Polynomial Algorithms for Single Item Lot Sizing Models with Bounded Inventory and Backlogging or Outsourcing

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Abstract

This paper addresses a real life single item dynamic lot sizing problem arising in a refinery for crude oil procurement. It can be considered as a lot sizing problem with bounded inventory. We consider two managerial policies. With one policy, a part of the demand of a period can be backlogged and with the other, a part of the demand of a period can be outsourced. We define actuated inventory bounds and show that any bounded inventory lot sizing model can be transformed into an equivalent model with actuated inventory bounds. The concept of actuated inventory bounds significantly contributes to the complexity reduction. In the studied models, the production capacity can be assumed to be unlimited and the production cost functions to be linear but with fixed charges. The results can easily be extended to piecewise linear concave production cost functions. The goal is to minimize the total cost of production, inventory holding and backlogging or outsourcing. We show that the backlogging model can be solved in $O(T^2)$ time with general concave inventory holding and backlogging cost functions where $T$ is the number of periods in the planning horizon. The complexity is reduced to $O(T)$ when the inventory/backlogging cost functions are linear and there is no speculative motives to hold either inventory or backlogging. When the outsourcing levels are unbounded, we show that the outsourcing model can be transformed into an inventory/backlogging model. As a consequence, the problem can be solved in $O(T^2)$ time, if the outsourcing cost functions are linear with fixed charges even if the inventory holding cost functions are general concave functions. When the outsourcing level of a period is bounded from above by the demand of the period, which is the case in many application areas, we show that the outsourcing model can be solved in $O(T^2 \log T)$ time if the inventory holding and the outsourcing cost functions are linear.
Keywords

Capacitated lot sizing, actuated inventory bounds, dynamic programming, backlogging, outsourcing, complexity, algorithms

1 Introduction

This paper is motivated by a real life lot sizing problem arising in a refinery for crude oil procurement. The problem can be described as follows. Based on the forecasted customer requirements on final products for future periods, the refinery can compute its own requirement on crude oil. The crude oil is mainly bought from international market and perhaps occasionally in local market. Clearly, this is a purchase planning problem whereas the terminology used in lot sizing literature is mainly for production planning context. We still use the same terminology as in the literature by keeping in mind that the shipment from international market is equivalent to in-house production in manufacturing.

We investigate two managerial policies. In one policy, a part of the requirement may be backlogged. In the other, a part of the requirement may be bought from local market even though the cost per barrel from international market is much lower than from local market. This policy is relevant because the shipment cost from international market is very high whereas the transportation cost from the local market is negligible. In this policy, purchasing from local market can be seen as outsourcing or subcontracting (see Atamtürk and Hochbaum 2001) in manufacturing companies. As in many refineries, the main difficulty is the limited storage capacity; i.e., the number of tanks, whereas the delivery capacity from the international market (i.e. production capacity) can be considered as unlimited. In addition, because of maintenance or cleaning operations, the storage capacity may vary over time. Therefore, our problem in hand can be considered as a lot sizing model with time-varying bounded inventory. The two managerial policies described above correspond to backlogging and outsourcing, respectively. In our models, the production cost function at a period can be considered to include a fixed setup cost, which is simply the shipment cost, and a linear part proportional to the production quantity to represent the cost per barrel. More generally, this function is a piecewise linear concave function. In spite of the real life application context, the result can be applied to general lot sizing problems arising in various other areas such as manufacturing, logistics, telecommunication, call centers (see Atamtürk and Hochbaum 2001).

The general single item capacitated dynamic lot size model can be described as follows. In a plant or a warehouse, there is a dynamic forecasted demand for a single item in each period of the $T$-period horizon. These demands can be satisfied by production and/or by inventory from previous periods and/or by backlogging to subsequent periods or partially or entirely outsourced. Some authors considered immediate lost sales models (Sandbothe and Thompson 1990 and 1993, Aksen, Altinkemer and Chand 2003). We prefer talking about outsourcing instead of lost sales, since in current
increasingly competitive environment, all companies make their best to satisfy customer requirements and to avoid lost sales. Deliberate lost sales are unrealistic especially when demands are assumed to be deterministic. However, from a modelling point of view, lost sales are equivalent to partially or entirely outsourcing the demand of customers, for various reasons (e.g. for small quantity, outsourcing subcontractors is cheaper than producing). Outsourcing which is becoming a common practice in industry needs to be addressed. In the real life application that motivated this work, outsourcing consists of purchasing required crude oil from local market. It must be kept in mind that the results of this paper can also be applied to actual lost sales models.

In a lot sizing model, there may be limitations on the production, inventory and backlogging or outsourcing levels. Three kinds of costs must be taken into account: Production cost, holding cost and backlogging or outsourcing cost. Setup cost can be included in the production cost. The problem consists of determining the quantity to be produced or purchased and/or to be outsourced in each period in order to satisfy each demand and to minimize the total cost of production, inventory holding and backlogging or outsourcing over the planning horizon.

In the literature, the following families of models have been investigated:

- No backlogging and no outsourcing models. The demand of every period must be entirely delivered; i.e., the demand must be satisfied by the production and/or by inventory from previous periods.

- Models with backlogging. Backlogging is allowed, and the demand of every period can be delivered at a later period at the expense of backlogging cost; i.e., the entire demand must be satisfied by the production and/or by inventory from previous periods and/or by backlogging to subsequent periods.

- Outsourcing models. The demand does not have to be entirely met by in-house production in all periods. It may be outsourced, even if the inventory level is strictly positive. Such a period is called conservation period. Conservation is possible when the outsourcing cost is very high at one of the subsequent periods. Conservation in the current period can also prove cost-saving when in-house production is too expensive in those subsequent periods whereas outsourcing is relatively cheap in the current period. In these cases, one may prefer conserving inventory at hand and outsourcing current period’s demand; i.e., keeping items in inventory to meet the demand of those subsequent periods instead of using them to meet the demand of the current period.

The first lot sizing model is un-capacitated and was studied by Wagner and Whitin (1958). In their model, backlogging is not permitted. The setup cost is constant. The production and inventory holding cost functions are linear and stationary. They provided an \( O(T^2) \) dynamic programming
method. Federgruen and Tzur (1991), Aggarwal and Park (1993), and Wagelmans et al. (1992) provided simple forward algorithms to solve the problem in $O(T \log T)$ time with linear but time-varying cost functions. Aggarwal and Park (1993), and Wagelmans et al. (1992) showed that the complexity is reduced to $O(T)$ if the cost functions are stationary or time-varying but verifying a special property that Atamtürk and Hochbaum (2001) called having no speculative motives to hold inventory.

For the un-capacitated setting with backlogging, Zangwill (1966) generalized the work of Wagner and Whitin (1958) by allowing backlogging. He proposed a strongly polynomial dynamic programming algorithm with general concave holding and backlogging cost functions and a limit on the number of periods that a backorder stays on the books.

A lot of work has been done in capacitated lot sizing models without backlogging. The computational complexity was investigated by Florian et al. (1980) and Bitran and Yanasse (1982). Florian et al. (1980) assumed that production and holding cost functions were continuous and nondecreasing and time-varying. They showed that this problem was NP-hard for quite general objective functions and provided a pseudo-polynomial dynamic programming algorithm. Some research has been done to identify polynomially solvable special cases. Bitran and Yanasse (1982) studied the computational complexity with linear production and holding cost functions with constant production capacity. Florian and Klein (1971) proposed an $O(T^4)$ dynamic programming algorithm with constant production capacity and concave production and inventory holding cost functions. Janannathan and Rao (1973) extended their results to a more general production cost function which was neither concave nor convex. Love (1973) investigated properties of problems with production capacity and inventory bounds and concave production and inventory/backlogging cost functions, introduced decomposition of an optimal production plan into subplans and gave an $O(T^3)$ algorithm for unlimited production capacity by assuming that the value of the concave functions can be computed in constant time. Gutiérrez et al. (2002) proposed necessary conditions of an optimal production plan to reduce the computational effort of Love’s procedure but with a constant inventory bound. The same authors ([?]) showed that the results remain valid with time-varying inventory bounds and backlogging. Van Hoesel and Wagelmans (1996) proposed an $O(T^3)$ algorithm for a model with constant production capacity, concave production cost and linear holding cost. Baker et al. (1978) exploited properties of an optimal plan in the case of time-varying production capacity and setup cost, time-varying holding and production cost. They devised an $O(2^T)$ algorithm. Chung and Lin (1988) obtained an $O(T^2)$ dynamic programming algorithm for problems with linear production and holding costs where setup costs and the unit production costs are nonincreasing in time and the capacities are nondecreasing in time. Chen et al. (1994) proposed an efficient pseudo-polynomial dynamic programming approach by a geometric argument for a capacitated model with linear production and holding cost functions and time-varying capacities.
For the capacitated lot sizing with backlogging decisions, Swoveland (1975) gave a pseudo-polynomial dynamic programming algorithm for the backlogging model with piecewise concave production and holding/backlogging cost functions. This model is obviously NP-hard, since its counterpart without backlogging is already NP-hard. Shaw and Wagelmans (1998) developed a pseudo-polynomial algorithm with running time linearly dependent on the magnitude of the data for piecewise linear production costs and general holding cost functions. Van Hoesel and Wagelmans (2001) developed fully polynomial approximation schemes for concave backlogging and production cost functions and arbitrary monotone holding cost functions.

In production planning context, concave cost functions arise much more frequently than convex cost functions. Problems with convex cost functions are also much more complicated. This perhaps explains why most of the papers deal with linear and/or concave functions. Lee and Nahmias (1993) indicated that concavity occurs when there are declining marginal production costs or other scale economies.

For the single-item lot sizing problem with outsourcing, very few papers have been published. Sandbothe and Thompson (1990) presented a capacitated model with stockouts. They proved necessary conditions for an optimal solution with constant costs and presented an $O(T^3)$ forward dynamic programming algorithm when production capacity is constant and an $O(2^T)$ algorithm for the model with time-varying production capacity for the first $k + 1$ periods with computable $k$. They considered later the same problem but with bounded inventory (Sandbothe and Thompson 1993). A forward algorithm was proposed and the amount of computational effort was multiplied by a factor of 4. Aksen et al. (2003) considered an un-capacitated single-item lot sizing problem with outsourcing and proposed an $O(T^2)$ forward recursive dynamic programming algorithm with time-varying linear production, inventory and outsourcing cost functions. Atamtürk and Hochbaum (2001) showed that the model with concave production and outsourcing cost functions and constant production capacity but with unbounded outsourcing level can be solved in $O(T^5)$ time. They also investigated other polynomially solvable models with nonspeculative cost functions.

As described earlier, this paper investigates real life single item lot sizing models with bounded inventory under backlogging or outsourcing managerial policies. In the backlogging model, outsourcing is not permitted whereas backlogging is forbidden in the outsourcing model.

We first introduce actuated inventory bounds which do not alter the set of feasible solutions but significantly simplify the analysis and problem solving. This property also holds for all bounded inventory models. Even though Atamtürk and Küçükyavuz (2005) implicitly assumed actuated inventory bounds but without backlogging, we explicitly give a formal definition and make clear the advantages to use actuated inventory bounds. In particular, we deduce an interesting relation which will play an very important role in the complexity reduction for both inventory/backlogging and outsourcing models.
We show that the inventory/backlogging model can be solved in $O(T^2)$ time with general concave inventory/backlogging cost functions. The complexity is reduced to $O(T)$ when the inventory holding/backlogging cost functions are linear and there is no speculative motives to hold inventory or backlogging. This has been possible due to an appropriate definition of matrices so that these matrices are Monge (see Aggarwal and Park 1993) even with inventory bounds and cumulative costs can be seen as column minima of these matrices. This avoids individually computing the cost of subplans. The cumulative cost to a period can be computed with an on-line algorithm based on the results of previous periods.

We then show that the outsourcing model is solvable in $O(T^2 \log T)$ time when the inventory holding and outsourcing cost functions are linear. The key points of this method are the decomposition of a subplan into two subproblems and then handling the remaining link between subproblems in a very simple way due to an analytical property. Another key point of the method is using the optimal solution of a subproblem to solve another subproblem. This makes it possible to use a same procedure to simultaneously solve several subproblems.

The remainder of the paper is organized as follows. Section 2 gives the mathematical formulation of the models studied in the paper. Section 3 defines actuated bounds and show related properties which significantly simplify the analysis. These results are valid for any bounded inventory model. Section 4 shows that the general model with linear production cost functions and fixed charges can be solved in $O(T^2)$ time, generalizes the result to piecewise linear concave production cost functions and addresses a linear time solvable special case. Section 5 addresses an outsourcing model. Section 6 ends the paper with some concluding remarks.

## 2 Mathematical Formulation

The following notation will be used throughout the paper.

- $T$: number of periods in the planning horizon
- $d_t$: demand of period $t$ ($d_t \geq 0, t = 1, \ldots, T$)
- $I_t$: inventory level at the end of period $t$
- $S_{t-1}$: lower limit of the storage level ($S_{t-1} \leq 0$, $-S_{t-1}$ is the upper limit of the backlogging level) at period $t$
- $S_{t,1}$: upper limit (capacity) of the storage level at period $t$ ($S_{t,1} \geq 0$)
- $p_t(\cdot)$: production cost function at period $t$
- $h_t(\cdot)$: inventory holding/backlogging cost function at period $t$
- $\lambda_t(\cdot)$: outsourcing cost function at period $t$
- $x_t$: production level at period $t$
- $L_t$: outsourcing level at period $t$
where \( x_t \) and \( L_t \) are decision variables and \( I_t \) are state variables, \( t = 1, 2, \ldots, T \).

We also make the following realistic assumptions

- Without loss of generality, the initial inventory level (at the end of period 0) is 0. Otherwise, we can always transform the problem into an equivalent one with zero initial inventory by reducing the demands of the first periods. This assumption is equivalent to \( S_{0,-1} = S_{0,1} = 0 \).

