Structural Optimization of Hexrotors Based on Dynamic Manipulability and the Maximum Translational Acceleration

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Abstract—This paper investigates a structural optimization problem of hexrotors. As structural evaluation indices, we introduce the maximum translational acceleration, in addition to general 6-DOF dynamic manipulability, taking account of payload flying against the gravity force. We next consider a special class of structure where 6-DOF dynamic manipulability can be approximated by the simple product of translational and rotational manipulability. Then, we consider the optimization problem and solve it by a nonlinear optimization technique. As a result, it is shown that the optimized structure has symmetric properties. We finally evaluate the optimality via simulations with simple feedback control.

I. INTRODUCTION

Practical applications of UAVs (Unmanned Aerial Vehicles) gain huge interest thanks to their usability. One of the main advantages of UAVs is obviously flying ability which enables them to move or track a target while avoiding obstacles on the ground. Thus, not only military usage but also civilian applications such as investigation of earth environment [1], regular inspection of constructions like bridges or dams [2], seeking the missing and monitoring farm products [3] are expected as their potential applications.

Among various kinds of UAVs, quadrotor control is eagerly studied in this decade. Quadrotors have 4 inputs in the Euclidean 6-dimensional (6D) space: 3D position and 3D attitude space. Therefore, they are underactuated systems, and the regulation/tracking control problem for 3D positions and only the yaw angle is mainly studied as in [4]-[7]. Due to this input limitation, it is well known that quadrotors have to turn the body around roll and pitch axes in order to move rapidly in the horizontal direction. This makes it hard to carry out some tasks such as moving in small space, monitoring a target by a pinhole type camera and touching something. To overcome these issues technically, as a new technology, hexrotors (Fig. 1) are just beginning to be studied which have the potential to control full 6 degrees of freedom (DOF), independently [8]-[10].

The common type of hexrotors has the special structure where the six rotors are placed on the vertices of a planar hexagon [8], [9]. These rotors often have different orientations each other so that the number of controllable DOF is not decreased. However, the structure is not optimally designed based on certain control objectives for technical applications, and few works investigate the structural optimization of hexrotors to the best of our knowledge. Although the work [10] tackles an optimization problem, the author imposes the constraint that each rotor is placed on a hexagon and only tilt angle parameters can be optimized. Namely, the class of hexrotor applications is narrow.

In view of these facts, this paper tackles a structural optimization problem of hexrotors applicable to a wider class. As structural evaluation indices, we introduce not only 6-DOF dynamic manipulability [11] for the basic movement index but also the maximum translational acceleration for payload flying against the force of gravity. Throughout this work, to make the optimization problem easy to be tackled, we introduce the special class of structure called 3-twin rotors structure. Then, 6-DOF dynamic manipulability can be approximately given by the simple product of 3-DOF translational and 3-DOF rotational manipulability. It should be noted here that the 3-twin rotors structure has a much wider class than those in [8]-[10] and includes them as special cases. We next consider the structural optimization problem and solve it by a nonlinear optimization technique. Then, as the main result of this paper, we get the interesting knowledge that symmetric structure is desirable for the present evaluation indices. Numerical simulations with simple feedback control finally demonstrate the validity of the present optimization method.

II. OPTIMIZATION INDEX AND HEXROTOR CLASS

A. Dynamic Manipulability and the Maximum Translational Acceleration

In this work, we introduce 6-DOF dynamic manipulability as a basic movement index. Here, dynamic manipulability is defined as the volume of the output acceleration generated by applying normalized force/torque inputs to the system [11]. We also consider the maximum translational acceleration as the evaluation index because hexrotors require thrust force exceeding the gravity force for payload flying.
Let us first consider general hexrotor structure shown in Fig. 1. We denote the position, thrust direction vector and thrust force of the i-th rotor, and the ratio between the antitorque and thrust force by ξ_i ∈ ℝ^3, s_i ∈ ℝ^3 (∥s_i∥ = 1), f_i ∈ ℝ and c_i ∈ ℝ, respectively (i ∈ {1, · · · , 6}). Then, the total force F ∈ ℝ^3 and moment M ∈ ℝ^3 applied to the center of mass (COM) are given as

\[
\begin{bmatrix}
F \\
M
\end{bmatrix} = 
\begin{bmatrix}
s_1 & \cdots & s_6 \\
\xi_1 \times s_1 + c_1 s_1 & \cdots & \xi_6 \times s_6 + c_6 s_6
\end{bmatrix}
\begin{bmatrix}
f_1 \\
f_2 \\
f_3 \\
f_4 \\
f_5 \\
f_6
\end{bmatrix} \text{.}
\]

