Continuous and discrete state reconstruction for nonlinear switched systems via high-order sliding-mode observers

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A class of nonlinear switched systems is studied, and a finite-time converging state observer is proposed. The observer strategy, based on the high-order sliding-mode approach, is able to reconstruct both the continuous and discrete states of the switched system based on output measurements and on the knowledge of the set of possible system's dynamics. All the 'operating modes' of the switched system are required to satisfy certain observability-like and boundedness restrictions. The observer provides a finite-time converging estimate and, after a switching of the active mode, it features an arbitrarily fast transient to recover the correct (continuous and discrete) state estimate. A numerical example illustrates the performance of the proposed observer.

Keywords: high-order sliding-mode observers; nonlinear observers; switched systems

1. Introduction

Hybrid systems represent a rich area of investigation in modern control theory (see e.g. the tutorial papers van der Schaft and Schumacher 2000; Liberzon 2003; Christofides and El-Farra 2005; Lin and Antsaklis 2009) due to many systems of engineering relevance that can be effectively modelled in the hybrid systems framework. Generally speaking, hybrid systems denote dynamic systems that mix real-time (continuous) and discrete event dynamics. The continuous and discrete dynamics may not only co-exist but can also interact, and changes occur in response to discrete events and/or continuous inputs, both internally or exogenously generated.

One of the possible representations of hybrid systems refers to the so-called ‘switched’ systems. A switched system is a dynamic system that consists of a finite number of operation modes (or locations) along with a ‘switching logic’ that sets the current mode of operation through instantaneous transitions from one mode to another. In these systems, a discrete and a continuous state are usually identified and distinguished, the first one being an integer number corresponding to the active operation mode, and the latter being the set of continuous variables that evolve in time according to the active mode’s dynamics. The reconstruction of the continuous and/or discrete states by means of output measurements is actively studied in all its possible versions (see e.g. Alessandri and Coletta 2001; Chen and Lagoa 2005; Baglietto, Battistelli, and Scardovi 2007). The problem of the simultaneous discrete and continuous state reconstruction appears to be of particular challenge.

Sliding-mode control techniques have been used to control switched systems as well, due to their robustness and finite-time convergence features. In Lian and Zhao (2009), an output feedback controller is designed for uncertain linear switched systems using a standard sliding-mode controller; an $H_\infty$ sliding-mode control is suggested to stabilise a class of uncertain switched delay system in Lian, Zhao, and Dimirovski (2009); an integral sliding-mode controller is applied to stabilise a switched nonlinear system in Lian, Zhao, and Dimirovski (2010). These enhanced control techniques could be used to design new types of observers with possibly better robustness features as compared with the observers based on standard algorithms.

In this field, sliding-mode-based observers (Utkin, Guldner, and Shi 1999) have been successfully implemented. In Saadaoui, Mammanin, Djemai, Barbot, and Floquet (2006), a second-order sliding-mode observer is implemented to reconstruct the continuous and discrete states of the switching system. In Saadaoui, Leon, Djemai, Mammanin, and Barbot (2006) a step-by-step second-order sliding-mode observer is applied for continuous state reconstruction in the presence of model uncertainty.

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The step-by-step approach to state observation by cascade differentiation (see e.g. Saadaoui et al. 2006) is closely related to the concept of differential flatness (Fließ, Levine, Martin, and Rouchon 1995). Indeed, if some measurable output of the system turns out to be a flat output then all state variables can be expressed by an algebraic function of the flat output and of a certain number of its derivatives, which makes it possible to recover the system’s state by differentiating the flat outputs in real time. The concept of flatness is being recently studied in the framework of switched systems too (Millerioux and Daafouz 2009; Vo Tan, Millerioux, and Daafouz 2010).

In this work, we study the problem of simultaneous discrete and continuous state observation for a class of nonlinear switched system by using several instances of the ‘quasi-continuous’ high order sliding-mode observer that was proposed in Davila, Fridman, Pisano, and Usai (2008, 2009) for some classes of nonlinear non-switched systems.

The features of the above-mentioned observers (Davila et al. 2008, 2009) are generalised in this study to the setting of nonlinear switched systems with uncertain discrete state (or, equivalently, ‘location’), and are exploited to reconstruct both the continuous and discrete states of the nonlinear switched system.

Here we study the case in which a scalar output is available, which is the most challenging situation. However, the method can be easily extended to the multiple output case. The multiple output case for nonlinear non-switched dynamics is addressed by one of the authors in the paper (Rios, Davila, and Fridman 2010). The extension can be quite simply derived from the scalar output result, and it would not bring new significant ideas. On the other hand, the mathematical treatment would be much more complicated, hence we prefer to limit the scope of this article to the scalar output case.