- Similarly, the holding level at the end of period \( T \) must be 0. This assumption is equivalent to \( S_{T,-1} = S_{T,1} = 0 \).

In the remainder, we set \( S_{t,0} = 0 \) for \( t = 0, 1, 2, \ldots, T \), in order to uniform the analysis. As a consequence, we have:

\[
S_{t,-1} \leq S_{t,0} \leq S_{t,1} \quad t = 0, 1, \ldots, T
\]  

(1)

If \( S_{t,-1} \leq I_t \leq S_{t,0} \), \( t \) is a backlogging period. If \( S_{t,0} \leq I_t \leq S_{t,1} \), \( t \) is a holding period.

For the sake of simplicity, we will use the following notation

\[
d_{i,j} = \sum_{t=i+1}^{j} d_t
\]

With this notation, the cumulative demand of periods \( i + 1 \) to \( j \) with \( 0 \leq i \leq j \leq T \) is simply \( d_{i,j} = d_{0,j} - d_{0,i} \). Furthermore, all \( d_{0,i} \)'s can be recursively computed as follows in \( O(T) \) time.

\[
\begin{cases}
  d_{0,0} = 0 \\
  d_{0,i} = d_{0,i-1} + d_i, \quad i = 1, \ldots, T
\end{cases}
\]

The general capacitated single item lot sizing model can be formulated as follows.

\[
\text{Minimize} \sum_{t=1}^{T} [p_t(x_t) + h_t(I_t) + \lambda_t(L_t)]
\]

(2)

subject to the following state equations and constraints

\[
I_t - I_{t-1} = x_t + L_t - d_t \quad t = 1, 2, \ldots, T
\]

(3)

\[
S_{t,-1} \leq I_t \leq S_{t,1} \quad t = 1, 2, \ldots, T - 1
\]

(4)

\[
x_t \geq 0 \quad t = 1, 2, \ldots, T
\]

(5)

\[
L_t \geq 0 \quad t = 1, 2, \ldots, T
\]

(6)

\[
I_0 = I_T = 0
\]

(7)

In fact, production and outsourcing are both sources of items. Therefore, \( x_t + L_t \) is the input quantity to the inventory at period \( t \) and \( d_t \) is the output quantity from the inventory. This explains state equation (3). An alternative explanation when the outsourcing level is less than the demand is that \( x_t \) is the input quantity whereas \( d_t - L_t \) is the quantity actually delivered to the customer, hence output quantity from the inventory.
Throughout the paper, we assume time-varying linear production cost functions with fixed charges (i.e., nonnegative setup costs), both for backlogging models and outsourcing models. To be more specific, the production cost function at period $t$ is

$$p_t(x) = \begin{cases} 
0 & \text{if } x = 0 \\
K_t + \pi_t x & \text{otherwise}
\end{cases}$$

where $K_t \geq 0$ is the setup cost at period $t$ and $\pi_t \geq 0$ is the unit production cost at period $t$. We will show that the result can be easily extended to general concave piecewise linear production cost functions. In the backlogging model, we first consider concave inventory holding and backlogging cost functions $h_t(\cdot)$ ($t = 1, 2, \ldots, T - 1$) which are concave over $(-\infty, 0)$ and over $(0, +\infty)$ with $h_t(0) = 0$.

Then we show that if the inventory/backlogging cost functions are linear and there is no speculative motives to hold either inventory or backlogging, then the problem can be solved in linear time. In the outsourcing model, we consider linear inventory and outsourcing cost functions.

3 Actuated inventory bounds and general properties

Due to various constraints, not all values between $S_{t-1}$ and $S_{t,1}$ are feasible for $I_t$. In order to avoid redundant computation related to infeasible states, we determine actuated lower and upper bounds on inventory levels denoted as $S'_{t-1}$ and $S'_{t,1}$ for period $t$, respectively. From the state equation (3), we have

$$I_{t-1} = I_t - x_t - L_t + d_t$$

Due to constraint (5) and (6), we obtain $I_{t-1} \leq I_t + d_t$. Therefore, if $S'_{t,1}$ is a valid upper bound of $I_t$, then $S'_{t-1,1} = \min(S'_{t,1} + d_t, S_{t-1,1})$ is a valid upper bound of $I_{t-1}$. These actuated upper bounds can be recursively obtained as follows.

$$
\begin{cases}
S'_{T,1} = 0; \\
S'_{t,1} = \min(S'_{t+1,1} + d_{t+1}, S_{t,1}), & t = 0, 1, \ldots, T - 1
\end{cases}
$$

Similarly, actuated lower bounds of the inventory level at the end of period $t$, denoted as $S'_{t,-1}$, $t = 1, 2, \ldots, T$ can be recursively obtained as follows.

$$
\begin{cases}
S'_{0,-1} = 0; \\
S'_{t,-1} = \max(S'_{t-1,-1} - d_t, S_{t,-1}), & t = 1, 2, \ldots, T
\end{cases}
$$

We now show a general property for all bounded inventory lot sizing models. This property simplifies the analysis of these models.

From (8), we must have $S'_{t,1} \leq S'_{t+1,1} + d_{t+1}$; i.e.,

$$S'_{t,1} + d_{0,t} \leq S'_{t+1,1} + d_{0,t+1}$$
Similarly, we can show that

\[ S'_{t,1} + d_{0,t} \leq S'_{t+1,1} + d_{0,t+1} \]

By definition, we have the following Property.

**Property 1** In a bounded inventory lot sizing problem, replacing \( S_{t,1} \) and \( S_{t,-1} \) with \( S'_{t,1} \) and \( S'_{t,-1} \), respectively, does not alter the set of feasible solutions and hence the optimal solutions.

In the remainder, we assume without loss of generality that the inventory bounds are actuated ones. All assumptions related to the original inventory bounds still hold for the actuated bounds. Firstly, we have \( S'_{t,1} \geq 0 \) for all \( t \) such that \( 0 \leq t \leq T \). This can be shown recursively. By the definition, we have \( S'_{T,1} = 0 \geq 0 \). Assume now that there is a \( t \) such that \( 1 \leq t \leq T \) and \( S'_{t,1} \geq 0 \). By the assumptions, we have \( S_{t-1,1} \geq 0 \) and \( S'_{t-1,1} + d_t \geq S_{t-1,1} \geq 0 \). Therefore, \( S'_{t-1,1} = \min(S_{t-1,1}, S'_{t-1,1} + d_t) \geq 0 \). Similarly, we can show that \( S'_{t,-1} \leq 0 \) for all \( t \) such that \( 0 \leq t \leq T \). As a consequence, (1) remains true.

Considering the fact that \( d_t \geq 0 \), we have \( S_{t,0} + d_t = d_t \geq 0 = S_{t+1,0} \) for any \( t = 0, \ldots, T-1 \).

With the actuated bounds, by using the following notation for the sake of simplification

\[ \sigma_{t,v} = S_{t,v} + d_{0,t}, \quad t = 0, 1, \ldots, T, v \in \{-1,0,1\} \]

we have the following interesting relation

\[ \sigma_{t,v} \leq \sigma_{t+1,v}, \quad t = 0, 1, \ldots, T-1, v \in \{-1,0,1\} \]

Therefore,

\[ \begin{align*}
\sigma_{\tau,v} & \leq \sigma_{t,v}, \quad 0 \leq \tau \leq t \leq T, v \in \{-1,0,1\} \\
\sigma_{t,-1} & \leq \sigma_{t,0} \leq \sigma_{t,1}, \quad t = 0, 1, \ldots, T
\end{align*} \]  \hspace{1cm} (9)

In fact, \( \sigma_{t,1} \) and \( \sigma_{t,-1} \) can be interpreted as upper and lower bounds of the cumulative production from period 1 to period \( t \), respectively. And \( \sigma_{t,0} \) is simply the cumulative demand from period 1 to period \( t \).

Relation (9) will play a crucial role in the remainder of this paper. With this relation, we have the following property proved in Appendix A which avoids many checks related to inventory bounds and hence leads to significant complexity reduction.

**Property 2** When there is no outsourcing, a production plan is feasible if and only if \( \sum_{t=1}^{T} x_t = d_{0,T} \) and for any \( t \) such that \( 1 \leq t \leq T \) and \( x_t > 0 \) (such a period is called production period), we have \( I_{t-1} \geq S_{t-1,-1} \) and \( I_t \leq S_{t,1} \).

From this property, it is unnecessary to check whether the inventory bounds are satisfied for every period.
Define, for each $u$, $v$ and $t$ with $u, v \in \{-1, 0, 1\}$ and $0 \leq t \leq T$, the following periods

$$a_u(t) = \begin{cases} \max\{\tau | t \leq \tau \leq T, S_{t,u} - d_{t,\tau} > S_{t,0}\} = \max\{\tau | t \leq \tau \leq T, \sigma_{t,0} < \sigma_{t,u}\} & \text{if } u = 1 \\ t - 1 & \text{otherwise} \end{cases}$$

$$a'_u(t) = \max\{\tau | t \leq \tau \leq T, S_{t,u} - d_{t,\tau} \geq S_{t,-1}\} = \max\{\tau | t \leq \tau \leq T, \sigma_{t,-1} \leq \sigma_{t,u}\}$$

$$b_{u,v}(t) = \begin{cases} \min\{\tau | 0 \leq \tau \leq t, S_{t,v} + d_{t,\tau} \leq S_{t,u}\} = \min\{\tau | 0 \leq \tau \leq t, \sigma_{t,v} \geq \sigma_{t,u}\} & \text{if } u \geq v \\ t + 1 & \text{otherwise} \end{cases}$$

$$b'_v(t) = \max\{\tau | 0 \leq \tau < t, S_{t,v} + d_{t,\tau} \geq S_{t,0}\} = \max\{\tau | 0 \leq \tau < t, \sigma_{t,0} \leq \sigma_{t,v}\}$$

If the inventory level of period $\tau$ is $S_{t,u}$, then the inventory level at a subsequent period $t$ is $S_{t,u} - d_{t,\tau}$ if there is no production or outsourcing between these two periods. Similarly, for the inventory level of a period $t$ to be $S_{t,v}$, the inventory level at an earlier period $\tau$ must be $S_{t,v} + d_{\tau,t}$ if there is no production or outsourcing between these two periods. As a consequence, $a_1(t)$ is the last holding period after (or at) period $t$ if the inventory level at period $t$ is $S_{t,1}$. Similarly, $b_{u,v}(t)$ is the earliest period $\tau$ such that an inventory level $S_{t,u}$ at period $\tau$ is enough to ensure that the inventory level at period $t$ is at least $S_{t,v}$ without any production between $\tau$ and $t$.

By the definition, all series $\{a_u\}$, $\{a'_u\}$, $\{b_{u,v}\}$ and $\{b'_v\}$ are nondecreasing from (9). All these periods can be computed in $O(T)$ time. The following procedure gives the computation of $a_1(t)$’s. The other periods can be computed in a similar way.

1. $t := \tau := 0$;
2. While $t \leq T$ do
   If $\tau > T$ or $\sigma_{t,1} \leq \sigma_{t,0}$ then $a_1(t) := \tau - 1$, $t := t + 1$
   Otherwise $\tau := \tau + 1$.
3. Endwhile

In this procedure, at each iteration, either $t$ or $\tau$ is incremented whereas the number of values of $\tau$ or $t$ is bounded by $T$. Therefore, the complexity is $O(T)$.

We now recall some properties derived from the result of Love (1973). These properties hold because of the concavity of the cost functions.

**Definition 1** If $x_t \neq 0$, then $t$ is a production period.

**Definition 2** If $I_t = S_{t,v}$, $v \in \{-1, 0, 1\}$, then $t$ is an inventory point.

Note that by the assumptions, periods 0 and $T$ are inventory points. The following property proved in Appendix B is the basis of the algorithms of the paper.
Property 3 There is an optimal solution such that between every pair of production periods \( j < j' \), there is at least one inventory point \( k, j \leq k < j' \). Equivalently, between any two adjacent inventory points, \( i < k \), there can be at most one production period \( j, i < j \leq k \).

Definition 3 In a production plan, a subplan starts with an inventory point and ends with another inventory point. A subplan containing no other subplan; i.e., if there is no other inventory points than the beginning and ending ones, is called elementary subplan.

From Property 3, an optimal production plan can be decomposed into a series of subplans. There are many ways to decompose a production plan into subplans, since each subplan may contain more than one elementary subplan. In the remainder, we will see different ways of decomposition according to the facility of analysis.

4 An inventory/backlogging model

In this section, we consider an inventory/backlogging model with inventory bounds. Love (1973) developed an \( O(T^3) \) algorithm when production cost functions are general concave functions. Using a polyhedral study, Atamtürk and Küçükyavuz (2005) showed that the model without backlogging but with fixed charged production cost functions and inventory holding cost functions can be solved in \( O(T^4) \) time and this complexity is reduced to \( O(T^2 \log T) \) if the inventory holding cost is linear. Blackburn and Kunreuther (1974), Lundin and Morton (1975) considered the same model of this section but with unbounded inventory and developed \( O(T^2) \) algorithms. We show that it can also be solved in \( O(T^2) \) time even if the inventory is bounded, as long as production cost functions \( p_t(\cdot) \) are linear with fixed charges. The result can be easily extended to concave piecewise linear functions. Obviously, the model considered by Atamtürk and Küçükyavuz (2005); i.e., without backlogging but with fixed charged linear (therefore concave) inventory holding cost functions, is a special case of the model studied here. In addition, it is not mentioned whether their results can be generalized to piecewise linear concave production cost functions. Using a capacitated network flow model, Ahuja and Hochbaum (2004) proved that the problem can be solved in \( O(T \log T) \) time even with production capacity, if all cost functions are linear without fixed charge. We will also show that if the inventory/backlogging cost functions are linear and there is no speculative motives to hold either inventory or backlogging, the problem can be solved in linear time. Table 1 summarizes the complexity result of different inventory/backlogging models with inventory bounds. In this table, \( N \) is the total number of segments in the piecewise linear concave production cost functions.
Table 1: Complexity results for bounded inventory/backlogging models without production capacity

<table>
<thead>
<tr>
<th>Inv./back. cost functions $h_t(\cdot)$</th>
<th>Production cost functions $p_t(\cdot)$</th>
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</thead>
<tbody>
<tr>
<td>Concave PWL Concave Lin. with fixed charge Linear</td>
<td></td>
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<tr>
<td>(PWL) Concave</td>
<td>$O(T^3)$</td>
</tr>
<tr>
<td>Linear with fixed charge</td>
<td>$O(T^3)$</td>
</tr>
<tr>
<td>Linear</td>
<td>$O(T^3)$</td>
</tr>
<tr>
<td>Nonspeculative</td>
<td>—</td>
</tr>
</tbody>
</table>

0: This paper
1: Love 1973
2: Atamtürk and Küçükyavuz 2005
3: Ahuja and Hochbaum 2004
*: without backlogging
**: even with production capacity
—: impossible case

4.1 An $O(T^2)$ algorithm for general concave inventory holding/backlogging cost functions

According to Property 3, an optimal production plan can be decomposed into subplans. In this section, an optimal production plan is decomposed into subplans in such a way that each subplan contains exactly one production period. Note that as soon as $d_{0,T} > 0$ (otherwise, there would be no demand and the problem would be trivial), there is at least one such subplan, since we must have $\sum_{t=1}^{T} x_t = d_{0,T} > 0$.

