\textbf{$\mathcal{L}_2$ stability for quantized linear systems with saturations}

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Abstract: This paper deals with ultimate bounded stability analysis and stabilization conditions for systems involving input saturation and quantized control law, which corresponds to the state quantization case. The state feedback control design problem is then addressed. Theoretical results to ensure the ultimate boundedness and the $\mathcal{L}_2$ stability of the closed-loop system are presented both in local as well as global contexts. The saturation and quantized nonlinearities are tackled through the use of some modified sector conditions. The proposed conditions are then cast in convex optimization problems aiming at maximizing the region of attraction of the closed-loop system and minimizing the set in which the closed-loop trajectories are ultimately bounded, maximizing the bound of admissible $\mathcal{L}_2$ disturbances or maximizing the $\mathcal{L}_2$-gain from the disturbance to the regulated output.

Keywords: Saturation, Quantized control law, Regional and global stability, $\mathcal{L}_2$ performance.

1. INTRODUCTION

Since at least on decade, we assist to a drastic change of control paradigm. Many controlled systems have to share a common information medium and are now interconnected by digital networks (Murray et al. (2003)). Apart from the transmission delays, the decentralized information structure, the phenomenon of quantization arises naturally since only a finite alphabet can be implemented in such networks to transmit the information. Thus, the quantization is a nonlinearity which converts a signal in $\mathbb{R}^n$ into a signal which values are belonging to a finite and discrete set. Recently, it has been proved that the introduction of such devices in the control loop often induces limit cycles and chaotic behavior (Delchamps (1990), Ceragioli et al. (2010b)) and thus deteriorates the overall closed-loop performances (Elia and Mitter (2001)). For all these reasons, this new research thematic has known an important echo in the control area. Effectively, the classical control theory, based the classical assumption of the signal infinite precision, has to be reformulated. In the literature, two main approaches have been developed (see Liberzon (2009) and references therein). The first approach considers memoryless finite quantizers. In that case, the value of the quantizer at time $t$ depends only on the value of the signal at time $t$. Generally, the quantizer is then modeled as an uncertainty (Cepeda and Astolfi (2008), Corradini and Orlando (2008)) which often leads to a robust control synthesis problem. Such approaches have been also extended to the problem of $H_\infty$ synthesis, $\mathcal{L}_2$ performances (Fu and Xie (2005)) or with input delay systems (Fridman and Dambrine (2009)). Nevertheless, in all these results, the asymptotic stability in closed-loop cannot be addressed and the local ultimate boundedness stability is generally looked for. The second research way considers a dynamic and time-varying quantizer (Gao and Chen (2008), Vu and Liberzon (2008)). In that case, we can take advantage of dynamic nature of the quantizer to propose local stabilization methodologies. Hence, in Vu and Liberzon (2008), Gao and Chen (2008), Ceragioli et al. (2010b) the authors propose to scale dynamically the quantizer in order to stabilize asymptotically the system. In this paper, we are interested in the stabilization problem and $\mathcal{L}_2$ performances of linear system subject to a uniform quantizer in the output and a saturation in the input. Compared to papers from the literature (see for example Wong and Brockett (1999), Liberzon (2009)), we do not assume that the saturation is a particular effect of a quantizer, that is we do not consider that the quantizer saturates. Hence, the controller gain is encapsulated between two nonlinearities, a uniform quantizer and a saturation. This new approach allows us to consider the two nonlinearities separately and to propose some modified sector conditions and appropriate variable changes to model them acutely. Hence, our main objective is twofold: Firstly, when the disturbance is set to zero, we aim at finding both an inner and an outer set such that the trajectories of the closed-loop system starting from the outer set converge to the inner set. Secondly, when the disturbance is present, one wants to ensure that the closed-loop trajectories of the system remain bounded for any disturbance satisfying an $\mathcal{L}_2$ bound i.e. the $\mathcal{L}_2$-stability should be ensured. Notice that the main idea behind the methodology is the use of quadratic stability concept but the technique proposed does not require the open-loop system to be stable. Synthesis conditions are expressed in terms of matrix inequalities and some optimization schemes are then proposed to maximize a measure of the size of the outer set (stability domain), whereas the inner set is minimized, for a fixed disturbance to output $\mathcal{L}_2$ gain.
Depending on the open-loop stability, the global stability context is also carried out. In this case, the outer set corresponds to the whole state space and the inner one reduces to the origin.