In the closely related study, Barbot, Saadaoui, Djemai, and Manamanni (2007), the considered system class was almost the same as that studied in this article. The main difference lies in the discrete mode function structure, and the way it is reconstructed. In Barbot et al. (2007), the actual mode was driven by guards in the state space, which were assumed to be known to the designer. Inside each closed region surrounded by the guards, a specific location is active. Then, once the continuous state has been reconstructed, one can reconstruct the current location by simple logical operations.

In this article we assume, more generally, that the current location is assigned by an external supervisor according to a logic (either state-driven or time-driven) which is unknown to the designer.

After the switching time instants at which the dynamics is abruptly changing, there is a finite-time transient, originated by the consequent abrupt changes in the high-order derivatives of the output variables, which is needed to recover the exact state estimated. The transient time can be reduced arbitrarily by appropriate observer tuning. The estimated observers’ equivalent control signals are used to reconstruct the switching signal that determines the current mode of operation.

The article is structured as follows. Section 2 introduces the main notions that will be used in this article. Section 3 studies the error dynamics for the non-switching dynamic. Section 4 presents the overall design of the observers input-injection signals, while Section 5 describes the proposed technique for the reconstruction of the switching signal. An academic simulation example is discussed in Section 6 to illustrate the observer performance.

2. Observation of nonlinear switched systems

We consider the following class of nonlinear autonomous switched systems:

\[ \dot{x} = f_i(x) \]

with the state vector \( x \in \mathcal{X} \subseteq \mathbb{R}^n \) and the so-called ‘switching signal’ (or ‘discrete state’) \( \tilde{s}(t) : [0, \infty) \to \{1, 2, \ldots, q\} \). The switching signal determines the current system dynamics among the possible \( q \) ‘operation modes’ \( f_1(x, u), f_2(x, u), \ldots, f_q(x, u) \). It can be either generated by an exogenous entity (e.g. an external supervisor) or driven by the current system state. Define as \( t_{\tilde{s},r} (r=1,2,\ldots) \) the ‘switching instants’ at which the system dynamics changes. Then, the piecewise-constant function \( \tilde{s}(t) \) undergoes discontinuous jumps at the switching instants.

Let a scalar output \( y \in \mathcal{Y} \subseteq \mathbb{R} \) be available for measurements such that

\[ y = h(x). \]

In this article, a multi-observer HOSM-based strategy is designed to reconstruct both, the continuous-time system state, and the operation mode of the system by means of switching function identification.

We consider \( q \) observers having the following structure:

\[ \dot{\hat{x}}_i = f_i(\hat{x}), \quad i = 1, 2, \ldots, q, \]

\[ \hat{y}_i = h(\hat{x}_i), \]

with observed state vectors \( \hat{x}_i \in \mathbb{R}^n \) \((i=1,2,\ldots,q)\), observed output variable \( \hat{y}_i \in \mathbb{R} \) and observer injection input \( u_i \). The vector fields \( g_i(\cdot) \in \mathbb{R}^p \) and the corrective injection terms \( u_i \in \mathbb{R} \) will be designed in this article. Let \( t_0 = 0 \) be the initial time instant. It is assumed that
all the possible solutions of system (1) exist in the whole semi-axis \( t \geq 0 \).

Under certain ‘observability conditions’ that are going to be specified, we shall select the vector functions \( g_i(\hat{x}_i) \) in such a way that the observer outputs \( \hat{y}_i \) have full relative degree \( n \) with respect to the associated observer input \( u_i \).

With reference to a generic scalar function \( \alpha \) with vector argument \( z \) defined on an open set \( \Omega \subset \mathbb{R}^n \), denote

\[
\frac{\partial \alpha(z)}{\partial z} = \begin{bmatrix}
\frac{\partial \alpha(z)}{\partial z_1} & \frac{\partial \alpha(z)}{\partial z_2} & \ldots & \frac{\partial \alpha(z)}{\partial z_n}
\end{bmatrix}
\]

Define the following \( n \) th-order square matrices

\[
M_i(z) = \begin{bmatrix}
\frac{d \alpha(z)}{d z}
\hline
\frac{d L_{f_i(z)}^{(1)} h(z)}{d z}
\hline
\vdots
\hline
\vdots
\hline
\frac{d L_{f_i(z)}^{(n-1)} h(z)}{d z}
\end{bmatrix}, \quad i = 1, 2, \ldots, q,
\]

where \( L_{f_i(z)} h(z) \) is sometimes called the Lie derivative (see the work by Isidori (1996)) of \( h(z) \) along \( f(z) \) and is defined as \( L_{f_i(z)} h(z) = \frac{d \alpha(z)}{d z} L_{f_i(z)} ^{1} h(z) \) and the \( k \) th derivative of \( h(z) \) along \( f(z) \) is defined as \( L_{f_i(z)}^{k} h(z) = \frac{d \alpha(z)}{d z} L_{f_i(z)}^{k-1} h(z) \).

Let the following assumptions be satisfied.