Let $c_{u,v}(i,j,k)$, with $0 \leq i < j \leq k \leq T$ and $u,v \in \{-1,0,1\}$, be the minimal cost of the subplan from $i$ to $k$ with a single production period $j$ in such a way that $I_i = S_i,u$ and $I_k = S_k,v$ (i.e., $i$ and $k$ are inventory points). This cost is the cumulative (production, inventory holding and backlogging) cost from period $i+1$ to $k$ with a single production period $j$. In this subplan, the quantity produced at period $j$ must be $x_j = \sigma_{k,v} - \sigma_{i,u}$. Obviously, this subplan is feasible if and only if the following three conditions are fulfilled:

- the production level at production period $j$; i.e., $x_j = \sigma_{k,v} - \sigma_{i,u}$, is strictly positive. This requirement is equivalent to $i < b_{u,v}(k)$.

- The upper limit of inventory is respected. This requirement is equivalent to $j \geq b_{1,v}(k)$ (see
• The lower limit of inventory is respected. This requirement is equivalent to $j - 1 \leq a'_u(i)$ (see Property 2).

If at least one of these requirements is not fulfilled, $c_{u,v}(i, j, k) = +\infty$, without loss of generality.

Let $E_w(t)$, with $0 \leq t \leq T$ and $w \in \{-1, 0, 1\}$, be the minimal cumulative (production, inventory and backlogging) cost from period 0 to period $t$ such that the inventory level at the end of period $t$ is $S_{t,w}$. According to Property 3, we have the following relations which respectively give the starting condition and the recursive equation for the dynamic programming approach.

$$
\begin{align*}
E_w(0) &= 0, \quad w = -1, 0, 1 \\
E_v(k) &= \min_{-1 \leq u \leq 1, 0 \leq t < b_{u,v}(k)} \left\{ E_u(i) + \min_{i < j \leq k} c_{u,v}(i, j, k) \right\} \quad 0 < k \leq T, v = -1, 0, 1
\end{align*}
$$

The critical step of this recursion comes from the computation of $c_{u,v}(i, j, k)$. Our idea consists of reducing the complexity to compute these quantities. The basic idea is to decompose this cost into the sum of two terms, each of which depends on only two periods. In fact, as soon as production period $j$ is known, $i$ and $k$ can be decoupled. For this purpose, define $A_u(i, j)$ (respectively $B_v(j, k)$) as the minimal cumulative inventory and backlogging cost from $i + 1$ to $j - 1$ (respectively from $j$ to $k$) such that the inventory level at $i$ (respectively $k$) is $S_{i,u}$ (respectively $S_{k,v}$).

By the definition, $A_u(i, j)$’s with $0 \leq i < j \leq T$, $u \in \{-1, 0, 1\}$ and $B_v(j, k)$’s with $1 \leq j \leq k \leq T$ and $v \in \{-1, 0, 1\}$ can be computed recursively as follows.

$$
\begin{align*}
A_u(\tau, \tau + 1) &= 0 \\
A_u(\tau, t + 1) &= \begin{cases} 
A_u(\tau, t) + h_t(\sigma_{\tau,u} - d_{0,t}), & \text{if } \tau < t \leq a'_u(\tau) \\
+\infty, & \text{otherwise}
\end{cases} \\
B_v(t, t) &= 0 \\
B_v(\tau, t) &= \begin{cases} 
B_v(\tau + 1, t) + h_\tau(\sigma_{t,v} - d_{0,\tau}), & \text{if } b_{1,v}(t) \leq \tau < t \\
+\infty, & \text{otherwise}
\end{cases}
\end{align*}
$$

From these relations, we can see that $A_u(i, j)$’s and $B_v(j, k)$’s can be computed in $O(T^2)$ time, if we consider that the computation of the inventory/backlogging cost of each period requires a constant time.

The cost of the subplan $c_{u,v}(i, j, k)$ can be expressed as follows:

$$
c_{u,v}(i, j, k) = A_u(i, j) + p_j(\sigma_{k,v} - \sigma_{i,u}) + B_v(j, k) = A'_u(i, j) + B'_v(j, k)
$$

(11)
where

\[ A'_u(i, j) = A_u(i, j) - \pi_j \sigma_{i,u} \]
\[ B'_v(j, k) = K_j + \pi_j \sigma_{k,v} + B_v(j, k) \]

By the definition, under the assumption that the computation of the inventory/backlogging cost of each period requires a constant time, \(A'_u(i, j)\) for given \((i, j)\) and \(u\) can be computed in constant time, knowing \(A_u(i, j)\) and \(\pi_{i,u}\). Therefore, all \(A'_u(i, j)\)'s can be computed in \(O(T^2)\) time. Similarly, all \(B'_v(j, k)\)'s can also be computed in \(O(T^2)\) time.

Therefore, (10) can be rewritten as

\[ E_v(k) = \min_{-1 \leq u \leq 1} \left\{ E_u(i) + A'_u(i, j) + B'_v(j, k) \right\} \]  \hspace{1cm} (12)

By changing minimization order in (12), we obtain

\[ E_v(k) = \min_{-1 \leq u \leq 1} \left\{ B'_v(j, k) + \min_{0 \leq i < \min[j, b_{u,v}(k)]} \left[ E_u(i) + A'_u(i, j) \right] \right\} \]  \hspace{1cm} (13)

In the following, for any \(u \in \{-1, 0, 1\}\), define

\[ F_u(j, l) = \min_{0 \leq i < l} \{ E_u(i) + A'_u(i, j) \} \] with \(l \leq j\)

We obtain

\[ E_v(k) = \min_{1 \leq j \leq k, -1 \leq u \leq 1} \left\{ B'_v(j, k) + F_u(j, \min[j, b_{u,v}(k)]) \right\} \]  \hspace{1cm} (14)

By recursion, for any \(j\) such that \(1 \leq j \leq T\), \(E_u(l)\) is known for all \(l\) such that \(0 \leq l < j\), \(F_u(j, l + 1)\)'s can be computed in a recursive way

\[
\begin{align*}
F_u(j, 0) &= +\infty \\
F_u(j, l + 1) &= \min\{F_u(j, l), E_u(l) + A'_u(l, j)\}, \quad \forall 0 \leq l < j
\end{align*}
\]  \hspace{1cm} (15)

Relations (14) and (15) give the recursive equations to compute alternatively \(E_v(k)\) and \(F_u(j, l)\) for all \(u, v \in \{-1, 0, 1\}\) and \(j, k = 1, 2, \ldots, T\) and \(l = 0, 1, \ldots, j\). The optimal value of the objective function is \(E_0(T)\). From the equations, we need to compute \(O(T^2)\) values of \(F_u(j, l)\)'s and \(O(T)\) values of \(E_v(k)\)'s. The computation of each value of \(F_u(j, l)\) requires \(O(1)\) time and the computation of each value of \(E_v(k)\) needs to consider all \(j\) such that \(1 \leq j \leq k\) and therefore \(O(k)\) additions and comparisons. It then requires \(O(T^2)\) time to compute all these quantities.
4.2 Generalization to concave piecewise linear production cost functions

In this subsection, we show that the results of the previous subsection can be easily extended to concave piecewise linear production cost functions. If $p_t(x)$ is a concave piecewise linear function then $p_t(x)$ can be expressed as follows.

$$p_t(x) = \min_{1 \leq q \leq n_t} (K_{t,q} + \pi_{t,q}x)$$

with

$$K_{t,1} < K_{t,2} < \ldots < K_{t,n_t}$$
$$\pi_{t,1} > \pi_{t,2} > \ldots > \pi_{t,n_t}$$

where $n_t$ is the number of segments.

Relation (11) becomes

$$c_{u,v}(i, j, k) = A_u(i, j) + p_j(\sigma_{k,v} - \sigma_{i,u}) + B_v(j, k)$$
$$= A_u(i, j) + \min_{1 \leq q \leq n_j} [K_{j,q} + \pi_{j,q}(\sigma_{k,v} - \sigma_{i,u})] + B_v(j, k)$$
$$= \min_{1 \leq q \leq n_j} [A'_u(i, j, q) + B'_v(j, k, q)]$$

(16)

where

$$A'_u(i, j, q) = A_u(i, j) - \pi_{j,q}\sigma_{i,u}$$

$$B'_v(j, k, q) = K_{j,q} + \pi_{j,q}\sigma_{k,v} + B_v(j, k)$$

Similarly as in the previous subsection, all $A'_u(i, j, q)$'s and $B'_v(j, k, q)$'s can be obtained in $O(NT)$ time where $N = \sum_{t=1}^{T} n_t$.

Relation (13) becomes

$$E_v(k) = \min_{-1 \leq u \leq 1 \atop 1 \leq i < j \leq T \atop 1 \leq q \leq n_j} \left\{ B'_v(j, k, q) + \min_{0 \leq i < \min[j,b_{u,v}(k)]} [E_u(i) + A'_u(i, j, q)] \right\}$$

Similarly to the previous subsection, define

$$F_u(j, q, l) = \min_{0 \leq i < l} \{E_u(i) + A'_u(i, j, q)\} \text{ with } l \leq j$$

We obtain

$$E_v(k) = \min_{-1 \leq u \leq 1 \atop 1 \leq j \leq T \atop 1 \leq q \leq n_j} \{B'_v(j, k, q) + F_u(j, q, \min[j, b_{u,v}(k)])\}$$

As in the previous subsection, $E_v(k)$ and $F_u(j, q, l)$ can be computed alternately and recursively. We need to compute $O(NT)$ values of $F_u(j, q, l)$'s and $O(T)$ values of $E_v(k)$'s. The computation of each value of $F_u(j, q, l)$ requires $O(1)$ time and that of each value of $E_v(k)$ needs to consider all $j$ such that $1 \leq j \leq k$ and all $q$ such that $1 \leq q \leq n_j$, therefore $O(N)$ additions and comparisons, where $N = \sum_{t=1}^{T} n_t$. It then requires $O(NT)$ time to compute these quantities.
4.3 A linear time solvable special case

We now consider a special case solvable in $O(T)$ time. In this model, the inventory/backlogging cost functions are linear; namely,

$$h_t(y) = \begin{cases} h_t y & \text{if } y \geq 0 \\ -g_t y & \text{otherwise.} \end{cases}$$

where $h_t \geq 0$ and $g_t \geq 0$ are unit inventory holding cost and unit backlogging cost of period $t$, respectively. Furthermore, we assume that there is no speculative motive to hold either inventory or backlogging; i.e., $\pi_t + h_t \geq \pi_{t+1} + g_t$ for all $t$ such that $1 \leq t \leq T - 1$. Note that stationary unit production cost is a special case of the model studied here. Morton (1978) considered an even more special case with unbounded inventory and stationary unit production cost and unit inventory holding and backlogging cost. He developed an $O(T^2)$ algorithm. Aggarwal and Park (1993) extended the model of Morton to the same cost structure as ours but still without inventory bounds. They showed that the problem can be solved in linear time. We show hereafter that it can still be solved in linear time even with bounded inventory.

For the sake of simplicity, we use the following notation

$$h_{r,t} = \sum_{l=\tau+1}^{t} h_l, \quad g_{r,t} = \sum_{l=\tau+1}^{t} g_l, \quad H_{r,t} = \sum_{l=\tau+1}^{t} h_0 d_{0,t}, \quad G_{r,t} = \sum_{l=\tau+1}^{t} g_0 d_{0,t}$$

By definition, $h_{0,t}$'s, $g_{0,t}$'s, $H_{0,t}$'s and $G_{0,t}$'s can be computed in $O(T)$ time.

With the assumptions, we have

$$\pi_{\tau} - h_{0,\tau-1} \geq \pi_{\tau} - h_{0,\tau-1}, \quad 1 \leq \tau < t \leq T$$

$$\pi_{\tau} + g_{0,\tau-1} \leq \pi_{\tau} + g_{0,\tau-1}, \quad 1 \leq \tau < t \leq T$$

(17) (18)

The problem can be formulated as follows.

$$\text{Minimize } \sum_{t=1}^{T} [K_t y_t + \pi_t x_t + h_t \max(I_t, 0) + g_t \max(-I_t, 0)]$$

subject to the following state equations and constraints

$$I_t - I_{t-1} = x_t - d_t \quad t = 1, 2, \ldots, T$$

(20)

$$S_{t-1} \leq I_t \leq S_{t-1} \quad t = 1, 2, \ldots, T - 1$$

(21)

$$0 \leq x_t \leq M y_t \quad t = 1, 2, \ldots, T$$

(22)

$$y_t \in \{0, 1\} \quad t = 1, 2, \ldots, T$$

(23)

$$I_0 = I_T = 0$$

(24)

where $y_t$ is a binary variable indicating whether $t$ is a production period, $M$ is an arbitrarily large number.
One of the bottlenecks of the method of Subsection 4.1 comes from the condition \( i < b_{u,v}(k) \) to guarantee nonnegative production level. Because of this condition, \( i \) and \( k \) cannot be decoupled. Therefore, to reduce the complexity, it is necessary to remove this condition which is equivalent to considering a relaxed problem where strictly negative production is allowed. We will see later that this relaxation does not alter an optimal solution. This relaxed problem is obtained by replacing constraints (22) with the following one.

\[
-M y_t \leq x_t \leq M y_t \quad t = 1, 2, \ldots, T
\]

Note that in this relaxed problem, \( x_t \neq 0 \) (either \( x_t > 0 \) or \( x_t < 0 \)) implies \( y_t = 1 \).