**Notation.** 1 and 0 denote respectively the identity matrix and the null matrix of appropriate dimensions. Furthermore, \(1_m\) denotes a vector of dimension \(m\) with all components equal to 1. The elements of a matrix \(A \in \mathbb{R}^{m \times n}\) are denoted by \(A_{i,j}\), \(i = 1, \ldots, m\), \(j = 1, \ldots, n\). \(A_i\) and \(A'\) denote the ith row and the transpose of matrix \(A\), respectively. \(\text{He}(A) = A + A'\). \(|A|\) is the matrix given by the absolute value of each element of \(A\). For two symmetric matrices, \(A\) and \(B\), \(A > B\) (resp. \(A \geq B\)) means that \(A - B\) is positive definite (resp. semi-definite positive). For two vectors \(x, y\) of \(\mathbb{R}^n\), the notation \(x \geq y\) means that \(x(i) - y(i) \geq 0\), \(\forall i = 1, \ldots, n\). \(\mathbb{Z}\) is the set of integers and \(\Delta, \mathbb{Z}\) is the set of numbers of the form \(\Delta r\) with \(r \in \mathbb{Z}\).

### 2. PROBLEM STATEMENT

Consider the following continuous-time system:

\[
\begin{align*}
\dot{x} &= Ax + B\text{sat}(u) + B_ww \\
z &= Cx + Dww
\end{align*}
\]

where \(x \in \mathbb{R}^n\) and \(u \in \mathbb{R}^m\) are the state and the input of the system. \(w \in \mathbb{R}^m\) is the additive disturbance. \(z \in \mathbb{R}^n\) is the regulated output. Matrices \(A, B, B_ww, C\) and \(Dww\) are real constant matrices of appropriate dimensions. Given any vector \(v \in \mathbb{R}^n\), the saturation map \(\text{sat}(v) \in \mathbb{R}^m\) is classically defined from the symmetric saturation function in having as level the positive vector \(v_0\):

\[
\text{sat}(v_i) = \text{sign}(v_i) \min \{ |v_0(i)|, |v(i)| \}, i = 1, \ldots, m
\]

The disturbance vector \(w\) is assumed to be limited in energy, i.e. \(w \in L_2\), and for some positive scalar \(\delta\), \(0 \leq \frac{1}{\delta} \leq \infty\), the disturbance \(w\) is bounded as follows:

\[
\|w\|^2_2 = \int_0^\infty w(t)'w(t)dt \leq \frac{1}{\delta}
\]

In the following, we consider that the input of the system is the result of a quantized control laws, called the state quantization case (see Liberzon (2003)) \(u = Kq(x)\), where \(K\) is real constant matrix of appropriate dimensions. The quantizer function \(q\) is a mapping defined as:

\[
q(.) : \mathbb{R}^l \rightarrow \Delta, \mathbb{Z}^l,
\]

where \(\Delta\) is positive for all \(i = 1, \ldots, l\) and represents the quantization error bound. By denoting \(q_i(y) = q(y(i))\), the quantizer \(q(y) \in \Delta, \mathbb{Z}^l\) is described through the following condition:

\[
-\Delta \leq q_i(y(i)) - y(i) \leq \Delta, i = 1, \ldots, l
\]

Note that in the input quantization case \(l = m\) with \(u = q(Kx)\) and in the state quantization case \(l = n\) with \(u = Kq(x)\).

The problem we intend to solve in the case of state quantization (i.e., \(u = Kq(x)\)) can be summarized as follows:

**Problem 1.** Determine a stabilizing gain \(K\) and characterize two sets \(S_0\) and \(S_\infty\) such that both internal stability and \(L_2\)-stability are guaranteed.