**Assumption 1:** The \( q \) matrices \( M_i(z) \) in (4) are non-singular for every possible value of \( z \).

Let us select the vector functions \( g_i(\hat{x}_i) \) in (3) as the unique solution of equations

\[
M_i(\hat{x}_i) g_i(\hat{x}_i) = [0, 0, \ldots, 1]^T, \quad i = 1, 2, \ldots, q.
\]

We get

\[
g_i(\hat{x}_i) = M_i^{-1} \hat{x}_i \cdot [0, 0, \ldots, 1]^T,
\]

i.e. \( g_i(\hat{x}_i) \) is the last column of matrix \( M_i^{-1}(\hat{x}_i) \). Note that \( g_i(\hat{x}_i) \) is well-defined, according to Assumption 1.

In light of (6) the observer input/output dynamics for the \( q \) observers are

\[
\frac{d}{dt} \begin{bmatrix}
\hat{y}_i \\
\hat{x}_i \\
\hat{x}_i^{(n-1)}
\end{bmatrix} = \begin{bmatrix}
L^{(1)}_{f_i(\hat{x}_i)} h(\hat{x}_i) \\
L^{(2)}_{f_i(\hat{x}_i)} h(\hat{x}_i) \\
\vdots \\
L^{(n)}_{f_i(\hat{x}_i)} h(\hat{x}_i)
\end{bmatrix} + \begin{bmatrix}
0 \\
\vdots \\
0 \\
1
\end{bmatrix} u_i,
\]

which means that all observer output variables \( \hat{y}_i \) have full relative degree \( n \) with respect to the corresponding observer input \( u_i \).

Define the output and state observation errors

\[
e_{y_i} = \hat{y}_i - y, \quad e_{x_i} = \hat{x}_i - x = \begin{bmatrix}
e_{x_{i1}} \\
e_{x_{i2}} \\
\vdots \\
e_{x_{in}}
\end{bmatrix}.
\]

Clearly, the output error \( e_{y_i} = \hat{y}_i - y \) possesses the same relative degree \( n \) with respect to \( u_i \).

Define also the output error vector \( e \), containing the output error \( e_i \) and its first \((n-1)\) derivatives:

\[
e_i = \begin{bmatrix}
e_{y_1} \\
e_{y_2} \\
\vdots \\
e_{y_n} \\
\end{bmatrix} = \begin{bmatrix}
e_{x_{11}} \\
e_{x_{12}} \\
\vdots \\
e_{x_{1(n-1)}}
\end{bmatrix}.
\]

**3. Error analysis for non-switching dynamics**

As a preliminary step, let us study a non-switching dynamics, i.e. let \( f(t) \equiv f^* = \text{const} \forall t \geq 0 \). The system dynamics is

\[
\dot{x} = f_p(x), \quad t \geq 0.
\]

To each of the \( q \) observers (3) can be associated with the corresponding output error \( e_{y_i} = \hat{y}_i - y \), state error \( e_{x_i} = \hat{x}_i - x \) and output error vector \( e_i \).

**Lemma 1:** Consider system (9) and the observers (3), (6), and let Assumption 1 be satisfied. Then the following implications hold:

\[
e_i = 0 \iff e_{x_i} = 0, \quad i = f^*.
\]

**Proof:** The observation error dynamics for the \( i \)th observer is given by

\[
\dot{e}_{x_i} = f_{p}(\hat{x}_i) - f_p(x) + g_i(\hat{x}_i)u_i,
\]

\[
e_{y_i} = h(\hat{x}_i) - h(x).
\]

Let us derive, the expression for the successive derivatives of the output observation error \( e_{y_i} \), up to the order \( n \). Considering (5), it yields

\[
e_{y_i}^{(j)} = L^{(j)}_{f_p(x)} h(\hat{x}_i) - L^{(j)}_{f_p(x)} h(x), \quad 1 \leq j \leq n - 1,
\]

\[
e_{y_i}^{(n)} = L^{(n)}_{f_p(x)} h(\hat{x}_i) - L^{(n)}_{f_p(x)} h(x) + u.
\]

Considering that \( x = \hat{x}_i - e_{x_i} \), Equations (12)–(14) allow for constructing explicitly the following diffeomorphic mappings (Isidori 1996):

\[
e_i = \Phi(e_{x_i}, \hat{x}_i) = \co1 \{ \Phi_{y}(e_{y_i}, \hat{x}_i) \}, \quad 1 \leq j \leq n.
\]
whose entries can be rewritten componentwise as follows:

\[
\Phi_{1k}(e_{x_k}, \hat{x}_i) = h(\hat{x}_i) - h(\hat{x}_i - e_{x_k}),
\]

(16)

\[
\Phi_{2k}(e_{x_k}, \hat{x}_i) = L_{f_j}(\hat{x}_i)h(\hat{x}_i) - L_{f_j}(\hat{x}_i - e_{x_k})h(\hat{x}_i - e_{x_k}),
\]