We will show that there is an optimal solution to the relaxed problem such that the production level is nonnegative, therefore, feasible to the original one. As a consequence, solving the relaxed problem is equivalent to solving the original one. This result is based on the following theorem, which is a common knowledge with the assumptions (see (17) and (18)). We still give the proofs in Appendix C for readers unfamiliar with these assumptions.

**Theorem 1** There is an optimal solution to the relaxed problem such that for any production period \( t, 1 \leq t \leq T \), we have \( I_{t-1} \leq 0 \) and \( I_t \geq 0 \).

From this theorem, we obtain the following corollary which indicates that there is an optimal solution to the relaxed problem which is also feasible (hence optimal as well) to the initial problem.

**Corollary 1** There is an optimal solution to the relaxed problem such that \( x_t \geq 0 \) for any \( t = 1, 2, \ldots, T \).

**Proof.** From Theorem 1, there is an optimal solution to the relaxed problem such that \( I_{t-1} \leq 0 \) and \( I_t \geq 0 \) for any production period \( t \) (1 \( \leq t \leq T \)). Therefore \( x_t = I_t - I_{t-1} + d_t \geq d_t \geq 0 \).

From this corollary, omitting \( b_{u,v}(k) \), or equivalently replacing \( b_{u,v}(k) \) with \( +\infty \), in (14) will not change the optimal solution.

Furthermore, from Theorem 1, \( A'_u(i, j) \) can be considered to be undefined if \( j \leq a_u(i) + 1 \). Otherwise; i.e., if \( a_u(i) + 1 < j \leq a'_u(i) + 1 \), \( A'_u(i, j) \) can be computed as follows.

\[
A'_u(i, j) = \sum_{m=i+1}^{a_u(i)} h_m (\sigma_{i,u} - d_{0,m}) + \sum_{m=a_u(i)+1}^{j-1} g_m (d_{0,m} - \sigma_{i,u}) - \pi_j \sigma_{i,u}
\]

\[
= \sigma_{i,u} \left( h_{i,a_u(i)} + g_{0,a_u(i)} - H_{i,a_u(i)} - G_{0,a_u(i)} + G_{0,j-1} - \sigma_{i,u} (\pi_j + g_{0,j-1}) \right)
\]

Therefore, \( A'_u(i, j) \), for any \( 0 \leq i, j \leq T \), can be defined as follows.

\[
A'_u(i, j) = \left[ \sigma_{i,u} \left( h_{i,a_u(i)} + g_{0,a_u(i)} - H_{i,a_u(i)} - G_{0,a_u(i)} \right) + G_{0,j-1} - \sigma_{i,u} (\pi_j + g_{0,j-1}) + M \max \left[ a_u(i) + 2 - j, j - a'_u(i) - 1, 0 \right] \right]
\]
Similarly, \( B'_v(j, k) \) for any \( 0 \leq j, k \leq T \) can be defined as follows.

\[
B'_v(j, k) = \left[ \sigma_{k,v}(h_{0,b'_v(k)} - g_{b'_v(k)},k) - H_{0,b'_v(k)} + G_{b'_v(k),k} \right] \\
- \sigma_{k,v}(h_{0,j-1} - \pi_j) + M \max \left[ b_{1,v}(k) - j, j - b'_v(k), 0 \right]
\]

Let

\[
F'_u(j) = \min_{0 \leq i < T} (E_u(i) + \Lambda'_u(i, j)) \\
E'_{u,v}(k) = \min_{1 \leq j \leq T} (F'_u(j) + B'_v(j, k))
\]

We have

\[
E_v(k) = \min_{-1 \leq u \leq 1} E'_{u,v}(k)
\]

Let \( E_{u,v} \) and \( F_u \) be the matrices with \( e_{u,v}(j, k) \) \((1 \leq j, k \leq T)\) and \( f_u(i, j) \) \((0 \leq i \leq T - 1, 1 \leq j \leq T)\) as entries, respectively, where

\[
e_{u,v}(j, k) = F'_u(j) + B'_v(j, k) \\
= \left[ F'_u(j) + H_{0,j-1} + K_j \right] + \left[ \sigma_{k,v}(h_{0,b'_v(k)} - g_{b'_v(k)},k) - H_{0,b'_v(k)} + G_{b'_v(k),k} \right] \\
- \sigma_{k,v}(h_{0,j-1} - \pi_j) + M \max \left[ b_{1,v}(k) - j, j - b'_v(k), 0 \right]
\]

\[
f_u(i, j) = E_v(i) + \Lambda'_u(i, j) \\
= \left[ E_v(i) + \sigma_{i,u}(h_{i,a_u(i)} - g_{0,a_u(i)}) - H_{i,a_u(i)} - G_{0,a_u(i)} \right] + G_{0,j-1} \\
- \sigma_{i,u}(\pi_j + g_{0,j-1}) + M \max \left[ a_u(i) + 2 - j, j - a'_u(i) - 1, 0 \right]
\]

By definition, \( E'_{u,v}(k) \) and \( F'_u(j) \) are column minima of \( E_{u,v} \) and \( F_u \), respectively. On the other hand, since \( \sigma_{k,v}, h_{0,j-1} - \pi_j, b_{1,v}(k), b'_v(k), \sigma_{i,u}, \pi_j + g_{0,j-1}, a_u(i) + 2, a'_u(i) - 1 \) are nondecreasing series, from Appendix D, matrices \( E_{u,v} \) and \( F_u \) are Monge (Aggarwal and Park 1993). Furthermore, any entry of the matrices can be computed on-line in constant time. Therefore, the column minima of the matrices can be computed in linear time by an on-line algorithm (see Larmore and Schieber 1991).

In this algorithm, due to the Monge\'t, only \( O(T) \) entries in a matrix are examined. Furthermore, these entries are computed (on-line) in constant time only if they are examined. Therefore, the computation of these entries and the examination require \( O(T) \) time.

5 An outsourcing model

In this section, we consider an outsourcing model where one can keep the holding of a period to meet the demand of a future period instead of the demand of the current period. The demand of the current period may be partially or entirely outsourced. In this model, for each period \( t \), in addition to determining the production level \( x_t \), we must also decide the outsourcing level \( L_t \). Atamürk and Hochbaum (2001) showed that the model with concave production and outsourcing cost functions
and constant production capacity but with unbounded outsourcing level can be solved in \(O(T^3)\) time. They also investigated other polynomially solvable models with nonspeculative cost functions.

Indeed, outsourcing models can be transformed into general inventory/backlogging models by considering virtual production level and virtual production cost functions. To be more specific, the actual production level at period \(t\) is \(z_t = x_t + L_t\). It is clear from (5) and (6) that \(z_t \geq 0\). Furthermore, let \(\eta_t(z)\) be the cost for a virtual production level \(z\) at period \(t\). The problem can be transformed into

\[
\text{Minimize } \sum_{t=1}^{T} [\eta_t(z_t) + h_t(I_t)]
\]

subject to the following state equations and constraints

\[
I_t - I_{t-1} = z_t - d_t \quad t = 1, 2, \ldots, T
\]
\[
S_{t-1} - I_t \leq S_{t,1} \quad t = 0, 1, \ldots, T
\]
\[
z_t \geq 0 \quad t = 1, 2, \ldots, T
\]

which is exactly an inventory/backlogging model. It can be solved in \(O(T^3)\) time if \(h_t(\cdot)\)’s and \(\eta_t(\cdot)\)’s are concave functions. The complexity is reduced to \(O(T^2)\) if \(\eta_t(\cdot)\)’s are linear with fixed charges, as shown in the previous section. However, the concavity of \(\eta_t(\cdot)\)’s does not always hold even if \(p_t(\cdot)\)’s and \(\lambda_t(\cdot)\)’s are concave or even linear functions. We discuss hereafter about \(\eta_t(\cdot)\)’s in different cases.

If the outsourcing levels are unbounded and can take any nonnegative values, by definition, we have, for each \(z\) such that \(z \geq 0\),

\[
\eta_t(z) = \min_{0 \leq y \leq z} \{p_t(z - y) + \lambda_t(y)\}
\]

If \(p_t(\cdot)\) and \(\lambda_t(\cdot)\) are concave, \(p_t(z - y) + \lambda_t(y)\) is a concave function of \(y\) and the minimum is obtained when either \(z = 0\) or \(z = y\). As a consequence, \(\eta_t(z)\) can be rewritten as

\[
\eta_t(z) = \min \{p_t(z), \lambda_t(z)\}
\]

and \(\eta(\cdot)\) is also a concave function. If the inventory/backlogging cost functions are concave, the problem can be solved in \(O(T^3)\) time according to the result of Love (1973). If furthermore the production cost functions \(p_t(\cdot)\)’s and outsourcing cost functions \(\lambda_t(\cdot)\)’s are linear with fixed charges, then the virtual production cost functions \(\eta_t(\cdot)\)’s are piecewise linear concave with two linear segments (see Figure 1). Such a model can be solved in \(O(T^2)\) time from the result of Section 4. To summarize, we have the following theorem.

**Theorem 2** When outsourcing levels are unbounded, the outsourcing model can be solved in \(O(T^3)\) time if the production cost functions, inventory cost functions and outsourcing cost functions are concave. If furthermore, the production cost functions and outsourcing cost functions are concave piecewise linear, the complexity is reduced to \(O(NT)\) where \(N\) is the number of segments in these
functions. In particular, when the production cost functions and outsourcing cost functions are linear with fixed charges, the problem can be solved in $O(T^2)$ time.

However, this complexity result does not hold when the outsourcing levels are bounded, for instance, by the demand level of the corresponding periods. In fact, in this case, the virtual production cost functions are given by, for each $z$ such that $z \geq 0$,

$$
\eta_t(z) = \min_{0 \leq y \leq \min(z, d_t)} \{ p_t(z - y) + \lambda_t(y) \}
$$

Figure 2 shows such a function, assuming linear actual production cost functions with fixed charges and linear outsourcing cost functions.

From the figure, we can see that it is possible that the virtual production cost functions are neither concave nor convex, even if the initial production cost functions and outsourcing cost functions are all concave or convex, even linear. On the other hand, the concavity or convexity of production cost functions is required in almost all polynomially solvable inventory/backlogging models.

Even if in many cases the outsourcing levels are unbounded, some real life problems can be considered as outsourcing models where outsourcing level at a period $t$ is bounded, often by $d_t$. This is
the case in our real life refinery problem. In refineries, the crude oil from international market must be precipitated in tanks for some time before utilization, whereas the crude oil (outsourced) from the local market, which is already precipitated, can be directly used. If the quantity outsourced is more than the demand of the period, the surplus must be put in tanks which are never completely empty for technical reasons. The crude oil in these tanks must be precipitated again. Outsourcing more than demand thus causes additional manipulations with the tanks, creating delay in production and additional costs. The same situation appears in many other environments. For instance, it becomes a common practice that the subcontractor ships the outsourced quantity directly to customers on the behalf of the producer without physically passing by the warehouse of the producer, especially in the case that the subcontractor is far from the producer. Actual lost sales models are also equivalent to outsourcing models with bounded outsourcing levels. In these models, lost sales or outsourcing level cannot be more than demand. All these cases can be considered as outsourcing models where the outsourcing level of a period is bounded by the demand of the period. Therefore, such models are relevant in practice and need to be addressed. This is the purpose of this section.

The fact that the outsourcing level is bounded from above by the demand makes the model complicated. Furthermore, since outsourcing and production are both sources of the product, in some extent, the boundedness of outsourcing level is similar to production capacity. As we know, many lot sizing models with production capacity is NP-hard whereas its uncapacitated counterpart is polynomially solvable. This explains why thorough mathematical analysis is necessary to show that the problem with linear cost functions is polynomially solvable with inventory bounds. Thorough mathematical analysis and sophisticated algorithms are also always necessary for the complexity to be as low as possible. The typical example is sorting n-entry arrays where sophisticated algorithms are necessary to reduce the complexity from $O(n^2)$ to $O(n \log n)$.

In this section, we still assume that the production cost functions are linear with fixed charges. We assume in addition that backlogging is forbidden, the inventory and outsourcing cost functions are linear; i.e.,

$$ h_t(I) = h_tI $$
$$ \lambda_t(L) = \lambda_tL $$

with $h_t \geq 0$ and $\lambda_t \geq 0$ for any $t$ such that $1 \leq t \leq T$. To simplify the notation, let

$$ \lambda'_t = \lambda_t - h_{0,t-1} $$
$$ \pi'_t = \pi_t - h_{0,t-1} $$

According to Property 3, there is an optimal production plan composed of subplans. We decompose an optimal production plan into subplans in the following way. Each subplan contains exactly one production period, starts by an inventory point and ends with an inventory point but there is no
period $t$ such that $I_t = S_{t,1}$ between the production period and the inventory point. However, it is possible that there is more than one period $t$ between the last production period and $T$ such that $I_t = S_{t,1}$ in an optimal solution. In that case, the last periods will be included in no subplan. To cope with this point, it is sufficient to add a dummy period $T + 1$ with $d_{T+1} = 0$, $K_{T+1} = 0$, $\pi_{T+1} = 0$.

