see Lagarias and Shor [9].

In Theorem 1.1, the n-cube $[0,1]^n$ appears in both conditions (i) and (ii), but in functorially different contexts. The n-cube in (i) lies in the space domain $\mathbb{R}^n$ while the n-cube in (ii) lies in the Fourier domain $(\mathbb{R}^n)^*$, so they transform differently under linear change of variables. Thus Theorem 1.1 is equivalent to the following result.

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THEOREM 1.2. *For any invertible linear transformation $A \in \mathrm{GL}(n, \mathbb{R})$, the following conditions are equivalent.*

(i) $\Lambda \subset \mathbb{R}^n$ *is a spectrum for* $\Omega_A := A([0,1]^n)$.

(ii) *The collection of sets* $\{\lambda + D_A : \lambda \in \Lambda\}$ *is a tiling of* $\mathbb{R}^n$, *where* $D_A = (A^T)^{-1}([0,1]^n)$.

Our main result in §3 gives a necessary and sufficient condition for a general set Λ to be a spectrum of Ω in terms of a tiling of $\mathbb{R}^n$ by $\Lambda + D$, where D is a specified auxiliary set in Fourier space. This result applies whenever a suitable auxiliary set D exists. In §4 we show that this is the case when Ω is an n-cube, with D also being an n-cube, and obtain Theorem 1.1.

Spectral sets were originally studied by Fuglede [1], who related them to the problem of finding commuting self-adjoint extensions in $L^2(\Omega)$ of the set of differential operators $-i\frac{\partial}{\partial x_1}, \dots, -i\frac{\partial}{\partial x_n}$ defined on the common dense domain $C_c^\infty(\Omega)$. Our definition of spectrum differs from his by a factor of 2π. Fuglede showed that for sufficiently nice connected open regions Ω, each spectrum Λ of Ω (in our sense) has $2\pi\Lambda$ as a joint spectrum of a set of commuting self-adjoint extensions of $-i\frac{\partial}{\partial x_1}, \dots, -i\frac{\partial}{\partial x_n}$, and conversely. He also showed that only very special sets Ω are spectral sets. In particular, Fuglede [1, p. 120] made the following conjecture.

SPECTRAL SET CONJECTURE. *A set Ω in $\mathbb{R}^n$ is a spectral set if and only if it tiles $\mathbb{R}^n$ by translations.*

Much recent work on spectral sets is due to Jorgenson and Pedersen (see [4]–[6], [13], and [14]), with additional work by Lagarias and Wang [10].

The spectral set conjecture concerns tilings by Ω in the space domain. In contrast, Theorem 1.2 describes spectra Λ for the n-cube in terms of tilings in the Fourier domain by an auxiliary set D. In general there does not seem to be any simple relation between sets of translations T used to tile Ω in the space domain and the set of spectra Λ for Ω (see [5], [10], and [14]). Our main results in §3 indicate a relation between the spectral set conjecture and tilings in the Fourier domain—this is discussed at the end of §3.

Theorem 1.2 also implies a result concerning sampling and interpolation of certain classes of entire functions. Given a compact set Ω of nonzero Lebesgue measure, let $B_2(\Omega)$ denote the set of band-limited functions on Ω, which are those entire functions $f : \mathbb{C}^n \to \mathbb{C}$ whose restriction to $\mathbb{R}^n$ is the Fourier transform of an L^2-function with compact support contained in Ω. A countable set Λ is a set of sampling for $B_2(\Omega)$ if there exist $A, B > 0$ such that for all $f \in B_2(\Omega)$,

$$A\|f\|^2 \le \sum_{\lambda \in \Lambda} |f(\lambda)|^2 \le B\|f\|^2. \tag{1.3}$$

A set of sampling is always a set of uniqueness for $B_2(\Omega)$, where a set Λ is a set of uniqueness if for each set of complex values $\{c_\lambda : \lambda \in \Lambda\}$ with $\sum |c_\lambda|^2 < \infty$ there is at most one function $f \in B_2(\Omega)$ with

$$f(\lambda) = c_\lambda, \quad \text{for each } \lambda \in \Lambda. \tag{1.4}$$

A set Λ is a set of interpolation for $B_2(\Omega)$ if for each such set $\{c_\lambda : \lambda \in \Lambda\}$ there is at least one function $f \in B_2(\Omega)$ such that (1.4) holds. It is clear that a spectrum Λ of a spectral set Ω is both a set of sampling and a set of interpolation for $B_2(\Omega)$, so Theorem 1.2 immediately yields the following theorem.