1. **Internal stabilization.** In the absence of disturbances, one wants to ensure that for every initial condition \(x(0)\) belonging to \(S_0\), the resulting trajectories of the closed-loop system (1) converge toward \(S_\infty\).

2. **\(L_2\)-stabilization.** One wants to ensure that the closed-loop trajectories of the system remain bounded for any disturbance satisfying (3) for a certain \(\delta\), i.e. the \(L_2\)-stability should be ensured. Moreover, the controller should ensure an upper bound for the \(L_2\) gain between the disturbance \(w\) and the regulated output \(z\), which corresponds to a disturbance rejection problem.

**Remark 2.** As a matter of fact, the closed-loop quantized system is modeled by a differential equation with a discontinuous right-hand side. Existence of solutions such a differential equation has to be carefully considered and several tools have been proposed in the literature to cope with this problem (see for example Cortès (2008) and references therein for an extensive explanation). In our cases, we focus on Carathéodory solutions since the closed-loop system can be viewed as a differential equation submitted to a measurable and bounded perturbation \(q(t)\). Nevertheless, as has been pointed in Ceragioli et al. (2010a), this case may be restrictive (no solution can occur) and the notion of Krasovskii solutions should be more convenient allowing a more general solution (like sliding mode motion).

### 3. MAIN RESULTS

By defining \(\phi(Kx) = \text{sat}(K(q(x)) - Kq(x))\), which is a dead-zone nonlinearity directly defined from (2) and \(\psi(x) = q(x) - x\), which is a nonlinearity directly defined from (5), system (1) in the input quantization case is described by:

\[
\begin{align*}
\dot{x} &= (A + BK)x + B\phi(Kx) + BK\psi(x) + B_ww \\
z &= Cx + Dww
\end{align*}
\]

Note that differently from Bullo and Liberzon (2006), Fridman and Dambrine (2009), the gain \(K\) is inside the saturation argument.

**Remark 3.** Let us underline that in a neighbourhood of the origin, system (6) operates in open-loop. Actually, when no saturation occurs (i.e., \(\text{sat}(Kq(x)) = Kq(x)\)) and \(q(x) = 0\) then one gets \(\phi(Kx) = 0\) and \(\psi(x) = -x\). Thus, system (6) reads:

\[
\dot{x} = Ax + Bww
\]

Hence, if the open-loop system is unstable, the closed-loop trajectories cannot converge to the origin in the absence of disturbance. The first problem under consideration is then to evaluate a domain, as small as possible, where it is guaranteed that the trajectories will be ultimately bounded. Such a domain is the set \(S_\infty\).

**Remark 4.** The vectors \(x\) such that both \(\text{sat}(Kq(x)) = Kq(x)\) and \(q(x) = 0\) satisfy:

\[\Delta \leq 1_{n(i)}x \leq \Delta, \forall i = 1, \ldots, n\]

The following lemma extends Lemma 1 in Tarbouriech et al. (2006) to the case of the nonlinearity \(\phi(Kx) = \text{sat}(K(q(x)) - Kq(x))\).

**Lemma 5.** Consider a matrix \(G \in \mathbb{R}^{m \times n}\). The nonlinearity \(\phi(Kx) = \text{sat}(K(q(x)) - Kq(x))\) satisfies

\[\phi^T_1(\text{sat}(K(q(x)) - Kq(x)) + Gx) \leq 0\]
for any diagonal positive matrix $T_1 \in \mathbb{R}^{m \times m}$, if $x \in S(u_0)$ defined by

$$S(u_0) = \{ x \in \mathbb{R}^n ; -u_0(i) \leq G(i)x \leq u_0(i), \forall i \in \{1, \ldots, m\} \}$$

(10)

Similarly the following lemma deals with the nonlinearity $\psi(x)$ directly issued from (5). Some details can be found in Tarbouriech and Gouaisbaut (2010).

**Lemma 6.** The nonlinearity $\psi(x) = q(x) - x$ satisfies conditions

$$\psi'(T_2 x + \Delta L' T_2 x) \geq 0, \quad x \in \mathbb{R}^n$$

(11)

$$\psi'(\psi(x) + x) \leq 0, \quad x \in \mathbb{R}^n$$

(12)

for any diagonal positive matrices $T_2, T_3, T_4 \in \mathbb{R}^{n \times n}$.