The last subplan ends at $T + 1$ which is both an inventory point with zero inventory and a production period. In the remainder, let $(i, u, j, k, v)$ denote a subplan involving a production period $j$ such that $i$ and $k$ are (not necessarily adjacent) inventory points. However, it is required that there be no period $t$ such that $j < t < k$ and $I_t = S_{t,1}$. As in the previous section, let $c_{u,v}(i, j, k)$ denote the minimal cost of this subplan and $E_u(k)$ the minimal cumulative cost from 0 through $k$ such that $I_k = S_{k,v}$ and $k$ is an inventory point after a production period and there is no period $t$ such that $I_t = S_{t,1}$ between $k$ and this production period. We have the following starting condition and recursive equation for the dynamic programming approach.

$$E_0(0) = 0$$
$$E_u(k) = \min_{-1 \leq u \leq 1} \min_{0 \leq i < j \leq k} \{E_u(i) + c_{u,v}(i, j, k)\}$$

The optimal value of the objective function is given by $E_0(T + 1)$.

Therefore, the key of this recursion is the computation of $c_{u,v}(i, j, k)$. We first show that a relaxed version of the subplan can be decomposed into two independent subproblems. Then we show how to solve each of these subproblems. Finally we show how to reintegrate the relaxed constraint by handling the remaining link between the subproblems in order to obtain an optimal solution. The problem is shown to be solvable in $O(T^2 \log T)$ time. As in the previous section, the result can be easily extended to concave piecewise linear production cost functions. In this case, the complexity becomes $O(NT \log T)$, where $N$ is the total number of segments in the concave piecewise linear production cost functions.

5.1 Definition of subproblems

The problem corresponding to the optimal subplan can be described as follows.

$$c_{u,v}(i, j, k) = \min \sum_{m=i+1}^{k} [\lambda_m L_m + h_m I_m] + K_j + \pi_j x_j$$

subject to the following state equations and constraints

$$I_m = \sigma_{i,u} - d_{0,m} + \sum_{t=i+1}^{m} L_t, \quad m = i + 1, \ldots, j - 1$$

$$I_m = x_j + \sigma_{i,u} - d_{0,m} + \sum_{t=i+1}^{m} L_t, \quad m = j, \ldots, k$$

$$0 \leq L_m \leq d_m, \quad m = i + 1, \ldots, k$$
\[ 0 \leq I_m \leq S_{m,1}, \quad m = i + 1, \ldots, j - 1 \]  
\[ 0 \leq I_m < S_{m,1}, \quad m = j, \ldots, k - 1 \]  
\[ I_k = S_{k,v} \]  
\[ x_j > 0 \]

where \( x_j \) and \( L_m \ (m = i+1, \ldots, k) \) are decision variables and \( I_m \ (m = i+1, \ldots, k) \) are state variables. Note that in (30), the inequalities are strict. This is because we require that there be no period \( t \) such that \( j \leq t < k \) and \( I_t = S_{t,1} \).

From constraints (27) and (31), we obtain

\[ x_j = \sigma_{k,v} - \sigma_{i,u} - \sum_{t=i+1}^{k} L_t \]

After eliminating \( x_j \) and state variables, the model above becomes

\[ c_{u,v}(i, j, k) = [H_{0,i} - \sigma_{i,u}(\pi'_j + h_{0,i})] + K_j + \sigma_{k,v}(\pi'_j + h_{0,k}) - H_{0,k} + \]

\[ \text{Min} \sum_{t=i+1}^{k} (\lambda'_t - \pi'_j)L_t \]  
subject to

\[ 0 \leq L_m \leq d_m, \quad m = i + 1, \ldots, k \]  
\[ d_{0,m} - \sigma_{i,u} \leq \sum_{t=i+1}^{m} L_t \leq \sigma_{m,1} - \sigma_{i,u}, \quad m = i + 1, \ldots, j - 1 \]  
\[ \sigma_{k,v} - \sigma_{m,1} < \sum_{t=m+1}^{k} L_t \leq \sigma_{k,v} - d_{0,m}, \quad m = j, \ldots, k - 1 \]  
\[ \left( \sigma_{k,v} - \sum_{t=j}^{k} L_t \right) - \left( \sigma_{i,u} + \sum_{t=i+1}^{j-1} L_t \right) > 0 \]

where \( L_m \ (m = i + 1, \ldots, k) \) are decision variables.

Clearly, this model is a linear program. We now show that due to its special structure, it can be solved efficiently in strongly polynomial time.

**Remark 1** Without loss of generality, we only consider optimal solutions in which \( L_t \) is as small as possible whenever \( \lambda'_t = \pi'_j \). In other words, whenever \( \lambda'_t = \pi'_j \), \( \lambda'_t \) is increased of \( \varepsilon > 0 \).

We first consider a relaxed problem where constraint (37) is omitted. In subsection 5.4, we show how to reintegrate this constraint. The relaxed problem can be decomposed into two subproblems \( P_u(i, j) \) and \( Q_v(j, k) \) defined as follows.

**Problem** \( P_u(i, j) \).

\[ A_u(i, j) = \text{Min} \sum_{t=i+1}^{j-1} (\lambda'_t - \pi'_j)L_t \]
subject to

\[
0 \leq L_m \leq d_m, \quad m = i + 1, \ldots, j - 1
\]
\[
d_{0,m} - \sigma_{i,u} \leq \sum_{t=i+1}^{m} L_t \leq \sigma_{m,1} - \sigma_{i,u}, \quad m = i + 1, \ldots, j - 1
\]

Problem \(Q_v(j, k)\).

\[
B_v(j, k) = \min \sum_{t=j}^{k} (\lambda'_t - \pi'_j)L_t
\]

subject to

\[
0 \leq L_m \leq d_m, \quad m = j, \ldots, k
\]
\[
\sigma_{k,v} - \sigma_{m,1} < \sum_{t=m+1}^{k} L_t \leq \sigma_{k,v} - d_{0,m}, \quad m = j, \ldots, k - 1
\]

5.2 Solving \(P_u(i, j)\)

Note that the unique feasible solution to \(P_0(i, j)\) is that \(L_t = d_t\) for all \(t = i + 1, \ldots, j - 1\). Therefore,

\[
A_0(i, j) = \sum_{t=i+1}^{j-1} (\lambda'_t - \pi'_j)d_t
\]

which can be computed recursively as follows.

\[
A_0(j - 1, j) = 0
\]
\[
A_0(t, j) = A_0(t + 1, j) + (\lambda'_{t+1} - \pi'_j)d_{t+1}, \quad t = 0, 1, \ldots, j - 2
\]

Therefore, we focus on solving \(P_1(i, j)\) to compute \(A_1(i, j)\). Note that in \(P_1(i, j)\), the constraints \(\sum_{t=i+1}^{m} L_t \geq d_{0,m} - \sigma_{i,1}\) are redundant for \(m = i + 1, \ldots, j - 2\). These constraints can be deduced from the constraints \(L_m \leq d_m\) and \(\sum_{t=i+1}^{j-1} L_t \geq d_{0,j-1} - \sigma_{i,1}\). Therefore, problem \(P_1(i, j)\) can be rewritten as

\[
A_1(i, j) = \min \sum_{t=i+1}^{j-1} (\lambda'_t - \pi'_j)L_t
\]

subject to

\[
0 \leq L_m \leq d_m, \quad m = i + 1, \ldots, j - 1
\]
\[
\sum_{t=i+1}^{j-1} L_t \geq R(i, j)
\]
\[
\sum_{t=i+1}^{m} L_t \leq \bar{R}(i, m) \quad m = i + 1, \ldots, j - 1
\]
where

\[ R(i, j) = d_{0, j-1} - \sigma_{i, 1} \]
\[ \bar{R}(i, m) = \sigma_{m, 1} - \sigma_{i, 1} \]

By definition, \( R(i, j) \) is the lower bound of the cumulative outsourcing level from period \( i+1 \) to period \( j - 1 \) whereas \( \bar{R}(i, m) \) is the upper bound of the cumulative outsourcing level from period \( i+1 \) to period \( m \).

We establish some properties.

**Property 4** There is an optimal solution to \( P_1(i, j) \) in which for any \( \tau \) such that \( i + 1 \leq \tau < j \) and \( \lambda'_\tau \geq \pi'_j \) we have \( L_\tau [\sum_{t=i+1}^{j-1} L_t - R(i, j)] = 0 \).

**Proof.** If \( \sum_{t=i+1}^{j-1} L_t > R(i, j) \) and \( L_\tau > 0 \), the solution can be improved by decreasing \( L_\tau \) while satisfying all constraints. ■

**Property 5** There is an optimal solution to Problem \( P_1(i, j) \) in which \( L_\tau (d_t - L_t) = 0 \) for any \( \tau \) and \( t \) such that \( i \leq \tau < t < j \) and \( \lambda'_\tau \geq \lambda_t \).

**Proof.** See Appendix E. ■

**Property 6** There is an optimal solution to \( P_1(i, j) \) in which for any \( \tau \) such that \( i + 1 \leq \tau < j \) and \( \lambda'_\tau < \pi'_j \), we have \( (L_\tau - d_\tau)(\sum_{t=i+1}^{m} L_t - \bar{R}(i, m)) = 0 \) for some \( m \) such that \( \tau \leq m < j \). In other words, either \( L_\tau = d_\tau \) or there is an \( m \) such that \( \tau \leq m < j \) and \( \sum_{t=i+1}^{m} L_t = \bar{R}(i, m) \).

**Proof.** It is obvious that, if \( L_\tau < d_\tau \) and \( \sum_{t=i+1}^{m} L_t < \bar{R}(i, m) \) for any \( m \) such that \( \tau \leq m < j \), the solution can be improved by increasing \( L_\tau \) while satisfying all constraints. ■

**Property 7** There is an optimal solution to Problem \( P_1(i, j) \) in which, for any \( \tau \) and \( t \) such that \( i < \tau < t < j \) and \( \lambda'_\tau < \lambda'_t \), we have \( (d_\tau - L_\tau)(\bar{R}(i, m) - \sum_{t=i+1}^{m} L_t) = 0 \) for some \( m \) such that \( \tau \leq m < t \).

**Proof.** See Appendix F. ■

In what follows, we only consider solutions verifying these properties. Such an optimal solution is denoted as \( L(i, j) = (\omega_{i+1}(i, j), \omega_{i+2}(i, j), \ldots, \omega_{j-1}(i, j)) \).

In fact, it is not necessary to independently solve all problems \( P_1(i, j) \) defined by (38) to (41). It is possible to use the solution of \( P_1(i + 1, j) \); i.e., \( L(i + 1, j) \), to solve problem \( P_1(i, j) \), according to the following theorem which is proved in Appendix G.

**Theorem 3** For any \( m \) such that \( i + 2 \leq m \leq j - 1 \), we have \( \omega_m(i, j) \geq \omega_m(i + 1, j) \).
From this theorem, it is only necessary to consider solutions to \( P_1(i, j) \) such that \( L_t \geq \omega_t(i + 1, j) \) for any \( t \) such that \( i + 1 \leq t \leq j - 1 \), where \( \omega_{i+1}(i + 1, j) = 0 \), without loss of generality. In other words, an additional constraint \( L_t \geq \omega_t(i + 1, j) \) for any \( t \) such that \( i + 1 \leq t \leq j - 1 \) can be added without missing an optimal solution.

By a variable substitution \( e_t = L_t - \omega_t(i + 1, j) \) to denote the required additional amount to be outsourced at period \( t \) in \( P_1(i, j) \) compared to \( P_1(i + 1, j), t = i + 1, \ldots, j - 1 \), and knowing an optimal solution to problem \( P_1(i + 1, j) \); namely \( L(i + 1, j) \), problem \( P_1(i, j) \) can be rewritten as follows.

\[
A_1(i, j) = \min \sum_{t=i+1}^{j-1} (\lambda'_t - \pi'_j) e_t + \sum_{t=i+1}^{j-1} (\lambda'_t - \pi'_j) \omega_t(i + 1, j) \quad (42)
\]

subject to

\[
0 \leq e_t \leq \bar{e}_t, \quad t = i + 1, \ldots, j - 1 \quad (43)
\]

\[
\sum_{t=i+1}^{j-1} e_t \geq R(i, j) - \sum_{t=i+1}^{j-1} \omega_t(i + 1, j), \quad (44)
\]

\[
\sum_{t=i+1}^{m} e_t \leq \bar{R}(i, m) - \sum_{t=i+1}^{m} \omega_t(i + 1, j), \quad m = i + 1, \ldots, j - 1 \quad (45)
\]

where \( e_t \) are decision variables \( (i + 1 \leq t < j) \) and

\[
\bar{e}_t \equiv d_t - \omega_t(i + 1, j) \geq 0
\]

which denotes the quantity not yet outsourced at period \( t \) in \( L(i + 1, j) \).

In this formulation, the second term of the objective function is independent of the decision variables. Therefore, it is dropped in the sequel of the optimization process. By definition, we have

\[
R(i, j) - \sum_{t=i+1}^{j-1} \omega_t(i + 1, j) \leq R(i, j) - R(i + 1, j) = \sigma_{i+1,1} - \sigma_{i,1} = \bar{R}(i, m) - \bar{R}(i + 1, m) \leq \bar{R}(i, m) - \sum_{t=i+1}^{m} \omega_t(i + 1, j), \quad m = i + 1, \ldots, j - 1 \quad (46)
\]

This relation leads to the following property proved in Appendix H.