THEOREM 1.3. *Given a linear transformation A in* $\mathrm{GL}(n, \mathbb{R})$, *set* $\Omega_A = A([0,1]^n)$ *and* $D_A = (A^T)^{-1}([0,1]^n)$. *If $\Lambda + D_A$ is a tiling of $\mathbb{R}^n$, then Λ is both a set of sampling and a set of interpolation for* $B_2(\Omega_A)$.

Here the set Λ has density exactly equal to the Nyquist rate $|\det(A)|$, as is required by results of Landau (see [11], [12]) for sets of sampling and interpolation.

In the appendix we apply Theorem 1.1 to show that in dimensions $n = 1$ and $n = 2$ any orthogonal set of exponentials in $L^2([0,1]^n)$ can be completed to a basis of exponentials of $L^2([0,1]^n)$ but that this is not always the case in dimensions $n \geq 3$.

We conclude this introduction with two remarks concerning the relation of spectral sets and tilings. First, in comparison with other spectral sets, the n-cube $[0,1]^n$ has an enormous variety of spectra Λ. It seems likely that a "generic" spectral set has a unique spectrum, up to translations.[1] Second, the tiling result in §3 applies to more general sets Ω than linearly transformed n-cubes $\Omega_A = A([0,1]^n)$; we give the one-dimensional example $\Omega = [0,1] \cup [2,3]$.

After completing a preprint of this paper in early 1998, we learned that A. Iosevich and S. Pedersen [3] simultaneously and independently obtained a proof of Theorem 1.1, by a different approach. M. Kolountzakis [8] has proved Conjecture 2.1 below, building on the approach of our paper.

Notation. For $x \in \mathbb{R}^n$, let $\|x\|$ denote the Euclidean length of x. We let

$$B(x;T) := \{y : \|y - x\| \leq T\}$$

denote the ball of radius T centered at x. The Lebesgue measure of a set Ω in $\mathbb{R}^n$ is denoted $m(\Omega)$. The Fourier transform $\hat{f}(u)$ is normalized by

$$\hat{f}(u) := \int_{\mathbb{R}^n} e^{-2\pi i \langle u, x\rangle} f(x)\, dx.$$

Throughout the paper we let

$$e_\lambda(x) := e^{2\pi i \langle \lambda, x\rangle}, \quad \text{for } x \in \mathbb{R}^n. \tag{1.5}$$

Some other authors (see [1], [6]) define $e_\lambda(x)$ differently, without the factor 2π in the exponent.

[1] It can be shown that a "generic" fundamental domain Ω of a full rank lattice L in $\mathbb{R}^n$ has a unique spectrum $\Lambda = L^*$, the dual lattice.

2. Orthogonal sets of exponentials and packings. We consider packings and tilings in $\mathbb{R}^n$ by compact sets Ω of the following kind.

Definition 2.1. A compact set Ω in $\mathbb{R}^n$ is a regular region if it has positive Lebesgue measure $m(\Omega) > 0$, is the closure of its interior Ω°, and has a boundary $\partial\Omega = \Omega \setminus \Omega^\circ$ of measure zero.

Definition 2.2. If Ω is a regular region, then a discrete set Λ is a packing set for Ω if the sets $\{\Omega + \lambda : \lambda \in \Lambda\}$ have disjoint interiors. It is a tiling set if, in addition, the union of the sets $\{\Omega + \lambda : \lambda \in \Lambda\}$ covers $\mathbb{R}^n$. In these cases we say $\Lambda + \Omega$ is a packing or tiling of $\mathbb{R}^n$ by Ω, respectively.

To a vector λ in $\mathbb{R}^n$, we associate the exponential function

$$e_\lambda(x) := e^{2\pi i \langle \lambda, x \rangle}, \quad \text{for } x \in \mathbb{R}^n. \tag{2.1}$$

Given a discrete set Λ in $\mathbb{R}^n$, we set

$$\mathcal{B}_\Lambda := \{e_\lambda(x) : \lambda \in \Lambda\}. \tag{2.2}$$

Now suppose that $\mathcal{B}_\Lambda$ restricted to a regular region Ω gives an orthogonal set of exponentials in $L^2(\Omega)$. We derive conditions that the points of Λ must satisfy. Let

$$\chi_\Omega(x) = \begin{cases} 1, & \text{for } x \in \Omega, \\ 0, & \text{for } x \notin \Omega \end{cases} \tag{2.3}$$

be the characteristic function of Ω, and consider its Fourier transform

$$\hat{\chi}_\Omega(u) = \int_{\mathbb{R}^n} e^{-2\pi i \langle u, x \rangle} \chi_\Omega(x)\, dx, \quad u \in \mathbb{R}^n. \tag{2.4}$$