Then by using Lemmas 5 and 6, the following proposition can be stated.

**Proposition 7.** If there exist two symmetric positive definite matrices $W, Q$ in $\mathbb{R}^{n \times n}$, a diagonal positive matrix $S_1$, in $\mathbb{R}^{n \times n}$, three diagonal positive matrices $S_2, S_3, S_4$ in $\mathbb{R}^{n \times n}$, two matrices $Y, Z$ in $\mathbb{R}^{n \times n}$, positive scalars $\tau_1, \tau_2, \mu, \beta$ satisfying

$$
\begin{bmatrix}
M_1 & M_2 & M_3 & M_4 & M_5 \\
* & -2S_1 & -Y & 0 & 0 \\
* & * & -W(2S_3^{-1} + S_4^{-1})W & 0 & 0 \\
* & * & * & \frac{(\tau_1 - \tau_2 \mu)}{\mu n} S_2^{-1} & 0 \\
* & * & * & * & -S_4 \\
* & * & * & * & * \\
M_6 & M_7 \\
0 & 0 & 0 & 0 & 0 \\
-1 & D' & * & * & * \\
\end{bmatrix} < 0
$$

(13)

$$Q - \mu W \geq 0$$

(14)

$$\begin{bmatrix}
W & Z_{(i)}^{(i)} \\
* & \mu a_{(i)}^{(i)} \\
\end{bmatrix} \geq 0, \quad i = 1, \ldots, m$$

(15)

$$\begin{bmatrix}
\delta - \mu & \delta \\
* & \delta + \beta \\
\end{bmatrix} \geq 0$$

(16)

and

$$L_1 = \begin{bmatrix}
\text{He}(AW + BY) - \tau_1 W + \tau_2 Q + WS_2^{-1} W \\
S_1 B' - Y - Z_2 \\
Y' B' + W(-S_3^{-1} + S_4^{-1}) W \\
\Delta D W \\
B_w \\
S_3^{-1} W - Y' \\
-2S_1 \\
- Y' \\
-2WS_3^{-1} W - WS_4^{-1} W \\
0 \\
0 \\
\Delta W D_j \\
B_w \\
0 \\
0 \\
(\tau_1 \mu^{-1} - \tau_2) S_2 \\
n 0 \\
0 \\
-1
\end{bmatrix}$$

(17)

(2) When $w \neq 0$, the closed-loop trajectories of system (6) remains bounded in $S_0$ for any initial condition $x(0)$ belonging to $S_1 = \{ x \in \mathbb{R}^n ; x' P_1 x \leq \mu^{-1} \}$ and $x' P_2 x \geq 1$. Furthermore, one gets:

$$\| z \|_2 \leq \gamma \| w \|_2 + x(0)' P_1 x(0)$$

(21)

**Proof.** By considering the quadratic Lyapunov function $V(x) = x' P_1 x$, with $P_1 = P_1^T > 0$, the time-derivative of $V(x)$ along the trajectories of system (1), or equivalently of system (6), reads:

$$\dot{V}(x) = x'(A + BK) P_1 + P_1 (A + BK) x + 2x' P_1 B \psi(K x) + 2x' P_1 B \phi(K x) + x' P_1 B w.$$

To address Problem 1, we want to verify that $J = V(x) + \frac{1}{2} z' z - w' w < 0$, for all $x$ such that $x' P_1 x \leq \mu^{-1}$ and $x' P_2 x \geq 1$, with $P_2 = P_2^T > 0$. By using Lemma 6 and the S-procedure, one can write:

$$L = \dot{V}(x) - \tau_1 (x' P_1 x - \mu^{-1}) - \tau_2 (1 - x' P_2 x) + 2(\psi(x)' S_2^{-1} x + \Delta V' S_2^{-1} x) - 2\psi(x)' S_2^{-1} (\psi(x) x) - (\psi(x) x)' S_2^{-1} (\psi(x) x) + 1 - z' z - w' w$$

with $\tau_1, \tau_2$ some positive scalars and positive diagonal matrices $S_2, S_3, S_4$.