(17)

\[
\vdots
\]

\[
\Phi_{dk}(e_{x_k}, \hat{x}_i) = L_{f_j}^{d-1}(\hat{x}_i)h(\hat{x}_i) - L_{f_j}^{d-1}(\hat{x}_i - e_{x_k})h(\hat{x}_i - e_{x_k}).
\]

(18)

If \( e_{x_k} = 0 \) then \( e_{x_k} \equiv 0 \), which means that all components of vector \( e_i \) are identically zero irrespectively of \( \hat{x}_i \), i.e. the mapping \( \Phi \) fulfills the condition \( \Phi(0, \hat{x}_i) = 0 \). In order to prove the reversed implication, let us consider the inverse mapping

\[
e_{x_k} = \Phi^{-1}(e_i, \hat{x}_i).
\]

(19)

It must be shown that

\[
\Phi^{-1}(0, \hat{x}_i) = 0 \quad \forall \hat{x}_i.
\]

(20)

In other words, the lemma holds provided that the mapping \( \Phi(e_{x_k}, \hat{x}_i) \) is locally bijective in the neighbourhood of \( e_{x_k} = 0 \). Considering the Jacobian matrix \( J(e_{x_k}, \hat{x}_i) = \frac{\partial \Phi(e_{x_k}, \hat{x}_i)}{\partial e_{x_k}} \), the bijectivity of the mapping corresponds to the following condition:

\[
\det J(0, \hat{x}_i) \neq 0 \quad \forall \hat{x}_i.
\]

(21)

It is now demonstrated that condition (21) is implied by Assumption 1. Since \( x = \hat{x}_i - e_{x_k} \), it makes sense to denote \( x = x(e_{x_k}, \hat{x}_i) \). Let us derive the Jacobian matrix \( J(e_{x_k}, \hat{x}_i) \). By (12)–(14) it derives that

\[
\frac{\partial \Phi_{1k}(e_{x_k}, \hat{x}_i)}{\partial e_{x_k}} = -\frac{\partial h(x)}{\partial x} \frac{\partial x(e_{x_k}, \hat{x}_i)}{\partial e_{x_k}},
\]

(22)

\[
\frac{\partial \Phi_{jk}(e_{x_k}, \hat{x}_i)}{\partial e_{x_k}} = -\frac{\partial L_{f_j}^{j-1}(\hat{x}_i)h(x)}{\partial x} \frac{\partial x(e_{x_k}, \hat{x}_i)}{\partial e_{x_k}}, \quad 2 \leq j \leq n.
\]

(23)

Substituting \( x = \hat{x}_i - e_{x_k} \) into (22) and (23), noticing that \( \frac{\partial x(e_{x_k}, \hat{x}_i)}{\partial e_{x_k}} = 1 \) and putting \( e_{x_k} = 0 \) in the resulting equations, we get

\[
\frac{\partial \Phi_{1k}(e_{x_k}, \hat{x}_i)}{\partial e_{x_k}} \bigg|_{e_{x_k}=0} = \frac{\partial h(\hat{x}_i)}{\partial \hat{x}_i},
\]

(24)

\[
\frac{\partial \Phi_{jk}(e_{x_k}, \hat{x}_i)}{\partial e_{x_k}} \bigg|_{e_{x_k}=0} = -\frac{\partial L_{f_j}^{j-1}(\hat{x}_i)h(\hat{x}_i)}{\partial \hat{x}_i}, \quad 2 \leq j \leq n.
\]

(25)

which can be resumed compactly as

\[
J(e_{x_k}, \hat{x}_i) \bigg|_{e_{x_k}=0} = M_f(\hat{x}_i).
\]

(26)

Thus, Assumption 1 implies the condition (21), which proves the Lemma.

\[\square\]

4. Observer input design

Lemma 1 can be fruitfully exploited for state observation purposes if we are able to find an appropriate set of observer inputs \( u_i \) \((i = 1, 2, \ldots, q)\), which can steer the vectors \( e_i \) to zero in finite time.

The dynamics of the \( i \)th observation output error takes the following Brunowsky chain-of-integrators canonical form:

\[
\begin{cases}
\dot{e}_{1i} = e_{2i}, \\
\dot{e}_{2i} = e_{3i}, \\
\quad \vdots \\
\dot{e}_{mi} = \Phi_{n+1,i}(e_{x_k}, \hat{x}_i) + u_i.
\end{cases}
\]

(27)

\[
\Phi_{n+1,i}(e_{x_k}, \hat{x}_i) = L_{f_j}^{n}(\hat{x}_i)h(\hat{x}_i) - L_{f_j}^{n}(\hat{x}_i - e_{x_k})h(\hat{x}_i - e_{x_k}).
\]

(28)

Make the following boundedness assumption.