Since Ω is compact, the function $\hat{\chi}_\Omega(u)$ is an entire function of $u \in \mathbb{C}^n$. We denote the set of real zeros of $\hat{\chi}_\Omega(u)$ by

$$Z(\Omega) := \left\{u \in \mathbb{R}^n : \hat{\chi}_\Omega(u) = 0\right\}. \tag{2.5}$$

LEMMA 2.1. *If Ω is a regular region in $\mathbb{R}^n$, then a set Λ gives an orthogonal set of exponentials $\mathcal{B}_\Lambda$ in $L^2(\Omega)$ if and only if*

$$\Lambda - \Lambda \subseteq Z(\Omega) \cup \{0\}. \tag{2.6}$$

Proof. For distinct $\lambda, \mu \in \Lambda$ we have

$$\begin{aligned} \hat{\chi}_\Omega(\lambda - \mu) &= \int_{\mathbb{R}^n} e^{-2\pi i \langle \lambda - \mu, x \rangle} \chi_\Omega(x)\, dx \\ &= \int_\Omega e^{-2\pi i \langle \lambda, x \rangle} e^{2\pi i \langle \mu, x \rangle}\, dx = \langle e_\lambda, e_\mu \rangle_\Omega. \end{aligned} \tag{2.7}$$

If (2.6) holds, then $\langle e_\lambda, e_\mu \rangle_\Omega = 0$, and conversely. □

This lemma implies that the points of Λ have the property of being "well spaced" in the sense of being uniformly discrete; that is, there is some positive R such that any two points are no closer than R. Indeed, since $\hat{\chi}_\Omega(0) = m(\Omega) > 0$, the continuity of $\hat{\chi}_\Omega(u)$ implies that there is some ball $B(0; R)$ around zero that includes no point of $Z(\Omega)$; hence, $\|\lambda - \mu\| \geq R$ for all $\lambda, \mu \in \Lambda, \lambda \neq \mu$.

Definition 2.3. Let Ω be a regular region in $\mathbb{R}^n$. A regular region D is said to be an orthogonal packing region for Ω if

$$(D^\circ - D^\circ) \cap Z(\Omega) = \emptyset. \tag{2.8}$$

LEMMA 2.2. *Let Ω be a regular region in $\mathbb{R}^n$, and let D be an orthogonal packing region for Ω. If a set Λ gives an orthogonal set of exponentials $\mathcal{B}_\Lambda$ in $L^2(\Omega)$, then Λ is a packing set for D.*

Proof. If $\lambda \neq \mu \in \Lambda$, then Lemma 2.1 gives $\lambda - \mu \in Z(\Omega)$. By definition of an orthogonal packing region we have $D^\circ \cap (D^\circ + u) = \emptyset$ for all $u \in Z(\Omega)$; hence,

$$D^\circ \cap (D^\circ + \lambda - \mu) = \emptyset,$$

as required. □

As indicated above, each regular region Ω has an orthogonal packing region D given by a ball $B(0; T)$ for small enough T. The larger we can take D, the stronger the restrictions imposed on Λ.

LEMMA 2.3. *If Ω is a spectral set and if D is an orthogonal packing region for Ω, then*

$$m(D)m(\Omega) \leq 1. \tag{2.9}$$

Proof. Let Λ be a spectrum for Ω. Then Λ is a set of sampling for $B_2(\Omega)$, so the density results of Landau [11] (see also Gröchenig and Razafinjatovo [2]) give

$$\mathbf{d}(\Lambda) = \liminf_{n \to \infty} \frac{1}{(2T)^n} \#\big(\Lambda \cap [-T, T]^n\big) \geq m(\Omega). \tag{2.10}$$

Now $\Lambda + D$ is a packing of $\mathbb{R}^n$; hence if $R = \operatorname{diam}(D)$, we have

$$\begin{aligned} \frac{m(D)}{(2T)^n} \#\big(\Lambda \cap [-T, T]^n\big) &= \frac{1}{(2T)^n}\, m\left(\left\{\bigcup_\lambda (\lambda + D) : \lambda \in \Lambda \cap [-T, T]^n\right\}\right) \\ &\leq \frac{m\big(\big[-T + R,\ T + R\big]^n\big)}{(2T)^n} = \left(1 + \frac{R}{2T}\right)^n. \end{aligned} \tag{2.11}$$