**Property 8** There is an optimal solution such that

\[
\sum_{t=i+1}^{j-1} e_t \leq \sigma_{i+1,1} - \sigma_{i,1}
\]

From this property and relation (46), constraint (45) can be replaced by

\[
\sum_{t=i+1}^{j-1} e_t \leq \sigma_{i+1,1} - \sigma_{i,1}
\]
Therefore, by defining, for the sake of simplification,

\[ C = R(i, j) - \sum_{t=i+1}^{j-1} \omega_t(i + 1, j) \]

Problem \( P_1(i, j) \) can be rewritten as

\[ A'_1(i, j) = \min \sum_{t=i+1}^{j-1} (\lambda'_t - \pi'_j) e_t \]

subject to

\[ 0 \leq e_t \leq \bar{e}_t, \quad t = i + 1, \ldots, j - 1 \]

\[ \sum_{t=i+1}^{j-1} e_t \geq C \quad (47) \]

\[ \sum_{t=i+1}^{j-1} e_t \leq \sigma_{i+1,1} - \sigma_{i,1} \quad (48) \]

This problem consists of determining quantities \( e_t \)'s bounded from below by 0 and from above by \( \bar{e}_t \) such that the sum is at least \( C \) and at most \( \sigma_{i+1,1} - \sigma_{i,1} \) and a linear cost function is minimized. We can observe that

1. if \( \bar{e}_t = 0 \), then \( e_t = 0 \).

2. if \( e_t > 0 \) then \( e_\tau = \bar{e}_\tau \) for any \( \tau \) such that \( \lambda'_\tau < \lambda'_t \).

From Observation 1, \( e_\tau \) may be different from 0 only if \( \bar{e}_t > 0 \). Let \( K \) be the number of such \( t \)'s. Let \( W = (w_1, w_2, \ldots, w_K) \) be a permutation of these \( t \)'s such that \( \lambda'_{w_1} \leq \lambda'_{w_2} \leq \cdots \leq \lambda'_{w_K} \). From the observations above, an optimal solution of the problem can be recursively computed as follows.

\[
e_{w_q} = \begin{cases} 
\min \left( \bar{e}_{w_q}, \sigma_{i+1,1} - \sigma_{i,1} - \sum_{l=1}^{q-1} e_{w_l} \right) & \text{if } \lambda'_{w_q} < \pi'_j \\
\min \left[ \bar{e}_{w_q}, \max \left( 0, C - \sum_{l=1}^{q-1} e_{w_l} \right) \right] & \text{otherwise} 
\end{cases} \quad (49)
\]

In fact, when \( \lambda'_{w_q} < \pi'_j \), one must choose \( e_{w_q} \) as large as possible. The largest value possible is \( \min \left( \bar{e}_{w_q}, \sigma_{i+1,1} - \sigma_{i,1} - \sum_{l=i+1}^{q-1} e_{w_l} \right) \). When \( \lambda'_{w_q} \geq \pi'_j \), one must choose \( e_{w_q} \) as small as possible. If \( C - \sum_{l=1}^{q-1} e_{w_l} \leq 0 \), constraints (47) and (48) are already satisfied and one must set \( e_{w_q} = 0 \). When \( 0 < C - \sum_{l=1}^{q-1} e_{w_l} \leq \bar{e}_{w_q} \), constraint (47) is not satisfied yet. It can be satisfied by setting \( e_{w_q} = C - \sum_{l=1}^{q-1} e_{w_l} \). Furthermore, this is the least value possible with the least marginal cost possible and therefore is the best solution. For the same reason, when \( C - \sum_{l=1}^{q-1} e_{w_l} > \bar{e}_{w_q} \), \( e_{w_q} \) must be exactly \( \bar{e}_{w_q} \). Relation (49) summarizes the different cases. From (49), \( e_{w_q} > 0 \) if and only if either \( \sum_{l=1}^{q-1} e_{w_l} < C \) or \( \lambda'_{w_q} < \pi'_j \) and \( \sum_{l=1}^{q-1} e_{w_l} < \sigma_{i+1,1} - \sigma_{i,1} \). Hereafter, let \( e_t(i, j) \) be the value of \( e_t \) in an optimal solution.
By the definition, we have \( \omega_t(i, j) = \epsilon_t(i, j) + \omega_t(i + 1, j) \). Therefore,
\[
A_1(i, j) = \sum_{t=i+1}^{j-1} (\lambda'_t - \pi'_t)\omega_t(i, j) = A_1(i + 1, j) + \sum_{t=i+1}^{j-1} (\lambda'_t - \pi'_t)\epsilon_t(i, j) = A_1(i + 1, j) + A'_1(i, j)
\]

For a given \( j \) (\( 2 \leq j \leq T \)), the following algorithm simultaneously solves all the problems \( P_t(i, j) \)'s for all \( 0 \leq i < j - 1 \). In this algorithm, \( w \) is the head of the list \( W \) of \( t \)'s arranged in nondecreasing order of \( \lambda'_t \)'s and such that \( \bar{e}_t > 0 \). \( TotalOSL \) is the cumulated outsourcing level \( \sum_{t=i+1}^{j-1} \omega_t(i + 1, j) + \sum_{l=1}^{j-1} \epsilon_{w_l}(i, j) = -C + R(i, j) + \sum_{l=1}^{j-1} \epsilon_{w_l}(i, j) \) (Therefore, \( R(i, j) - TotalOSL = C - \sum_{l=1}^{j-1} \epsilon_{w_l}(i, j) \)). \( SumE \) is the current value of \( \sum_{l=1}^{j-1} \epsilon_{w_l}(i, j) \) and \( TotalCost \) is the current value of \( A_1(i, j) \).

1. \( W := \emptyset, TotalOSL := 0, TotalCost := 0, \bar{e}_t := d_t \) for any \( t = 1, \ldots, j - 1 \).
2. \( i := j - 2 \)
3. While \( i \geq 0 \) do
   /*Solving Problem \( P_t(i, j) \)*/
   (a) \( SumE := 0 \)
   (b) If \( \bar{e}_{i+1} > 0 \) then insert \( i + 1 \) into \( W \) in nondecreasing order of \( \lambda'_t \)'s.
   (c) While \( (\lambda'_w < \pi'_j \) and \( SumE < \sigma_{i+1,1} - \sigma_{i,1} \) or \( TotalOSL < R(i, j) \)) do
      i. If \( \lambda'_w < \pi'_j \) then \( \epsilon_w(i, j) := \min(\bar{e}_w, \sigma_{i+1,1} - \sigma_{i,1} - SumE) \).
         Otherwise \( \epsilon_w(i, j) := \min(\bar{e}_w, R(i, j) - TotalOSL) \).
      ii. \( \bar{e}_w := \bar{e}_w - \epsilon_w(i, j); TotalOSL := TotalOSL + \epsilon_w(i, j), SumE := SumE + \epsilon_w(i, j), TotalCost := TotalCost + (\lambda'_w - \pi'_j)\epsilon_w(i, j) \).
      iii. If \( \bar{e}_w = 0 \) then remove \( w \) from \( W \);
   (d) Endwhile
   (e) \( i := i - 1 \)
4. Endwhile

At step 3(c)i, we have always \( \epsilon_w(i, j) > 0 \). Since \( W \) is the list of \( t \)'s such that \( \bar{e}_t > 0 \), we have \( \bar{e}_w > 0 \). If \( \lambda'_w < \pi'_j \), we have either \( \sigma_{i+1,1} - \sigma_{i,1} - SumE > 0 \) or \( TotalOSL < R(i, j) \). When \( TotalOSL < R(i, j) \), due to the fact that \( \sigma_{i+1,1} - \sigma_{i,1} - SumE > R(i, j) - TotalOSL \), we must have \( \sigma_{i+1,1} - \sigma_{i,1} - SumE > 0 \). In either cases, we have \( \sigma_{i+1,1} - \sigma_{i,1} - SumE > 0 \). Therefore, \( \epsilon_w(i, j) > 0 \).

As a consequence, \( \epsilon_w(i, j) \) is either \( \bar{e}_w \) or \( \sigma_{i+1,1} - \sigma_{i,1} - SumE \) (if \( \lambda'_w < \pi'_j \)) or \( R(i, j) - TotalOSL \) (if \( \lambda'_w \geq \pi'_j \)).
If $\epsilon_w(i, j) = \bar{\epsilon}_w$, the new value of $\bar{\epsilon}_w$ is 0 and $w$ will be removed from $W$ forever. Therefore, the number of such $\epsilon_w(i, j)$’s is at most $j$. The removal of each $w$ from $W$ needing $O(\log j)$ time, the time due to these removals can be bounded by $O(j \log j)$.

If $\epsilon_w(i, j) = \sigma_{i+1,1} - \sigma_{i,1} - SumE$ and $\lambda'_t < \pi'_j$ or $\epsilon_w(i, j) = R(i, j) - TotalOSL$ and $\lambda'_t \geq \pi'_j$, the values of TotalOSL and SumE are updated and at least one of the conditions TotalOSL $< R(i, j)$ and SumE $< \sigma_{i+1,1} - \sigma_{i,1}$ becomes false. Problem $P_1(i, j)$ is solved. As a consequence, the number of such $\epsilon_w(i, j)$’s is at most $j$.

As a consequence, the number of iterations, which is equal to the number of $\epsilon_w(i, j)$’s such that $\epsilon_w(i, j) > 0$ is at most $2j$, each requiring at most $O(\log j)$ time. The insertion of each new $i + 1$ into $W$ requires $O(\log j)$ time. All insertions require $O(j \log j)$ time.

As a matter of result, $O(T^2 \log T)$ time is required to compute all the $A_1(i, j)$’s for all $0 \leq i < j$ and $1 \leq j \leq T$.

5.3 Solving $Q_v(j, k)$

Recall the definition of problem $Q_v(j, k)$ with $v \in \{0, 1\}$ as follows

$$B_v(j, k) = \text{Minimize } \sum_{t=j}^{k} (\lambda'_t - \pi'_j)L_t$$

subject to

$$0 \leq L_t \leq d_t \quad t = j, \ldots, k$$

$$\sigma_{k,v} - \sigma_{t,1} < \sum_{m=t+1}^{k} L_m \leq \sigma_{k,v} - d_{0,t} \quad t = j, \ldots, k - 1$$

From constraints (51), we obtain $\sum_{m=t+1}^{k} L_m \leq d_{0,k} - d_{0,t} \leq S_{k,v} + d_{0,k} - d_{0,t} = \sigma_{k,v} - d_{0,t}$. Therefore, constraints (52) can be rewritten as

$$\sum_{m=t+1}^{k} L_m > \sigma_{k,v} - \sigma_{t,1}, \quad t = j, \ldots, k - 1$$

Note that due to this strict inequality, it is possible that there is no optimal solution. Nevertheless, if there is an optimal solution, then $L_t = d_t$ for any $t$ such that $\lambda'_t < \pi'_j$.

If $\sum_{t<m \leq k, \lambda'_m < \pi'_j} d_m \leq \sigma_{k,v} - \sigma_{t,1}$ for some $t$ such that $j \leq t < k$, there must be a $\tau$ such that $t < \tau < k$, $L_{\tau} > 0$ and $\lambda'_\tau \geq \pi'_j$. Due to the strict inequality, and by considering Remark 1, there is no optimal solution verifying Remark 1. There is such an optimal solution if and only if $\sum_{t < m \leq k, \lambda'_m < \pi'_j} d_m > \sigma_{k,v} - \sigma_{t,1}$ for any $t$ such that $j \leq t < k$. As a consequence, we must have

$$\sigma_{k,v} < g(j, k)$$

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where

\[ g(j, k) = \min_{j \leq t < k} \left[ \sigma_{t,1} + \sum_{t < m \leq k, \lambda'_m < \pi'_j} d_m \right] \]

In this case, we have

\[ B_v(j, k) = \sum_{m=j}^{k} \min(\lambda'_m - \pi'_j, 0)d_m \]

Therefore, it is crucial to compute \( g(j, k) \) and check (54). All \( g(j, k) \)'s can be obtained in \( O(T^2) \) time with the following recursive equations.

\[ g(j, j) = +\infty; \]
\[ g(j, l) = \begin{cases} 
\min [g(j, l - 1), \sigma_{l-1,1}] + d_l & \text{if } \lambda'_l < \pi'_j \\
\min [g(j, l - 1), \sigma_{l-1,1}] & \text{otherwise} 
\end{cases} \quad j \leq l \leq T \]

Therefore, the subplan is undefined when \( g(j, k) - d_{0,k} \) becomes nonpositive since we would have \( S_{k,v} = \sigma_{k,v} - d_{0,k} < g(j, k) - d_{0,k} \leq 0 \) which is impossible. Therefore, for a given \( j \), considering the fact that \( g(j, k) - d_{0,k} \) is decreasing with \( k \), all \( Q_v(j, k) \)'s can be solved in \( O(T) \) time as follows.

1. \( g(j, j) := +\infty, \text{TotalCost} := 0 \)

2. \( k := j \)

3. While \( k \leq T \) and \( g(j, k) - d_{0,k} > 0 \) do

   (a) If \( \sigma_{k,v} \geq g(j, k) \) then \( B_v(j, k) := +\infty \)

      Otherwise if \( \lambda'_k < \pi'_j \) then \( \text{TotalCost} := \text{TotalCost} + (\lambda'_k - \pi'_j)d_k \) and \( B_v(j, k) := \text{TotalCost} \).

   (b) If \( \lambda'_k < \pi'_j \) then \( g(j, k + 1) = \min[g(j, k), \sigma_{k,1}] + d_{k+1} \)

      Otherwise \( g(j, k + 1) = \min[\sigma_{k,1}, g(j, k)] \)

   (c) \( k := k + 1 \)

4. Endwhile

5.4 Handling the link between subproblems and solving the global problem

In this section, we show how to integrate the contraint (37). Let \( \rho_u(i, j) \) and \( r_v(j, k) \) denote the value of \( I_{j-1} = \sigma_{i,u} - d_{0,j-1} + \sum_{t=i+1}^{j-1} L_t \) in the optimal solution of \( P_u(i, j) \) and the value of \( \sigma_{k,v} - d_{0,j-1} - \sum_{t=j}^{k} L_t \) in an optimal solution of \( Q_v(j, k) \), respectively. When \( B_v(j, k) = +\infty \), let \( r_v(j, k) = -\infty \), without loss of generality. By definition,

\[ \rho_1(i, j) = \sum_{t=i+1}^{j-1} \omega_t(i, j) - R(i, j) \]
Clearly, if $\rho_u(i,j) < r_v(j,k)$ then constraint (37) is satisfied even in the optimal solution of the relaxed problem. In this case, we have $c_{u,v}(i,j,k) = A''_u(i,j) + B''(j,k)$ where (see (33))

$A''_u(i,j) = H_{0,i} - \sigma_{i,u}(\pi_j' + h_{0,i}) + A_u(i,j)$

$B''(j,k) = K_j + \sigma_{k,v}(\pi_j' + h_{0,k}) - H_{0,k} + B_v(j,k)$

The following theorem proved in the Appendix I allows us to only consider subplans $(i,u,j,k,v)$ such that $\rho_u(i,j) < r_v(j,k)$.