Assumption 2: For any \( i = 1, 2, \ldots, q \) and whatever \( j^* \) is, there is a known constant \( \Gamma_i \) such that function \( \Phi_{n+1,i}(e_{x_k}, \hat{x}_i) \) satisfies

\[
|\Phi_{n+1,i}(e_{x_k}, \hat{x}_i)| < \Gamma_i.
\]

(29)

Assumption 2 is a boundedness assumption involving the higher order derivatives of the actual and observed outputs, as it follows from the definition of \( \Phi_{n+1,i} \).

The boundedness of signal (28) can be realistically assumed by requiring that the system dynamics is sufficiently smooth and by also imposing that the state of the observed system does not leave some compact, possibly large, domain. With this in force, the trajectories of the observer’s state will also be bounded. Overall, Assumption 2 gives the considered convergence domain a local nature.

A recently proposed method based on the so-called ‘arbitrary order’ sliding-mode approach (Levant 2005) is now outlined in order to provide for the finite-time stabilisation of system (27), (29).


The ‘quasi-continuous’ arbitrary-order sliding-mode controller was suggested in Levant (2005, 2006)
in order to stabilise the dynamics (27), (29) in finite time. Let \( i = 1, \ldots, q \), \( h = 1, \ldots, n - 1 \), and denote
\[
\varphi_{0,i} = \varepsilon_{1i}, \quad N_{0,i} = |\varphi_{1i}|,
\]
(30)
\[
\Psi_{0,i} = \varphi_{0,i}/N_{0,i} = \text{sign } \varepsilon_{1i},
\]
(31)
\[
\varphi_{h,i} = \frac{\varepsilon_{1i}}{N_{h-1,i}} + \beta_{h,i}N_{h-1,i} \varepsilon_{(n-h)/(n-h+1)},
\]
(32)
\[
N_{h,i} = \frac{|\varepsilon_{1i}|}{N_{h-1,i}} + \beta_{h,i}N_{h-1,i} \varepsilon_{(n-h)/(n-h+1)},
\]
(33)
\[
\Psi_{h,i} = \frac{\varphi_{h,i}}{N_{h,i}},
\]
(34)
where \( \beta_{1,i}, \ldots, \beta_{n-1,i} \) are positive numbers. The quasi-continuous \( i \)-sliding controller for system (27), (29) is
\[
u_i = -a_i \Psi_{h-1,i}(\xi_{1j}, \ldots, \theta_{nj}^{(n-1)}).
\]
(35)

It was shown in Levant (2005, 2006) that, provided the tuning parameters \( \beta_{1,i}, \ldots, \beta_{n-1,i} \), \( a_i \) are chosen sufficiently large in the given order, the control law defined by (30)–(35) stabilises the system (27), (29) in finite time.

Note that control defined by (30)–(35) is globally bounded \( |u_i| \leq a_i \) and continuous everywhere but the origin of the \( n \)-dimensional error space. An example of the second-order quasi-continuous controller follows:
\[
u_i = -a_i \frac{\varepsilon_i+|\varepsilon_i|^{1/2} \text{sign } \varepsilon_i}{|\varepsilon_i|+|\varepsilon_i|^{1/2}}.
\]
(36)

Taking into account Lemma 1 and the stability properties of the quasicontinuous HOSM controller, the following theorem can be demonstrated:

**Theorem 1:** Consider system (1) with the \( j \)-th dynamics being the active one, and let Assumptions 1 and 2 be satisfied. Consider the set of observers (3), (6) with the observer inputs \( u_i \) designed according to (30)–(35). Then, provided that the tuning parameters \( \beta_{1,i}, \ldots, \beta_{n-1,i} \), \( a_i \) are chosen sufficiently large in the given order, the state estimation errors \( e_{j} = \hat{x}_j - x \) converge to zero in finite time.

**Remark 1:** Let the functions \( f_j(x) \) and \( h(x) \) be such that, for each \( i = 1, \ldots, q \) and for each \( j = 1, \ldots, q \), \( i \neq j \),
\[
\begin{bmatrix}
L_{f_j(x)} h(x) \\
L_{f_j(x)} L_{h(x)} h(x) \\
\vdots \\
\end{bmatrix} = 0 \\
\begin{bmatrix}
0 \\
0 \\
\vdots \\
0 \\
\end{bmatrix},
\]
(37)
where \( \tilde{f}_j = f_j - f_i \). Then, all the estimation errors \( e_i = \hat{x}_i - x \) converge to zero in finite time.

The example that will be discussed in Section 6 actually fulfils the above condition (37), which demonstrates its feasibility. The above results considered the non-switching dynamics (9). For the switching case, it is important to remark that, after each discrete state switching, the observers could lose the correct estimate (due to the discontinuities in the higher order output derivatives). The differentiator device (described in the next section) that provides the required number of output derivatives in finite time will also feature a convergence transient after the discrete state switching times.

Then, after an overall differentiator/observer transient time, which can be made arbitrarily small by taking sufficiently large values of the differentiator and observers parameters \( a_i \) and \( C_i \), the correct state estimation is recovered. Thus, a design requirement is that the dwell time will be large enough to let the observer recover the correct state estimate well before the next mode switching. From a concrete point of view, given a known lower bound to the dwell time, the parameters of the proposed scheme can be tuned to provide a sufficiently fast convergence.