The satisfaction of relation (16) implies that the ellipsoid $S_0 = \mathcal{E}(P_1, \mu)$, with the change of variables $P_1 = W^{-1}, \quad G = ZW$, is included in the polyhedral set $S(u_0)$ defined in (10). Thus, for any $x \in S_0$ from Lemma 5 one can write:

$$\dot{L} \leq -2\phi(K x) S_2^{-1} (\phi(K x) + K \psi(x) + K z + G z)$$

(22)

By noting that the term $|x|$ can take $2^n$ values, one can replace in (22) this term by $D_j x, j = 1, \ldots, 2^n$, by considering the $2^n$ diagonal matrices $D_j, j = 1, \ldots, 2^n$, which diagonal elements are 1 or $-1$. Thus, by noting that $\tau_1 \mu^{-1} - \tau_2 = (\tau_1 \mu^{-1} - \tau_2) 1_n 1_n$, and developing the right-hand term of (22), one gets:

$$\dot{V}(x) - \tau_1 (x' P_1 x - \mu^{-1}) - \tau_2 (1 - x' P_2 x) + \frac{1}{2} z' z - w' w \leq \xi' L_1 \xi + \frac{1}{2} z' z, \quad j = 1, \ldots, 2^n,$n with
where $Q = WPW$. By definition $D_j^j = D_1^1 = 1$, then by pre- and post-multiplying $L_1$ by $\text{diag}(1; 1; 1; D_j; 1)$, and next by applying the Schur complement, one obtains the matrix of the right-hand side in relation (14). Hence, the satisfaction of relation (14) implies that $V(x) - \tau_1(x'P_1x - \mu^{-1}) - \tau_2(1 - x'P_2x) + \frac{1}{2}z'z = w'w < 0$, or equivalently $J < 0$ for all $x$ such that $x'P_1x \leq \mu^{-1}$ and $x'P_2x \geq 1$, with $P_2 = P_2^1 > 0$. Hence, it follows that item 1 of Proposition 7 holds. Furthermore, one has to prove that the set $S_0 = \mathcal{E}(P_1, \mu)$ contains the set $S_{\infty} = \mathcal{E}(P_2, 1)$. This is verified thanks to the satisfaction of relation (15).

The satisfaction of relation (15) means that $\beta^{-1} + \delta^{-1} \leq \mu^{-1}$ by using the fact that $(\beta^{-1} + \delta^{-1})^{-1} = \delta - \delta(\delta + \beta)^{-1}\delta$.

By integrating $J$, it follows:

$$V(x(T)) - V(x(0)) + \frac{1}{\gamma} \int_0^T w'wdt < 0, \forall T > 0$$

which implies that

$$V(x(T)) < V(x(0)) + \int_0^T w'wdt \leq V(x(0)) + \|w\|_2^2 \leq \delta^{-1} \leq \mu^{-1}$$

Then the closed-loop trajectories of system (6) do not leave $S_0$ for any initial condition $x(0)$ belonging to $S_1 = \{x \in \mathbb{R}^n; x'P_1x \leq \beta^{-1} \text{ and } x'P_2x \geq 1\}$ provided that $w$ satisfies (3). Furthermore, it follows that:

$$\|z\|_2^2 \leq \gamma \|w\|_2^2 + \gamma x(0)'P_1x(0)$$

Thus, item 2 of Proposition 7 holds.

Remark 8. In Proposition 7, the inner ellipsoid $S_{\infty} = \mathcal{E}(P_2, 1)$ is not an invariant set. Nevertheless, it is possible to determine the smallest invariant ellipsoid whose shape is characterized by $W$ and containing $S_{\infty} = \mathcal{E}(P_2, 1)$. For this, it suffices to compute the greatest scalar $\eta$ such that the resulting ellipsoid $E(P_1, \eta) = \{x \in \mathbb{R}^n; x'P_1x \leq \eta^{-1}\}$ is invariant and contains $E(P_2, 1)$; in other words one has to compute the maximal scalar $\eta \geq \mu$ such that $Q - \eta W \geq 0$.