Letting $T \to \infty$ and taking the inferior limit yields

$$m(D)\mathbf{d}(\Lambda) \leq 1, \tag{2.12}$$

and now (2.10) yields (2.9). □

In §3 we give a self-contained proof of Lemma 2.3. The inequality of Lemma 2.3 does not hold for general sets Ω. In fact the set $\Omega = [0,1] \cup [2, 2+\theta]$ for suitable irrational θ has a Fourier transform $\hat{\chi}_\Omega(\xi)$ that has no real zeros; so $Z(\Omega) = \emptyset$, and any regular region D, of arbitrarily large measure, is an orthogonal packing region for Ω.

In view of Lemma 2.3 we introduce the following terminology.

Definition 2.4. An orthogonal packing region D for a regular region Ω is tight if

$$m(D) = \frac{1}{m(\Omega)}. \tag{2.13}$$

This definition transforms in the Fourier domain under linear transformations: If D is a tight orthogonal packing region for a regular region Ω, then for any $A \in \mathrm{GL}(n, \mathbb{R})$ the set $(A^T)^{-1}(D)$ is a tight orthogonal packing region for $A(\Omega)$.

There are many spectral sets that have tight orthogonal packing regions. In §4 we show that $D = [0,1]^n$ is a tight orthogonal packing region for $\Omega = [0,1]^n$. Another example in $\mathbb{R}^1$ is the region

$$\Omega = [0,1] \cup [2,3]. \tag{2.14}$$

In this case we can take

$$D = \left[0, \frac{1}{4}\right] \cup \left[\frac{1}{2}, \frac{3}{4}\right]. \tag{2.15}$$

Indeed, $\chi_\Omega(x)$ is the convolution of $\chi_{[0,1]}(x)$ with the sum of two delta functions $\delta_0 + \delta_2$. Thus

$$\hat{\chi}_\Omega(x) = \left(1 + e^{-4\pi i x}\right) \hat{\chi}_{[0,1]}(x). \tag{2.16}$$

From this it is easy to check that the zero set is given by

$$Z(\Omega) = \left(\mathbb{Z} \setminus \{0\}\right) \cup \left(\frac{1}{4} + \mathbb{Z}\right) \cup \left(-\frac{1}{4} + \mathbb{Z}\right), \tag{2.17}$$

that D is an orthogonal packing region for Ω, and, since $m(D) = 1/2 = 1/m(\Omega)$, that D is tight. A spectrum for Ω is $\Lambda = \mathbb{Z} \cup (\mathbb{Z} + (1/4))$.

Lemma 2.3 together with the spectral set conjecture leads us to propose the following.

CONJECTURE 2.1. *If Ω tiles $\mathbb{R}^n$ by translations and if D is an orthogonal packing region for Ω, then*

$$m(\Omega) m(D) \le 1. \tag{2.18}$$

This conjecture has now been proved by Kolountzakis [8, Theorem 7].

3. Spectra and tilings. A main result of this paper is the following criterion that relates spectra to tilings in the Fourier domain.

THEOREM 3.1. *Let Ω be a regular region in $\mathbb{R}^n$, and let Λ be such that the set of exponentials $\mathcal{B}_\Lambda$ is orthogonal for $L^2(\Omega)$. Suppose that D is a regular region with*

$$m(D)m(\Omega)=1 \tag{3.1}$$

such that $\Lambda+D$ is a packing of $\mathbb{R}^n$. Then Λ is a spectrum for Ω if and only if $\Lambda+D$ is a tiling of $\mathbb{R}^n$.