### 4.2. Output error derivatives estimation

The \( n \)-th-order quasi-continuous controller requires the availability of the successive derivatives of the output estimation error up to the order \( n - 1 \). In order to reconstruct such derivatives exactly and in finite time, the well-known arbitrary-order sliding-mode differentiator by Levant (2003) can be used.

Under the assumption that a constant \( C_i \) exists such that
\[
|\varepsilon_{j}^{(i)}| \leq C_i,
\]
(38)
then the nth-order differentiator can be expressed in the following non-recursive form:

\[
\begin{align*}
\dot{z}_0 &= v_0 = z_0 - \kappa_0 \dot{C}^i(0) |\dot{z}_0 - e_{p,i}|^m \text{sign}(z_0 - e_{p,i}), \\
\dot{z}_1 &= v_1 = z_1 - \kappa_1 \dot{C}^i(1) |\dot{z}_1 - v_0|^m \text{sign}(z_1 - v_0), \\
&\vdots \\
\dot{z}_i &= v_i = z_i - \kappa_i \dot{C}^i(i) |\dot{z}_i - v_{i-1}|^m \text{sign}(z_i - v_{i-1}), \\
&\vdots \\
\dot{z}_n &= -\kappa_n \dot{C}_i \text{sign}(z_n - v_{n-1}) 
\end{align*}
\]

(39)

for suitable positive constants \(\dot{C}_i > C_i, \kappa_j\) to be chosen recursively large in the given order (Levant 2003). Then, after a finite-time transient process, the following equalities are true in the absence of measurement noise:

\[
|z_i - e_{p,i}| = 0, \quad i = 0, \ldots, n. \quad (40)
\]

Clearly, for the considered closed-loop system (27), (29) the \(\Gamma_i\) constant in Assumption 2 exists, and, taking into account that \(\dot{e}_{ni} = e_{ni}^{(0)}\), it is overestimated by \(C_i + \alpha e\). The separation and robustness results relevant to the combined use of the above differentiator and any \(n\)-sliding homogeneous controller were discussed in Levant (2003). It was demonstrated by Levant (2003, 2005, 2006) that non-idealities, such as measurement noise and finite frequency commutation cause a bounded error in the estimated derivatives and, as a result, a bounded loss of accuracy for the controller that uses the ‘noisy’ derivative estimates.

5. Identification of the switching signal

The method for reconstructing the switching signal \(\hat{f}(t)\), namely the discrete state of the switched system, is now outlined. By generalising some results presented in Saadaoui et al. (2006), let us define the conditions for which the discrete state of the switched system is distinguishable. Define the sets:

\[
\mathcal{M}_{i,j} = \{x \in \mathbb{H} : L^p_{f_i}(x) = L^p_{f_j}(x)\}, \quad (41)
\]

\[
\mathcal{S}_{i,j} = \left\{x \in \mathbb{H} : \frac{\partial}{\partial x} \left[ L^p_{f_i}(x) f_i(x) \right] = \frac{\partial}{\partial x} \left[ L^p_{f_j}(x) f_j(x) \right] \right\}. \quad (42)
\]

**Assumption 3:** One of the next two conditions is satisfied for all \(i, j = 1, 2, \ldots, q\):

1. \(\mathcal{M}_{i,j}\) is an empty set for all \(i \neq j\).
2. \(\mathcal{S}_{i,j}\) is an empty set for all \(i \neq j\).

The observation output error dynamics takes the Brunovsky canonical form (27) and (28). In the steady state, all entries of vector \(e_{p,i}\) and \(e_{c,i}\) (\(f^\prime\) being the current active location) are identically zero, while the error derivative of order \(n\), signal \(\dot{e}_{p,i}\), which result from being directly affected by the discontinuous control \(u_{p,i}\), is zero in the ‘average’ (or ‘Filippov’) sense (Filippov 1988). Thus, we are in a position to exploit one of the main features of sliding-mode observers, the equivalent output-injection principle.

The expression of \(\dot{e}_{ni}\) is

\[
\dot{e}_{ni} = L^p_{f_i}(x) \dot{h}(\tilde{x}_i) - L^p_{f_i}(x) h(x) + u_i. \quad (43)
\]

Thereby, in the steady state, the observer input \(u_i\) will take the value of the equivalent output injection \(u_{eq,i}\), i.e.

\[
u_{eq,i} = L^p_{f_i}(x) h(x) - L^p_{f_i}(x) \dot{h}(\tilde{x}_i), \quad (44)
\]

which derives from imposing the zeroing of \(\dot{e}_{ni}\) (equivalent control method, see Utkin et al. 1999). Consider the \(j^\prime\)th equivalent control \(\dot{u}_{eq,j}\) as

\[
\dot{u}_{eq,j} = L^p_{f_j}(x) h(x) - L^p_{f_j}(x) \dot{h}(\tilde{x}_j). \quad (45)
\]

Since, from Lemma 1 and Theorem 1, \(e_{eq,j} = 0\) then one can substitute the corresponding equation \(\tilde{x}_i = x\) into (45), which yields

\[
u_{eq,j} = 0. \quad (46)
\]

The above equation means that among the \(q\) observers (3), (6) there is one of them, namely that associated the current mode of operation \(j^\prime\), with the associated equivalent control value being identically zero according to (46).