Remark 9. Similarly to Castelan et al. (2008), relations of Proposition 7 could be reformulated by using the formalism developed in Hu and Lin (2001) and the LDI-based framework. Such an alternative solution imposes to test the stability condition at $2^n$ vertices, which may increase the numerical complexity (which is closely related to the number of lines and of decision variables in an LMI optimisation setup).

The conditions given in Proposition 7 concern Problem 1 in a local context. If the matrix $A$ is Hurwitz, the global asymptotic stabilization problem can be addressed: in this case the set $S_{\infty}$ will be reduced to the origin. The following global stabilization result can then be stated.

Proposition 10. If there exist a symmetric positive definite matrix $W$ in $\mathbb{R}^{n \times n}$, a diagonal positive matrix $S_1$, in $\mathbb{R}^{m \times m}$, three diagonal positive matrices $S_2, S_3, S_4$ in $\mathbb{R}^{m \times m}$, two matrices $Y, Z$ in $\mathbb{R}^{m \times n}$, positive scalars $\tau, \mu, \gamma$ satisfying

$$\begin{bmatrix} M_1 & M_2 & M_3 & M_4 \\ M_2 & -\tau & -\tau & -\tau \\ M_3 & -\tau & -\tau & -\tau \\ M_4 & -\tau & -\tau & -\tau \end{bmatrix} \begin{bmatrix} \mu^2 & 0 & 0 & 0 \\ 0 & \mu^2 & 0 & 0 \\ 0 & 0 & \mu^2 & 0 \\ 0 & 0 & 0 & \mu^2 \end{bmatrix} \begin{bmatrix} \mu^2 & 0 & 0 & 0 \\ 0 & \mu^2 & 0 & 0 \\ 0 & 0 & \mu^2 & 0 \\ 0 & 0 & 0 & \mu^2 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \mu & 0 & 0 \\ 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & \mu \end{bmatrix} < 0$$

$$\begin{bmatrix} W & \mathbf{0} \\ \mathbf{0} & \mu^2 \end{bmatrix} \geq 0, \quad i = 1, ..., n$$

with matrices $M_i, i = 3, 4, 5, 6, 7$ defined in (18) and

$$M_1 = \text{He}(AW + BY) + \tau W; M_2 = BS_{\infty} - Y'$$

then $K = W^{-1}$ is a stabilizing gain and

(1) When $w = 0$, the global asymptotic stability of the origin is ensured for the closed-loop system (6).

(2) When $w \neq 0$, the closed-loop trajectories of system (6) remains bounded for any initial condition $x(0) \in \mathbb{R}^n$ and any disturbance $w$ satisfying (3). Furthermore, relation (21) holds, that is:

$$\|z\|_2^2 \leq \gamma \|w\|_2^2 + \gamma x(0)'P_1x(0)$$

Proof. Let us consider $G = 0$. Then the resulting condition (9) is globally satisfied, i.e., is satisfied for any $x \in \mathbb{R}^n$.

Let us consider the quadratic Lyapunov function $V(x) = x'P_1x$, with $P_1 = P_1^1 > 0$. The time-derivative of $V(x)$ along the trajectories of system (1), or equivalently of system (6), reads: $\dot{V}(x) = x'(A + BK)P_1 + P_1(A + BK)x + 2x'P_1B\phi(Kx) + 2x'P_1BK\psi(x) + 2x'P_1Bw$. To prove the global stability of the origin, we want to verify that $J = \dot{V}(x) + \frac{1}{\gamma}z'z - w'w < 0$, for all $x$. To do this, we separate the study of $J$ into two zones: one zone in which saturation and quantization are active and the second one in which neither saturation nor quantization occur. The first zone corresponds to the set of $x$ such that $x'P_1x \geq \mu^{-1}$, whereas the second one is the set $S_{\infty} = \mathcal{E}(P_1, \mu) = \{x \in \mathbb{R}^n; x'P_1x \leq \mu^{-1}\}$. $P_1 = W^{-1}$. Furthermore, we impose that the set $S_{\infty} = \mathcal{E}(P_1, \mu) = \{x \in \mathbb{R}^n; x'P_1x \leq \mu^{-1}\}$, $P_1 = W^{-1}$, is included in the region of linearity $S(\Delta) = \{x \in \mathbb{R}^n; -\Delta \leq \mu \leq \Delta, \forall \mu \in [1, ..., n]\}$ as defined in Remark 4.