Proof ($\Rightarrow$). Suppose first that Λ is a spectrum for Ω. Pick a "bump function" $\gamma(x)\in C_c^\infty(\Omega)$, and set

$$\gamma_t(x)=e^{-2\pi i\langle t,x\rangle}\gamma(x),\quad \text{for } t\in\mathbb{R}^n.$$

By hypothesis $\mathcal{B}_\Lambda=\{e_\lambda(x):\lambda\in\Lambda\}$ is orthogonal and complete for $L^2(\Omega)$. Thus, on Ω, we have

$$\gamma_t(x)\sim\sum_{\lambda\in\Lambda}\frac{\left\langle e^{2\pi i\langle\lambda,x\rangle},\gamma_t(x)\right\rangle_\Omega}{\|e_\lambda\|_2^2}\,e^{2\pi i\langle\lambda,x\rangle}, \tag{3.2}$$

with coefficients

$$\begin{aligned}\frac{\left\langle e^{2\pi i\langle\lambda,x\rangle},\,\gamma_t(x)\right\rangle_\Omega}{\|e_\lambda\|_2^2}&=\frac{1}{m(\Omega)}\int_\Omega e^{-2\pi i\langle\lambda,x\rangle}\gamma_t(x)\,dx\\&=\frac{1}{m(\Omega)}\int_{\mathbb{R}^n}e^{-2\pi i\langle\lambda+t,x\rangle}\gamma(x)\,dx\\&=\frac{1}{m(\Omega)}\,\hat{\gamma}(\lambda+t),\end{aligned} \tag{3.3}$$

where $m(\Omega)$ is the Lebesgue measure of Ω. The rapid decrease of $\hat{\gamma}$ with increasing radius $\|x\|$ and the well-spaced property of Λ show that the right side of (3.2) converges absolutely and uniformly on $\mathbb{R}^n$. Since $\gamma_t(x)$ is continuous, we have

$$\gamma_t(x)=\frac{1}{m(\Omega)}\sum_{\lambda\in\Lambda}\hat{\gamma}(\lambda+t)e^{2\pi i\langle\lambda,x\rangle},\quad \text{for all } x\in\Omega. \tag{3.4}$$

This yields, for all $t\in\mathbb{R}^n$, that

$$\gamma(x)=e^{2\pi i\langle t,x\rangle}\gamma_t(x)=\frac{1}{m(\Omega)}\sum_{\lambda\in\Lambda}\hat{\gamma}(\lambda+t)e^{2\pi i\langle\lambda+t,x\rangle},\quad \text{for all } x\in\Omega. \tag{3.5}$$

The series on the right side of (3.5) converges absolutely and uniformly for all $x\in\mathbb{R}^n$ and t in any fixed compact subset of $\mathbb{R}^n$, but it is only guaranteed to agree with $\gamma(x)$ for $x\in\Omega$.

We now integrate both sides of (3.5) in t over all $t \in D$ to obtain

$$\begin{aligned} m(D)\gamma(x) &= \gamma(x)\int_{\mathbb{R}^n} \chi_D(t)\,dt \\ &= \frac{1}{m(\Omega)}\sum_{\lambda\in\Lambda}\int_D \hat{\gamma}(\lambda+t)e^{2\pi i\langle\lambda+t,x\rangle}\,dt \\ &= \frac{1}{m(\Omega)}\int_{\Lambda+D}\hat{\gamma}(u)e^{2\pi i\langle u,x\rangle}\,du, \quad \text{for all } x\in\Omega. \end{aligned} \tag{3.6}$$

In the last step we used the fact that the translates $\lambda + D$ overlap on sets of measure zero, because $\Lambda + D$ is a packing of $\mathbb{R}^n$. Since $m(D) = 1/m(\Omega)$, (3.6) yields

$$\gamma(x) = \int_{\mathbb{R}^n} h(u)\hat{\gamma}(u)e^{2\pi i\langle u,x\rangle}\,du, \quad \text{for all } x \in \Omega, \tag{3.7}$$

where

$$h(u) = \begin{cases} 1, & \text{if } u \in \Lambda + D, \\ 0, & \text{otherwise.} \end{cases}$$

Define $k \in L^2(\mathbb{R}^n)$ by $\hat{k} = h\hat{\gamma}$, so (3.7) asserts that $\gamma(x) = k(x)$ for almost all $x \in \Omega$. Plancherel's theorem on $L^2(\mathbb{R}^n)$ applied to k, together with (3.7), gives

$$\|\hat{\gamma}\|_2^2 \geq \|h\hat{\gamma}\|_2^2 = \|k\|_2^2 \geq \int_\Omega |k(x)|^2\,dx = \int_\Omega |\gamma(x)|^2\,dx = \|\gamma\|_2^2. \tag{3.8}$$