Since \(u_i\) is actually discontinuous, the equivalence \(u_i = u_{eq,i}\) holds only in the Filippov sense (Filippov 1988) so that the recovery of the equivalent output injection \(u_{eq}\) from the discontinuous control signal \(u_i\) requires filtration. Let us define the following equivalent control estimate \(\tilde{u}_{eq}\),

\[
\tau \dot{\tilde{u}}_{eq} = u_i - \tilde{u}_{eq}. \quad (47)
\]

The proposed scheme for the reconstruction of the switching signal takes the structure depicted in Figure 1.

The continuous signals \(\tilde{u}_{eq}(t = 1, 2, \ldots, q)\) must be analysed in order to extract the information about the current value of the switching signal, namely the active mode of operation. It was previously shown that the observer associated the active mode of operation (say, \(j\)) features an equivalent control which is identically zero. For the purpose of reconstructing the value of the active mode it must be guaranteed that
the remaining observers have a non-zero equivalent control. This would correspond to the following condition:

\[ u_{eq} \neq 0 \quad \forall i \neq j \]  \hspace{1cm} (48)

By considering (44), condition (48) holds true if the vector fields \( f() \) and the output function \( h() \) of the original switched system (1) and (2) fulfill the condition 1 in Assumption 3.

Theoretically, if both conditions (46) and (48) are satisfied, a simple threshold-based decision criterion could be applied to the signals \( \hat{u}_{eq} \) produced by the bank of filters in Figure 1 in order to derive a discrete state estimate \( \hat{j} \). Such an estimation rule could be represented as follows, which computes the value of \( i \) for which \( |\hat{u}_{eq}(k)| \) is closer to the zero value:

\[ \hat{j} = \arg \min_i |\hat{u}_{eq}(k)|, \]  \hspace{1cm} (49)

where \( \hat{u}_{eq}(k) \) represents the sampled value of \( \hat{u}_{eq} \) at the ‘sampling instant’ \( t = kT_s \), \( k \geq k_1 \), \( k_1 \) being a sufficiently large value that corresponds to the initial observation error transient time, during which the reconstruction formula is clearly not reliable.

However, condition (48) (or, equivalently, condition 1 in Assumption 3) might not be fulfilled in some cases. In that condition, signals \( \hat{u}_{eq} \), \( i \neq j \), can occasionally cross the zero value when the active mode is \( j \), which would make the decision logic (49) unreliable.

Condition (48) can be relaxed. For our purposes, it would be enough that signals \( \hat{u}_{eq}, i \neq j \) cannot be identically zero when the active mode is \( j \). It appears to be a more sensible requirement, which can be formalised as follows:

\[ \frac{du_{eq}}{dt} \Bigg|_{u_{eq}=0} \neq 0, \quad i \neq j \]  \hspace{1cm} (50)

The above restriction indeed allows the signals \( \hat{u}_{eq}, i \neq j \), to occasionally cross the zero, but guarantees that the zero value cannot be held permanently. After some manipulations, condition (50) can be expressed in terms of the vector fields \( f() \) and the output function \( h() \) of the original system (1) and (2) as in condition 2 of Assumption 3.

Therefore, a modified discrete state estimation logic should be implemented that looks for the signal \( \hat{u}_{eq} \) (or, equivalently, its magnitude \( |\hat{u}_{eq}| \)) being closer to zero over a suitable receding-horizon time interval of finite length. This can be done easily via the numerical method described below.

At the sampling time \( t = kT_s \), define the following non-negative quantities

\[ \delta_i(k) = \sum_{r=0}^{N} |\hat{u}_{eq}(k-r)|, \quad i = 1, 2, \ldots, q \]  \hspace{1cm} (51)

with an integer value \( N > 0 \). \( N \) should be selected in such a way that \( NT_s \ll T_d \), with \( T_d \) being the dwell time of the switched system (i.e. \( |\tau_{i+1} - \tau_i| \leq T_d, \quad r \geq 1, \quad \tau_0 = 0 \)).

The value of \( i \) for which \( \delta_i \) is minimum is evaluated, and this value will be the discrete state estimate \( \hat{j}(t) \).

This corresponds to the next discrete state decision rule

\[ \hat{j} = \arg \min_i |\delta_i(k)|. \]  \hspace{1cm} (52)

It is worth noting that the above logic (51), (52) is robust against a bounded error (unavoidably caused by noises and discretisation issues) affecting the estimated equivalent control signals \( \hat{u}_{eq} \).