**Theorem 4** There is an optimal solution in which for any subplan $(i,u,j,k,v)$ with a production period $j$, we have $\rho_u(i,j) < r_v(j,k)$.

As a consequence, we have

$$E_v(k) = \min_{1 \leq j \leq k} \left\{ B''_v(j,k) + \min_{-1 \leq u \leq 1} \min_{0 \leq i < j} \min_{\rho_u(i,j) < r_v(j,k)} [E_u(i) + A''_u(i,j)] \right\}$$

Because of the requirement that $\rho_u(i,j) < r_v(j,k)$, we must consider all $i$ such that $0 \leq i < j$ for each $(j,k)$ to check whether this requirement is fulfilled. The complexity will be $O(T^3)$. In fact, the computation of $E_v(k)$’s can be reduced to $O(T^2 \log T)$, as soon as $A''_u(i,j)$’s and $B''_v(j,k)$’s are known.

The following theorem is the key to reduce the complexity.

**Theorem 5** For any $i$ and $j$ such that $0 \leq i < j-1 < T$ and $u \in \{0,1\}$, we have $\rho_u(i,j) \leq \rho_u(i+1,j)$.

**Proof.** See Appendix J.

Let us define, for any $(j,k)$ ($1 \leq j \leq k \leq T$) and $u,v \in \{0,1\}$,

$$\beta_{u,v}(j,k) = \max \{ i | 0 \leq i < j, \rho_u(i,j) < r_v(j,k) \}$$

If $\rho_u(i,j) \geq r_v(j,k)$ for all $i$ such that $0 \leq i < j$, then $\beta_{u,v}(j,k)$ is set to $-1$, without loss of generality.

From Theorem 5, if $r_v(j,k_1) \leq r_v(j,k_2)$ then $\beta_{u,v}(j,k_1) \leq \beta_{u,v}(j,k_2)$. Therefore, for given $u,v \in \{0,1\}$ and a given $j$, all $\beta_{u,v}(j,k)$’s with $k \geq j$ can be computed in $O(T \log T)$ time using the following procedure.

1. Let $(k_1,k_2,\ldots,k_{T-j+1})$ be the permutation of $\{j,j+1,\ldots,T\}$ in such a way that $r_v(j,k_1) \leq r_v(j,k_2) \leq \ldots \leq r_v(j,k_{T-j+1})$.
2. $i := j - 1$; $l := T - j + 1$
3. While $l \geq 1$ do
   - If $i < 0$ or $\rho_u(i,j) < r_v(j,k_l)$ then $\beta_{u,v}(j,k_l) := i$ and $l := l - 1$
   - Otherwise $i := i - 1$
4. Endwhile

In this procedure, for a given \( j \), the computation of the permutation \((k_1, k_2, \ldots, k_{T-j+1})\) requires \(O(T \log T)\) time. At each of the subsequent iterations, either \( l \) or \( i \) is decremented. Since the number of \( l \)'s is upper-bounded by \( T - j + 1 \) and that of \( i \)'s is upper-bounded by \( j \), there are at most \( T \) iterations. Therefore, all \( \beta_{u,v}(j,k) \)'s can be computed in \(O(T^2 \log T)\) time.

From Theorems 4 and 5, for the subplan \((i, u, j, k, v)\) to belong to an optimal solution, we must have

\[ i \leq \beta_{u,v}(j,k) \]

As a consequence, we have

\[ E_v(k) = \min_{1 \leq j \leq k} \left\{ B''_v(j,k) + \min_{u \in \{0,1\}, 0 \leq i \leq \beta_{u,v}(j,k)} \left[ E_u(i) + A''_u(i,j) \right] \right\} \]

By defining

\[ F_u(j,l) = \min_{0 \leq i \leq l} [E_u(i) + A''_u(i,j)] \]

we have the following recursive equations

\[ F_u(j,l) = \min \left\{ F_u(j,l-1), E_u(l) + A''_u(l,j) \right\} \]

\[ E_v(k) = \min_{1 \leq j \leq k} \left[ B''_v(j,k) + \min_{u \in \{0,1\}} \left( F_u(j, \beta_{u,v}(j,k)) \right) \right] \]

with \( F_u(j,-1) = +\infty \) for any \( u \in \{0,1\} \) and \( j \) such that \( 1 \leq j \leq T \) and \( E_0(0) = 0 \) as starting conditions.

The equations above show that as soon as \( A''_u(i,j) \)'s, \( B''_v(j,k) \)'s and \( \beta_{u,v}(j,k) \)'s are known, it requires an additional amount of \(O(T^2)\) time to obtain the optimal solution. Since all \( \beta_{u,v}(j,k) \)'s can be computed in \(O(T^2 \log T)\) time and \( A''_u(i,j) \)'s, \( B''_v(j,k) \)'s can be computed in \(O(T^2 \log T)\) time, the overall complexity is \(O(T^2 \log T)\).

6 Conclusion

This paper investigated real life crude oil procurement problems that arise in a refinery. They can be modelled as single item lot sizing problems with bounded inventory and backlogging or outsourcing. Polynomial dynamic programming algorithms are proposed. Due to the concept of actuated inventory bounds and equation (9) which played a crucial role in this paper, we showed that the computational complexity obtained for unbounded inventory/backlogging models almost holds for bounded inventory/backlogging models. When the inventory holding/backlogging cost functions are concave and the production cost functions are linear with fixed charges or concave piecewise linear, the problem can be solved in \(O(T^2)\) time whereas its unbounded inventory counterpart can be solved in \(O(T \log T)\) time. This is due to the requirement that \( i < b_{u,v}(k) \) in (14) to guarantee positive production levels. When the inventory holding and the outsourcing cost functions are linear, the outsourcing model is shown
to be solvable in $O(T^2 \log T)$ time. Our future research consists of investigating models with limited production capacity and models where both backlogging and outsourcing are allowed.

A Proof of Property 2

The conditions are obviously necessary for the production plan to be feasible. We now show that these conditions are sufficient.

Without loss of generality, we consider periods 0 and $T + 1$ as production periods. For these two dummy production periods, we have $I_0 = 0 \leq S_{0,1} = 0$ and $I_T = 0 \geq S_{T,-1} = 0$ due to the fact that $\sum_{t=1}^{T} x_t = d_{0,T}$.

We now show that for any two adjacent production periods $\tau$ and $t$ with $\tau < t$, if $I_{\tau} \leq S_{\tau,1}$ and $I_{t-1} \geq S_{t-1,-1}$, then $S_{t,-1} \leq I_t \leq S_{t,1}$ for any $i$ such that $\tau \leq i \leq t - 1$. In fact, when there is no outsourcing, for any $i$ such that $\tau \leq i \leq t - 1$, from relation (9), we have $I_i = I_{\tau} - d_{\tau,i} \leq S_{\tau,1} - d_{\tau,i} \leq S_{i,1}$ and $I_i = I_{t-1} + d_{i,t-1} \geq S_{t-1,-1} + d_{i,t-1} \geq S_{t,-1}$.

B Proof of Property 3

Consider an optimal solution where $j$ and $j'$ such that $1 \leq j < j' \leq T$ are two adjacent production periods; i.e., for any period $t$ such that $j < t < j'$, we have $x_t = 0$, and there is no inventory points between $j$ and $j'$. Consider any feasible solution identical to this optimal solution except the production levels of periods $j$ and $j'$ such that the inventory level at period $j'$ remains unchanged. The difference between such a solution and the considered optimal solution is the total production, inventory/backlogging and outsourcing cost from period $j$ to period $j'$ which is given by

$$p_j(x_j) + \sum_{t=j}^{j'-1} \left[ \lambda_t(L_t) + h\left(I_{j-1} - d_{j-1,t} + x_j + \sum_{\tau=j}^{t} L_{\tau}\right) \right] + \lambda_{j'}(L_{j'}) + h_{j'}(I_{j'})$$

$$+ p_{j'}\left( I_{j'} - I_{j-1} + d_{j-1,j'} - x_j - \sum_{\tau=j}^{j'} L_{\tau} \right)$$

This cost is function of a single variable $x_j$. Due to the fact that $p_j(\cdot)$ and $p_{j'}(\cdot)$ are concave over $(0, +\infty)$ and $h_t(\cdot)$ is concave over $(S_{t,-1}, 0)$ and over $(0, S_{t,1})$ for any $t$ such that $j \leq t < j'$, the cost is a concave function about $x_j$. According to concave minimization theory, the minimal cost is obtained at the boundary of the interval on which the function is concave. As a consequence, the cost is minimized when either $x_j = 0$, or $x_{j'} = I_{j'} - I_{j-1} + d_{j-1,j'} - x_j - \sum_{\tau=j}^{j'} L_{\tau} = 0$, or $I_t = I_{j-1} - d_{j-1,t} + x_j + \sum_{\tau=j}^{t} L_{\tau} \in \{S_{t,-1}, 0, S_{t,1}\}$ for some $t$ such that $j \leq t < j'$. In other words, in this new optimal solution, either $j$ or $j'$ is no more a production period, or some period $t$ such that $j \leq t < j'$ becomes an inventory point.
C Proof of Theorem 1

Consider an optimal solution in which there is a production period \( t \) such that \( 1 \leq t \leq T \) and \( I_{t-1} > 0 \). Since \( I_{t-1} = \sum_{j=1}^{t-1} x_j - d_{0,t-1} \), we obtain \( \sum_{j=1}^{t-1} x_j > d_{0,t-1} \geq 0 \), which implies that there is at least one production period before \( t \). Let \( \tau \) be the last production period before \( t \). It is clear that \( I_i = I_{t-1} + d_{i,t-1} \geq I_{t-1} > 0 \), for any \( i = \tau, \tau + 1, \ldots, t - 1 \). Consider another solution by decreasing the production level at period \( \tau \) by \( I_{t-1} \) and increasing the production level at period \( t \) by \( I_{t-1} \). The production cost of period \( \tau \) is decreased at least by \( \pi_t I_{t-1} \). The cumulative inventory cost from \( \tau \) to \( t - 1 \) is decreased by \( h_{t-1,t-1}I_{t-1} = (h_{0,t-1} - h_{0,\tau-1})I_{t-1} \). The production cost at period \( t \) is increased by \( \pi_t I_{t-1} \). The other cost remains the same. The total cost is reduced at least by \( [\pi_t - h_{0,\tau-1} - (\pi_t - h_{0,t-1})]I_{t-1} \geq 0 \) (see (17)), which means that either the initial solution is not optimal or the new solution is optimal as well.

Similarly, we can prove that if in an optimal solution there is some production period \( t \) \((1 \leq t \leq T)\) such that \( I_t < 0 \), then there is another solution at least as good such that the inventory level at \( t \) is nonnegative.

D Mongé of matrices

**Definition 4** An \( n \times n \) matrix \( E = [e_{i,j}] \) is said to be Monge, if for any \( 1 \leq i < k \leq n \) and \( 1 \leq j < l \leq n \), we have \( e_{i,j} + e_{k,l} \leq e_{i,l} + e_{k,j} \).

**Property 9** An \( n \times n \) matrix \( E = [e_{i,j}] \) is Monge if \( e_{i,j} = X_i + Y_j - U_iV_j + M \max(\alpha'_i - \beta'_j, \alpha_j - \beta_i, 0) \) for any \( i, j = 1, 2, \ldots, n \), with \( \{U\} \), \( \{V\} \), \( \{\alpha'\} \), \( \{\beta'\} \), \( \{\alpha\} \) and \( \{\beta\} \) being all nondecreasing series.

**Proof.** By definition, for any \( 1 \leq i < k \leq n \) and \( 1 \leq j < l \leq n \), we have

\[
e_{i,l} + e_{k,j} - (e_{i,j} + e_{k,l}) = (U_k - U_i)(V_l - V_j) + M \max(\alpha'_i + \alpha'_k - \beta'_j - \beta'_l, \alpha_j + \alpha_l - \beta_k, 0, \alpha'_i - \beta_k + \alpha_j - \beta'_l, \alpha'_k - \beta_i + \alpha_l, \alpha'_i - \beta'_j + \alpha_l, \alpha'_k - \beta'_l, \alpha_j - \beta_k) - M \max(\alpha'_i + \alpha'_k - \beta'_j - \beta'_l, \alpha_j + \alpha_l - \beta_k, 0, \alpha'_i - \beta_k + \alpha_j - \beta'_l, \alpha'_k - \beta_i + \alpha_l, \alpha'_i - \beta'_j + \alpha_l, \alpha'_k - \beta'_l, \alpha_j - \beta_k)
\]

\[
\alpha'_i - \beta'_j + \alpha_l \geq \alpha'_i - \beta_k + \alpha_l - \beta'_j \]  
\[
\alpha'_k - \beta_i + \alpha_l \geq \alpha'_k - \beta_i + \alpha_j - \beta'_l
\]

Since \( i < k, j < l, \) and \( \{U\}, \{V\}, \{\alpha'\}, \{\beta'\}, \{\alpha\} \) and \( \{\beta\} \) are all nondecreasing, we have

\[
(U_k - U_i)(V_l - V_j) \geq 0 
\]

\[
\alpha'_k - \beta_i + \alpha_l \geq \alpha'_i - \beta_k + \alpha_l - \beta'_j 
\]

\[
\alpha'_k - \beta_i + \alpha_l \geq \alpha'_k - \beta_i + \alpha_j - \beta'_l
\]

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In other words, in (55), the first term is nonnegative and for any term in the second maximization operation, there is at least one term at least as large in the first maximization operation. Therefore, we have $e_{i;l} + e_{k;j} - (e_{i,j} + e_{k,l}) \geq 0$. ■

E  Proof of Property 5

Assume that $L_t(d_t - L_t) \neq 0$ in an optimal solution. From the constraints, we must have $L_t > 0$ and $L_t < d_t$. Construct another solution by modifying $L_t$ and $L_t$ into $L'_t = L_t - \Delta$ and $L'_t = L_t + \Delta$, respectively, with $\Delta = \min[L_t, d_t - L_t] > 0$. This new solution also verifies all the constraints, considering the fact that $\tau < t$. Furthermore, due to the fact that $\lambda'_t \geq \lambda_t$, the cost does not increase, which means that either the new solution is also optimal or the initial solution is not optimal which is in contradiction with the assumption.