By following the same reasoning as in the proof of Proposition 7, one can verify that the satisfaction of relation (23) guarantees that $J < 0$ for all $x$ such that $x'P_1x \geq 1$. Moreover, the inclusion of the set $S_{\infty} = \mathcal{E}(P_1, \mu)$ in $S(\Delta)$ is ensured thanks the satisfaction of (25). Hence, in $\mathcal{E}(P_1, \mu)$, the closed-loop trajectories are those of system (7). The satisfaction of relations (24) means that $J < 0$ along the trajectories of system (7). Therefore by integrating $J$, it follows:

$$V(x(T)) < V(x(0)) + \int_0^T w'wdt \leq V(x(0)) + \|w\|_2^2 \leq \delta^{-1} \leq \mu^{-1}$$

and

$$\|z\|_2^2 \leq \gamma \|w\|_2^2 + \gamma x(0)'P_1x(0)$$

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Thus, one can conclude that the second item of Proposition 10 holds.

By the same way, in the case \( w = 0 \), one can prove that the satisfaction of relations (23), (24) and (25) guarantees that \( V(x) + \frac{1}{2} z'z < 0 \), or equivalently \( \dot{V}(x) < 0 \) along the trajectories of system (6). Hence, the first item of Proposition 10 holds. □

**Remark 11.** To address Problem 1 with respect to system (1) or (6), one can observe that three cases can be handled. Actually, these cases arise directly from the definition of saturation and quantizer nonlinearities and can be described as follows:

- **Case 1.** Saturation and quantizer are active (\( sat(Kq(x)) \)); one has to study the system (6);
- **Case 2.** Saturation non active (\( sat(Kq(x)) = Kq(x) \)) but quantizer is active; one has to study the behavior of the system \( \dot{x} = (A + BK)x + BK\psi(x) + B_w w \), which is valid in the set
  \[
  S(w, \Delta) = \{ x \in \mathbb{R}^n; -u_{0(i)} \leq Kq(x) \leq u_{0(i)} \text{ and } -\Delta \leq \psi(j)(x) \leq \Delta, \forall i = 1, ..., m, j = 1, ..., n \}
  \]
- **Case 3.** Neither saturation nor quantizer are active: this case corresponds to Remark 3 and therefore the behavior of the open-loop system (7) has to be studied

In each case, one could define different \( L_2 \) gains.

### 4. NUMERICAL ISSUES

Proposition 7 exhibits the inclusion of one inner ellipsoid into an outer one which is proved to be contractive provided that the state \( x \) does not belong to the inner ellipsoid. An optimization problem may be then stated to simultaneously evaluate the smallest inner ellipsoid, the largest outer ellipsoid and the smallest \( \gamma \) such that Proposition 7 is satisfied.

For Proposition 10, as the stabilization is global, the optimization scheme to be followed is to find the smallest \( \gamma \) which satisfied inequalities (23), (24), (25). Nevertheless, due to the product of several unknown variables namely \(-W(2S_1^{-1} - S_2^{-1}) W, S_2^{-1} W(-S_1^{-1} + S_3^{-1}) W, \) inequalities (14), (23) are not Linear Matrix Inequalities and thus the optimization schemes are difficult to solve. In this part, we aim at giving some ways to solve efficiently such inequalities. Indeed, by choosing \( S_2 = S_3 = \tau_3^{-1}1 \), \( S_4 = \tau_4^{-1}1 \) and by imposing the constraint

\[
\begin{bmatrix}
\delta 1 & 1 \\
1 & W
\end{bmatrix} > 0, \tag{27}
\]

to be satisfied with a scalar \( \delta > 0 \), the two inequalities (14), (23) are modified as follows.