Since Plancherel's theorem also gives $\|\hat{\gamma}\|_2^2 = \|\gamma\|_2^2$, we must have

$$\|\hat{\gamma}\|_2^2 = \|h\hat{\gamma}\|_2^2. \tag{3.9}$$

We next show that this equality implies that $h(u) = 1$ almost everywhere on $\mathbb{R}^n$. To do this we show that $\hat{\gamma}(u) \neq 0$ a.e. in $\mathbb{R}^n$. Since γ has compact support, the Paley-Wiener theorem states that $\hat{\gamma}(u)$ is the restriction to $\mathbb{R}^n$ of an entire function on $\mathbb{C}^n$ that satisfies an exponential growth condition at infinity; see Stein and Weiss [16, Theorem 4.9]. Thus $\hat{\gamma}(u)$ is real analytic on $\mathbb{R}^n$ and is not identically zero; hence

$$Z := \left\{u \in \mathbb{R}^n : \hat{\gamma}(u) = 0\right\}$$

has Lebesgue measure zero. Together with (3.9) this yields

$$h(u) = 1, \quad \text{a.e. in } \mathbb{R}^n. \tag{3.10}$$

Thus, $\Lambda + D$ covers all of $\mathbb{R}^n$ except a set of measure zero.

Finally we show that $\Lambda + D$ covers all of $\mathbb{R}^n$. By the well-spaced property of Λ and the compactness of D, the set $\Lambda + D$ is locally the union of finitely many translates of D; hence $\Lambda + D$ is closed. Thus, the complement of $\Lambda + D$ is an open set. But the complement of $\Lambda + D$ has zero Lebesgue measure; hence, it is empty, so $\Lambda + D$ is a tiling of $\mathbb{R}^n$.

($\Leftarrow$). Suppose $\Lambda + D$ tiles $\mathbb{R}^n$. By hypothesis, $\mathcal{B}_\Lambda$ is an orthogonal set in $L^2(\Omega)$, and to show that Λ is a spectrum it remains to show that it is complete in $L^2(\Omega)$. Let S be the closed span of $\mathcal{B}_\Lambda$ in $L^2(\Omega)$. We show that $C_c^\infty(\Omega)$ is contained in S. Since $C_c^\infty(\Omega)$ is dense in $L^2(\Omega)$, this implies $S = L^2(\Omega)$.

For each $\gamma \in C_c^\infty(\Omega)$, set

$$\gamma_t(x) = e^{-2\pi i \langle t, x\rangle}\gamma(x), \quad \text{for } t \in \mathbb{R}^n.$$

Since the elements of $\mathcal{B}_\Lambda$ are orthogonal, Bessel's inequality gives

$$\|\gamma_t\|^2 \geq \sum_{\lambda\in\Lambda} \frac{|\langle e_\lambda, \gamma_t\rangle|^2}{\|e_\lambda\|^2} = \frac{1}{m(\Omega)} \sum_{\lambda\in\Lambda} |\hat{\gamma}(\lambda + t)|^2, \tag{3.11}$$

where the last series converges uniformly on compact sets by the rapid decay of $\hat{\gamma}$ at infinity. Integrating this inequality over $t \in D$ yields

$$\int_D \|\gamma_t\|^2\, dt \geq \frac{1}{m(\Omega)} \int_D \sum_{\lambda\in\Lambda} |\hat{\gamma}(\lambda + t)|^2\, dt.$$

Since $\|\gamma_t\| = \|\gamma\|$ for all t and since $\Lambda + D$ is a tiling, we obtain $m(D)\|\gamma\|^2 \geq \|\hat{\gamma}\|^2/m(\Omega)$. But $m(D) = 1/m(\Omega)$ and $\|\gamma\|^2 = \|\hat{\gamma}\|^2$, so equality must hold in (3.11) for almost all t:

$$\|\gamma\|^2 = \sum_{\lambda\in\Lambda} \frac{|\langle e_\lambda, \gamma_t\rangle|^2}{\|e_\lambda\|_2^2}. \tag{3.12}$$

Now the right side of (3.12) converges uniformly on compact sets, so (3.12) holds for all t, including $t = 0$. Hence,

$$\|\gamma\|^2 = \sum_{\lambda\in\Lambda} \frac{|\langle e_\lambda, \gamma\rangle|^2}{\|e_\lambda\|_2^2},$$

and so $\gamma \in S$. □

At first glance, the first half of this proof of Theorem 3.1 appears too good to be true because it only uses functions $\gamma_t(x)$ supported on a fixed subset of Ω. But the relevant fact is that $\operatorname{supp} \hat{\gamma}_t$ is dense in Fourier space $\mathbb{R}^n$.