F  Proof of Property 7

Assume that there is an optimal solution such that $i < \tau < t < j$, $\lambda'_t < \lambda_t$ and $(d_t - L_t)[\bar{R}(i, m) - \sum_{l=i+1}^{m} L_l L_t] \neq 0$ for any $m$ such that $\tau \leq m < t$. From the constraints, we must have $L_t < d_t$, $L_t > 0$ and $\sum_{l=i+1}^{m} L_l < \bar{R}(i, m)$ for any $m$ such that $\tau \leq m < t$. Construct another solution by modifying $L_t$ and $L_t$ into $L'_t = L_t + \Delta$ and $L'_t = L_t - \Delta$, respectively, with

$$\Delta = \min \left\{ d_t - L_t, L_t, \min_{\tau \leq m < t} \left[ \bar{R}(i, m) - \sum_{l=i+1}^{m} L_l \right] \right\} > 0$$

This new solution also verifies all the constraints. Furthermore, due to the fact that $\lambda'_t < \lambda_t$, the cost decreases, which means that the initial solution is not optimal which is in contradiction with the assumption.

G  Proof of Theorem 3

In this proof, we respectively use $\omega_t$ and $\omega'_t$ instead of $\omega(i, j)$ and $\omega(i+1, j)$, to simplify the notation.

Assume that there is some $\tau$ such that $i + 2 \leq \tau < j$ and $0 \leq \omega_\tau < \omega'_\tau \leq d_\tau$. Let $\tau^*$ be the largest one of such $\tau$'s; namely,

$$0 \leq \omega_{\tau^*} < \omega'_{\tau^*} \leq d_{\tau^*}$$  (56)
\[ \omega_t \geq \omega_t^*, \quad t = \tau^* + 1, \ldots, j - 1 \]  

Let \( \tau^* \) be the largest one of those \( t \)'s such that \( i + 1 \leq t < j \) and \( \omega_t > \omega_t^* \), if any; namely,

\[ 0 \leq \omega_t^* < \omega_t^* \leq d_{\tau^*} \]  
\[ \omega_t \leq \omega_t^*, \quad t = \tau^* + 1, \ldots, j - 1 \]

We examine all possible cases related to \( \tau^* \).

**Case 1.** \( \tau^* \) is undefined. In this case, \( \omega_t \leq \omega_t^* \) for any \( t \) such that \( i + 1 \leq t < j \) with at least one inequality being strict. We have

\[
\begin{align*}
\sum_{t=i+2}^{j-1} \omega_t^* &= \sum_{t=i+1}^{j-1} \omega_t^* > \sum_{t=i+1}^{j-1} \omega_t 
&\geq R(i, j) \geq R(i + 1, j) \\
\sum_{t=i+1}^{m} \omega_t &< \sum_{t=i+1}^{m} \omega_t^* = \sum_{t=i+1}^{m} \omega_t^* \leq R(i + 1, m) \leq R(i, m) 
&\quad m = \tau^*, \ldots, j - 1
\end{align*}
\]

The fact that \( 0 \leq \omega_{\tau^*} < \omega_{\tau^*} \leq d_{\tau^*} \) (see (56)) implies that \( L(i + 1, j) \) does not verify Property 4 if \( \lambda_{\tau^*} \geq \pi_j \) and \( L(i, j) \) does not verify Property 6 if \( \lambda_{\tau^*} < \pi_j \).

**Case 2.** \( \tau^* > t^* \). In this case, the fact that \( 0 \leq \omega_{\tau^*} < \omega_{\tau^*} \leq d_{\tau^*} \) (see (56)) and \( 0 \leq \omega_{\tau^*} < \omega_{\tau^*} \leq d_{t^*} \) (see (58)) implies, from Property 5, that \( \lambda_{\tau^*} < \lambda_{t^*} \). Therefore, from Property 7, there must be an \( m^* \) such that \( t^* \leq m^* < \tau^* \) and

\[
\sum_{t=i+2}^{m^*} \omega_t^* = R(i + 1, m^*)
\]

For any \( m \) such that \( \tau^* \leq m < j \), we have

\[
\sum_{t=i+1}^{m} \omega_t = \sum_{t=i+1}^{m} \omega_t + \sum_{t=m^*+1}^{m} \omega_t \leq R(i, m^*) + \sum_{t=m^*+1}^{m} \omega_t 
= R(i + 1, m^*) + \sigma_{i+1,1} - \sigma_{i,1} + \sum_{t=m^*+1}^{m} \omega_t 
< R(i + 1, m^*) + \sigma_{i+1,1} - \sigma_{i,1} + \sum_{t=m^*+1}^{m} \omega_t^* 
= \sum_{t=i+2}^{m^*} \omega_t^* + \sigma_{i+1,1} - \sigma_{i,1} + \sum_{t=m^*+1}^{m} \omega_t^* 
\leq R(i + 1, m) + \sigma_{i+1,1} - \sigma_{i,1} = R(i, m)
\]

Therefore, from Property 6, we must have \( \lambda_{t^*} \geq \pi_j \). From Property 4, we have

\[
\sum_{t=m^*+1}^{j-1} \omega_t^* = \sum_{t=i+2}^{j-1} \omega_t^* - \sum_{t=i+2}^{m^*} \omega_t^* = R(i + 1, j) - \sum_{t=i+2}^{m^*} \omega_t^* 
= R(i + 1, j) - R(i + 1, m^*) = d_{0,j-1} - \sigma_{m^*,1} 
= R(i, j) - \sum_{t=i+1}^{j-1} \omega_t - R(i, m^*) 
\leq \sum_{t=i+1}^{j-1} \omega_t - \sum_{t=i+1}^{m^*} \omega_t = \sum_{t=m^*+1}^{j-1} \omega_t
\]

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This is in contradiction with the fact that \( t^* \leq m^* < \tau^* \) which implies, from (56) and (59),
\[
\sum_{t=m^*+1}^{j-1} \omega_t < \sum_{t=m^*+1}^{j-1} \omega_t.
\]

Case 3. \( \tau^* < t^* \). This case can be proved to be impossible in a symmetrical manner to case 2.

## H Proof of Property 8

Let \( m' = \max \{ m | i + 1 \leq m < j, \sum_{t=i+1}^{m} \omega_t(i+1, j) = \bar{R}(i+1, m) \} \). Note that \( m' \) necessarily exists, since \( \sum_{t=i+1}^{m} \omega_t(i+1, j) = 0 = \bar{R}(i+1, i+1) \).

If \( m' = j - 1 \), from the constraint (45), we would have
\[
\sum_{t=i+1}^{j-1} e_t = \sum_{t=i+1}^{j-1} \omega_t(i, j) - \sum_{t=i+1}^{j-1} \omega_t(i+1, j) \leq \bar{R}(i, j - 1) - \sum_{t=i+1}^{j-1} \omega_t(i+1, j)
\]
\[
= \bar{R}(i, j - 1) - \sum_{t=i+1}^{j-1} \omega_t(i+1, j) = \bar{R}(i, j - 1) - \bar{R}(i + 1, j - 1) = \sigma_{t+1,1} - \sigma_{i,1}
\]

When \( m' < j - 1 \), assume that there is an optimal solution such that \( \sum_{t=i+1}^{j-1} e_t > \sigma_{i+1,1} - \sigma_{i,1} \). Then from (46), we have \( \sum_{t=i+1}^{j-1} e_t > \bar{R}(i, j) - \sum_{t=i+1}^{j-1} \omega_t(i+1, j) \). Taking into account the constraint (45), we have
\[
\sum_{t=i+1}^{j-1} e_t = \sum_{t=i+1}^{j-1} \omega_t(i, j) - \sum_{t=i+1}^{j-1} \omega_t(i+1, j) \leq \bar{R}(i, m') - \sum_{t=i+1}^{m'} \omega_t(i+1, j)
\]
\[
= \bar{R}(i, m') - \bar{R}(i + 1, m') = \sigma_{i+1,1} - \sigma_{i,1}
\]

there must be some \( t \) such that \( m' < t \leq j - 1 \) and \( 0 < e_t \leq \bar{e}_t \); i.e., \( \omega_t(i+1, j) < d_t \). On the other hand, from Property 6, we must have \( \lambda'_t \geq \pi'_j \), since there is no \( m \) such that \( t \leq m < j \) and \( \sum_{t=i+2}^{m} \omega_t(i+1, j) = \bar{R}(i + 1, m) \). Reducing the value of \( e_t \) will not increase the cost while satisfying all the constraints.

## I Proof of Theorem 4

Consider an optimal solution \((x^*, L^*, I^*)\) verifying Property 3, where \( x^* = (x^*_1, \ldots, x^*_T) \), \( L^* = (L^*_1, \ldots, L^*_T) \) and \( I^* = (I^*_1, \ldots, I^*_T) \) are respectively the optimal values of \( x_t \)'s, \( L_t \)'s and \( I_t \)'s. Assume that in this optimal solution there is a subplan \((i, u, j, k, v)\) with a production period \( j \) such that \( \rho_u(i, j) \geq r_v(j, k) \).

Let \( V = \sigma_{k,v} - d_{0,j-1} - \sum_{t=j}^{k} L^*_t \). We would have \( x^*_j = V - I^*_{j-1} > 0 \) from (37). As a consequence, at least one of the relations \( I^*_{j-1} = \sum_{t=i+1}^{j-1} L^*_t - R(i, j) < \rho_u(i, j) \) or \( V > r_v(j, k) \) is true.

Consider the case \( I^*_{j-1} < \rho_u(i, j) \). This case can occur only if \( u = 1 \), since when \( u = 0 \) we would have \( \rho_u(i, j) = I^*_{j-1} = 0 \). When \( u = 1 \) and \( 0 \leq I^*_{j-1} = \sum_{t=i+1}^{j-1} L^*_t - R(i, j) < \rho_u(i, j) = \sum_{t=i+1}^{j-1} \omega_t(i, j) - R(i, j) \), we must have
\[
R(i, j) \leq \sum_{t=i+1}^{j-1} L^*_t < \sum_{t=i+1}^{j-1} \omega_t(i, j)
\]
(60)
From this relation, we can deduce two observations.

1. There must be a \( \tau \) such that \( i < \tau < j \) and \( 0 \leq L^*_\tau < \omega_\tau(i, j) \leq d_\tau \). In the remainder of the proof, let
   \[
t^* = \max\{\tau | i < \tau < j, L^*_\tau < \omega_\tau(i, j)\}
   \]

2. From property 4, we have \( \omega_\tau(i, j) = 0 \) for any \( t \) such that \( i < t < j \) and \( \lambda'_t \geq \pi'_j \).

These observations imply that

\[
\lambda^*_t < \pi'_j
\]

Let \( m' = \max\{m | i \leq m < j, \sum_{i=1}^{m} L^*_t = \tilde{R}(i, m)\} \). Note that \( m' \) necessarily exists, since \( \sum_{i=1}^{j} L^*_t = 0 = \tilde{R}(i, i) \). Since \( \sum_{t=i+1}^{m'} \omega_t(i, j) \leq \tilde{R}(i, m') = \sum_{t=i+1}^{m'} L^*_t \) whereas \( \sum_{t=i+1}^{j} \omega_t(i, j) > \sum_{t=i+1}^{j-1} L^*_t \) (see (60), we have \( \sum_{t=m'+1}^{j-1} \omega_t(i, j) > \sum_{t=m'+1}^{j-1} L^*_t \). Therefore, by the definition of \( t^* \), we have \( t^* > m' \).

The fact that \( \lambda^*_t < \pi'_j \), \( t^* > m' \) and \( L^*_t < d_\tau' \) imply that the cost of the subplan can be strictly reduced by increasing the value of \( L^*_t \), which is in contradiction with the optimality of the solution.

As a matter of result, we must have \( I^*_{j-1} \geq \rho_u(i, j) \).

It can be proved in a similar way that the subplan is not optimal if \( V > r_v(j, k) \).

\[ \textbf{J Proof of Theorem 5} \]

When \( u = 0 \), we have \( \rho_u(i, j) = \rho_u(i + 1, j) = 0 \). Therefore, \( \rho_u(i, j) \leq \rho_u(i + 1, j) \) holds.

From Property 8, when \( u = 1 \), we have

\[
\rho_1(i, j) = \sum_{t=i+1}^{j-1} \omega_t(i, j) - R(i, j)
\]

\[
= \left( \sum_{t=i+1}^{j-1} \omega_t(i, j) - \sum_{t=i+2}^{j-1} \omega_t(i + 1, j) \right) + \sum_{t=i+2}^{j-1} \omega_t(i + 1, j) - R(i, j)
\]

\[
\leq \sigma_{i+1,1} - \sigma_{i,1} + \sum_{t=i+2}^{j-1} \omega_t(i + 1, j) - R(i, j)
\]

\[
= -R(i + 1, j) + \sum_{t=i+2}^{j-1} \omega_t(i + 1, j)
\]

\[
= \rho_1(i + 1, j)
\]