- **Proposition 7.** Inequality (14) is replaced by

\[
\begin{bmatrix}
M_1 & M_2 & \delta M_3 & \tau_3 M_4 & \tau_4 M_5 \\
* & -S_1 & -\delta Y & 0 & 0 \\
* & * & -\delta(2\tau_3 + \tau_4) W & 0 & 0 \\
* & * & * & \frac{(\tau_1 - \tau_2\mu)}{\mu} & 1 \\
* & * & * & * & -\tau_1 \end{bmatrix} < 0, \tag{28}
\]

where \( \tau_1, \tau_2, \tau_3, \tau_4 \) are now tuning parameters and are chosen a priori.

- **Proposition 10.** Inequality (23) is replaced by

\[
\begin{bmatrix}
M_1 & M_2 & \delta M_3 & \tau_1 M_4 & \tau_2 M_5 & M_6 & M_7 \\
* & -S_1 & -\delta Y & 0 & 0 & 0 & 0 \\
* & * & -\delta(2\tau_3 + \tau_4) W & 0 & 0 & 0 & 0 \\
* & * & * & \frac{1}{\mu} & -\tau_1 & 0 & 0 \\
* & * & * & * & * & -1 & \tau_1 \]
\]

\[
< 0, \tag{29}
\]

where \( \tau_1, \tau_2, \tau_3, \tau_4 \) are now tuning parameters and are chosen a priori.

Notice that the parameters \( \tau_i \), with \( i = \{1, ..., 4\} \) are the result of the use of the S-procedure. In the forthcoming example, these parameters are usually fixed. Thus, we can then conduct some optimization strategies to obtain a set \( S_0 \) as large as possible and a set \( S_\infty \) as small as possible (for proposition 7) and all the while minimising the \( L_2 \) gain \( \gamma \).

### 5. EXAMPLE

To illustrate Proposition 7, let us consider system (1) described by the following data:

\[
A = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}; B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}; B_w = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix} \tag{30}
\]

\[
C = [0.1 \ 0.1]; D_w = 0.1
\]

The additive disturbance input \( w \) has a limited energy bound as defined in (3) given by \( \delta = 3 \). Furthermore, the level of saturation is given by \( u_0 = 1 \), whereas the state is quantized with \( \Delta = 0.2 \).

Using the optimization procedure associated to Proposition 10 with

\[
\tau_1 = 10^{-7}, \tau_2 = 0.2, \tau_3 = 1, \tau_4 = 10^{-7}, \delta = 0.9
\]

one gets:

\[
K = [-1.3417 \ -0.8605] \text{ and } \gamma = 0.61
\]

Figure 1 shows several closed-loop trajectories issued from \( x(0) \in S_0 \) which converge to a limit cycle contained in the inner set \( S_\infty \), when the disturbance is set to zero. Some
trajectories starting from outside $S_0$ are also steered to $S_{\infty}$. This illustrates the conservatism of our method.

![Ellipsoid $S_0$ (solid line), Ellipsoid $S_{\infty}$ (dashed line) and convergent/ divergent trajectories](image)

6. CONCLUSION

This paper presented theoretical conditions to deal with stabilization problem for systems involving input saturation and state quantization control law. Constructive conditions in the sense that numerical procedure was associated, guaranteed the ultimate boundedness and the $L_2$ stability of the closed-loop system both in local as well as global contexts. The saturation and quantized nonlinearities were tackled through the use of some modified sector conditions. The proposed conditions were then cast in convex optimization problems aiming at maximizing the region of attraction of the closed-loop system and minimizing the set in which the closed-loop trajectories are ultimately bounded, maximizing the bound of admissible $L_2$ disturbances or maximizing the $L_2$-gain from the disturbance to the regulated output.

Beyond the improvement of the conditions presented, there are still several open questions. The first one concerns the fact to decrease the conservatism by investigating more complex control law using in particular by adding an independent dither signal to the quantizer input. Furthermore, different classes of quantizers should be investigated with the goal to link the stability condition to the transmission capacity of the channel allowing to define the controller.

REFERENCES


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