The proof of Theorem 3.1 yields a direct proof of Lemma 2.3. If D is an orthogonal packing set, then (3.6) holds for it; hence $m(D)m(\Omega)\gamma(x)$ agrees with $k(x)$ on Ω, and hence

$$m(D)m(\Omega)\|\gamma\|_2 \leq \|k\|_2 \leq \|\gamma\|_2,$$

which shows that (2.9) holds.

The following result is an immediate corollary of Theorem 3.1, which we state as a theorem for emphasis.

THEOREM 3.2. *Let Ω be a regular region in $\mathbb{R}^n$, and suppose that D is a tight orthogonal packing region for Ω. If Λ is a spectrum for Ω, then $\Lambda + D$ is a tiling of $\mathbb{R}^n$.*

Proof. The assumption that D is a tight orthogonal packing region guarantees that $\Lambda + D$ is a packing for all spectra Λ, so Theorem 3.1 applies. □

Theorem 3.2 sheds some light on Fuglede's conjecture that every spectral set Ω tiles $\mathbb{R}^n$.

Definition 3.1. A pair of regular regions $(\Omega, \hat{\Omega})$ is a tight dual pair if each is a tight orthogonal packing region for the other.

In §4 we show that $([0,1]^n, [0,1]^n)$ is a tight dual pair of regions; it follows that if $A \in \mathrm{GL}(n,\mathbb{R})$, then $(A([0,1]^n), (A^T)^{-1}([0,1]^n))$ is also a tight dual pair of regions. The sets $([0,1] \cup [2,3], [0,1/4] \cup [1/2,3/4])$ are a tight dual pair in $\mathbb{R}^1$.

If $(\Omega, \hat{\Omega})$ is a tight dual pair, then Theorem 3.1 states that if one of $(\Omega, \hat{\Omega})$ is a spectral set, say, Ω, then the other set $\hat{\Omega}$ tiles $\mathbb{R}^n$. If $\hat{\Omega}$ were also a spectral set (as the spectral set conjecture implies), then Theorem 3.1 would show that Ω tiles $\mathbb{R}^n$. This raises the question whether the current evidence in favor of Fuglede's conjecture is mainly based on sets Ω that appear in a tight dual pair $(\Omega, \hat{\Omega})$. At present we can only say that there are many nontrivial examples of tight dual pairs.

To clarify matters, we formulate two conjectures.

CONJECTURE 3.1 (Spectral set duality conjecture). *If $(\Omega, \hat{\Omega})$ is a tight dual pair of regular regions and if Ω is a spectral set, then $\hat{\Omega}$ is also a spectral set.*

In this case Theorem 3.2 would imply that both Ω and $\hat{\Omega}$ tile $\mathbb{R}^n$. The corresponding tiling analogue of this conjecture is as follows.

CONJECTURE 3.2 (Weak spectral set conjecture). *If $(\Omega, \hat{\Omega})$ is a tight dual pair of regular regions and if one of them tiles $\mathbb{R}^n$, then so does the other, and both Ω and $\hat{\Omega}$ are spectral sets.*

4. Spectra for the n-cube and cube tilings. We prove Theorem 1.1, using the results of §3. We use the following basic result of Keller [7], which gives a necessary condition for a set Λ to give a cube tiling.

PROPOSITION 4.1 (Keller's criterion). *If $\Lambda + [0,1]^n$ is a tiling of $\mathbb{R}^n$, then each $\lambda, \mu \in \Lambda$ has*

$$\lambda_i - \mu_i \in \mathbb{Z} \setminus \{0\} \quad \textit{for some } i, 1 \le i \le n. \tag{4.1}$$

Proof. This result was proved by Keller [7] in 1930. A detailed proof appears in Perron [15, Satz 9]. □

The following lemma shows that Keller's necessary condition for a cube tiling is the same as orthogonality of exponentials in the set Λ.

LEMMA 4.1. *$\mathcal{B}_\Lambda := \{e^{2\pi i\langle\lambda,x\rangle} : \lambda \in \Lambda\}$ gives a set of orthogonal functions in $L^2([0,1]^n)$ if and only if, for any distinct $\lambda, \mu \in \Lambda$,*

$$\lambda_j - \mu_j \in \mathbb{Z} \setminus \{0\}, \quad \textit{for some } j, 1 \le j \le n. \tag{4.2}$$