Conditions (48) and (50) are consequences of Conditions 1 and 2 of Assumption 3, respectively, which guarantees the possibility of recovering the discrete state. The observation scheme is summarised by the following theorem.

**Theorem 2:** Consider system (1) and let Assumptions 1–3 be satisfied along with the condition (44), (50). Implement the set of observers (3), (6), (30)–(35), and the high-order differentiator (39), (40). Implement the set of filters (47), with sufficiently small time constant \( \tau > 0 \). Then, the switching state reconstruction rules (51), (52) and the continuous state estimate

\[ \hat{x}(t) = \hat{x}_{\hat{j}(t)} \]  \hspace{1cm} (53)

provide the finite-time reconstruction of the continuous and discrete states of the system (1) after an arbitrarily small transient following the switching instants.

**Proof:** The proof follows from the combination of Lemma 1, Theorem 1, the properties of the high-order differentiator (39) (see Levant 2003) and the discursive treatment developed in Section 5.
6. Simulation example

Let us consider the nonlinear switched system with three operating modes described by the equations:

\[
\dot{x} = f(x) \Rightarrow \begin{cases} 
\dot{x}_1 = -x_2 - x_3, \\
\dot{x}_2 = x_1 + ax_2, \\
\dot{x}_3 = b_1 + x_3(x_1 - c_1),
\end{cases}
\]

where \( a_i, b_i \) and \( c_i \) are constant parameters set as

- \( a_1 = b_1 = 0.2 \) and \( c_1 = 6 \),
- \( a_2 = 0.2, b_2 = 1 \) and \( c_2 = 6 \),
- \( a_3 = 0.2, b_3 = 1.5 \) and \( c_2 = 8 \).

The above system represents a switched version of Rossler’s chaotic dynamics. Let the measurable system output be \( y = x_2 \). The matrices \( M_i(x) \) are

\[
M_i = \begin{bmatrix} 0 & 1 & 0 \\
1 & a_i & 0 \\
a_i & -1 + a_i^2 & -1 \end{bmatrix}.
\]

Then each individual system satisfies Assumption 1. The column vectors \( g_i \) are easily derived as \( g_i = [0, 0, -1]^T \).

Note that for every pair of systems \( (f_i, f_k) \), \( i, k = 1, 2, 3 \), the next equality holds,

\[
\begin{bmatrix} L_{f_1(x)}h(x) \\
L_{f_2(x)}h(x) \\
L_{f_3(x)}h(x) \end{bmatrix} = \begin{bmatrix} 0 \\
0 \\
0 \end{bmatrix},
\]

then the assumption made in the Remark 1 is fulfilled and all the designed observers will provide the correct reconstruction of the continuous state. The observers are designed as follows:

\[
\begin{align*}
\dot{\hat{x}}_1 &= f(\hat{x}) + g(\hat{x})u_i, \\
\dot{\hat{x}}_2 &= \hat{x}_1 - \hat{x}_2, \\
\dot{\hat{x}}_3 &= \hat{x}_1 + a_1\hat{x}_2, \\
\hat{x}_1 &= b_1 + \hat{x}_3(\hat{x}_1 - c_1), \\
\end{align*}
\]

where

\[
u_i = -\alpha \frac{\hat{e}_1 + 2(\hat{e}_1 + |e_1|^{2/3})^{-1/2}(\hat{e}_1 + |e_1|^{2/3})\text{sign}(e_1)}{|\hat{e}_1| + 2(|\hat{e}_1| + |e_1|^{2/3})^{1/2}}.
\]

The switching signal \( j(t) \) is selected as shown in Figure 2. The system initial conditions are set as \( x(0) = [3.9, -3.2, 0.03]^T \), which leads to the continuous state evolution depicted in Figure 3. The observers initial conditions are taken as \( \hat{x}(0) = [0.1, 0.1, 0.1] \), \( i = 1, 2, 3 \). Simulations have been done in the Matlab-Simulink environment, with the fixed-step Runge-Kutta method (discretisation step \( T_s = 0.0001 \) s). The constants \( \alpha_i \) and \( \bar{C}_i \) are all set to the value of 100.

7. Conclusions

In this article, a technique for the reconstruction of both the continuous and discrete state in a class of
Figure 4. Estimation error convergence for the three observers.

Figure 5. Equivalent controls for the observers.

Figure 6. Actual and reconstructed discrete state of the switching system.
nonlinear switched systems is suggested. The proposed technique requires that all the system dynamics fulfill some ‘observability-like’ and boundedness requirements. After the mode switches, there is a transient during which the correct estimation needs to be restored. The duration of this transient can be made arbitrarily small by properly tuning the observers parameters. An academic simulation example illustrates the performance of the observer and confirms the expected convergence features.

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