Proof. For $\Omega = [0,1]^n$ and $u \in \mathbb{R}^n$,

$$\hat{\chi}_\Omega(u) = \int_{[0,1]^n} e^{-2\pi i\langle u,x\rangle}\, dx = \prod_{j=1}^{n} h_0(u_j),$$

where $h_0(\omega) := (1 - e^{-2\pi i\omega})/(2\pi i\omega)$, $\omega \in \mathbb{R}$, and $h_0(0) := 1$. Note that $h_0(\omega) = 0$ if and only if $\omega \in \mathbb{Z} \setminus \{0\}$. Hence, $\hat{\chi}_\Omega(u) = 0$ if and only if $u_j \in \mathbb{Z} \setminus \{0\}$ for some j, $1 \le j \le n$. The lemma now follows immediately from Lemma 2.1. □

Proof of Theorem 1.1. The set $D = [0,1]^n$ is a tight orthogonal packing region for $\Omega = [0,1]^n$. To see this, note that Lemma 4.1 implies that D is an orthogonal packing region for Ω, and since each of Ω and D has measure 1, it is tight.

(i)⇒(ii). By hypothesis, $\mathcal{B}_\Lambda$ is an orthogonal set in $L^2([0,1]^n)$. We showed above that D is a tight orthogonal packing region for Ω. Now Theorem 3.2 applies to conclude that $\Lambda + D$ is a tiling of $\mathbb{R}^n$.

(ii)⇒(i). By hypothesis, $\Lambda + D$ is a cube tiling, so by Proposition 4.1, $\mathcal{B}_\Lambda$ is an orthogonal set in $L^2([0,1]^n)$. Clearly, $m(\Omega)m(D) = 1$, and since $\Lambda + D$ is a cube tiling, it is a fortiori a cube packing. So by Theorem 3.1, Λ is a spectrum. □

APPENDIX: EXTENDING ORTHOGONAL SETS OF EXPONENTIALS TO ORTHOGONAL BASES

This appendix determines in which dimensions n every orthogonal set of exponentials on the n-cube can be extended to an orthogonal basis of $L^2([0,1]^n)$.

THEOREM A.1. *In dimensions $n = 1$ and $n = 2$, any orthogonal set of exponentials can be completed to an orthogonal basis of exponentials of $L^2([0,1]^n)$. In dimensions $n \ge 3$, this is not always the case.*

Proof. We say that a cube packing $\Gamma + [0,1]^n$ is orthogonal if for distinct $\gamma, \mu \in \Gamma$,

$$\gamma_j - \mu_j \in \mathbb{Z} \setminus \{0\}, \quad \text{for some } j,\ 1 \le j \le n. \tag{A.1}$$

Now Proposition 4.1 (Keller's criterion) and Lemma 4.1 together imply that an orthogonal set of exponentials $\{e^{2\pi i\langle\gamma,x\rangle} : \gamma \in \Gamma\}$ in $L^2([0,1]^n)$ corresponds to an orthogonal cube packing using Γ. By Theorem 1.1, the question of whether an orthogonal set of exponentials in $L^2([0,1]^n)$ can be extended to an orthogonal basis of exponentials of $L^2([0,1]^n)$ is equivalent to asking whether the associated orthogonal cube packing in $\mathbb{R}^n$ can be completed to a cube tiling by adding extra cubes.

Using the known structure of one- and two-dimensional cube tilings, it is straightforward to check that a completion of any orthogonal cube packing is always possible.

(Two-dimensional cube tilings always partition into either all horizontal rows of cubes or all vertical columns of cubes.) We omit the details.

To show that extendability is not always possible in dimension 3, consider the set of four cubes $\{v^{(i)}+[0,1]^3 : 1 \le i \le 4\}$ in $\mathbb{R}^3$, given by

$$v^{(1)} = \left(-1, 0, -\frac{1}{2}\right),$$
$$v^{(2)} = \left(-\frac{1}{2}, -1, 0\right),$$
$$v^{(3)} = \left(0, -\frac{1}{2}, -1\right),$$
$$v^{(4)} = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right).$$

The orthogonality condition (A.1) is easily verified. The cubes corresponding to $v^{(1)}$ through $v^{(3)}$ contain $(0,0,0)$ on their boundary and create a corner $(0,0,0)$. Any cube tiling that extended $\{v^{(i)}+[0,1]^3 : 1 \le i \le 3\}$ would have to fill this corner by including the cube $[0,1]^3$. However, $[0,1]^3$ has nonempty interior in common with $v^{(4)}+[0,1]^3$.

This construction easily generalizes to $\mathbb{R}^n$ for $n \ge 3$. □

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