Here, f = [f_1 f_2 · · · f_6]^T ∈ ℝ^6. Similarly to most of related works on quadrotors/hexrotors, we neglect each rotor dynamics and directly consider the thrust force f_i derived from the rotor torques as the hexrotor inputs in this work.

We now consider the case that the total mass is normalized (i.e. m = 1) without loss of generality. Then, the 6D acceleration a ∈ ℝ^6 is given as

\[
a = \begin{bmatrix}
F \\
I^{-1}M
\end{bmatrix} = 
\begin{bmatrix}
s_1 & \cdots & s_6 \\
I^{-1}[\xi_1 \times s_1 + c_1 s_1 & \cdots & \xi_6 \times s_6 + c_6 s_6]
\end{bmatrix}
\begin{bmatrix}
f_1 \\
f_2 \\
f_3 \\
f_4 \\
f_5 \\
f_6
\end{bmatrix} =: Af
\]

with the inertia matrix of the airframe I ∈ ℝ^{3×3}. We thus get the following dynamic manipulability v_{all} ≥ 0 [11].

\[
v_{all} = \sqrt{\det(AA^T)} = |\det(A)| \geq 0 \quad (1)
\]

On the other hand, when we consider the same kinds of 6 rotors for a hexrotor (i.e. |c_i| = |c_j| and max|f_i| = max|f_j| hold for all i, j ∈ {1, · · · , 6}), the maximum translational acceleration L_F ≥ 0 is defined as

\[
L_F = \max|f_i| \leq 1\|f_1 s_1 + \cdots + f_6 s_6\|. \quad (2)
\]

Here, we suppose that |f_i| ≤ 1 without loss of generality.

B. 3-twin Rotors Structure

We next introduce 3 twin rotors structure as a hexrotor class to make the optimization problem easy to be dealt with. The definition is given as follows.

Definition 1: The hexrotor satisfying the following three conditions is said to have 3-twin rotors structure.

\[
\xi_{i+3} = -\xi_i, \quad s_{i+3} \parallel s_i, \quad c_{i+3} = -c_i \forall i \in \{1, 2, 3\}.
\]

Here, the positive and negative signs of c_i describe the rotation directions as shown in Fig. 2.

We now suppose that the determinant of the inertia matrix I is fixed and |c_i| ≪ 1 holds. Then, we get the following result for 3-twin rotors structure.

Proposition 1: Under Definition 1, the 6-DOF dynamical manipulability v_{all} given in (1) can be expressed by the product of translational and rotational manipulability.

Proof: This is proved by direct calculations as follows.

\[
v_{all} = \left|\det\left(I^{-1}(m_1 + c_1 s_1) \cdots I^{-1}(m_6 + c_6 s_6)\right)\right| = \left|\det\left(\begin{array}{cccccc}
2s_1 & 2s_2 & 2s_3 & 0 & 0 & 0 \\
0 & 0 & 0 & -2I^{-1}(m_1 + c_1 s_1) & 0 & 0 \\
-2I^{-1}(m_2 + c_2 s_2) & -2I^{-1}(m_3 + c_3 s_3) & 0 & 0 & 0 & 0 \\
\end{array}\right)\right|
\]

\[
= 2^6|\det(I)|^{-1}|\det(s_1 s_2 s_3)|
\]

\[
= 2^6|\det(m_1 m_2 m_3)|\approx 2^6|\det(m_1 m_2 m_3)| = 2^6v_{EF M} \text{.}
\]

Here, \( m_i = \xi_i \times s_i \) (‘×’ means the cross product), and \( v_F, v_M \geq 0 \) respectively mean the translational and rotational dynamic manipulability. We also suppose without loss of generality that \|det(I)\| = 1 holds for simplicity.

Note here that the condition |c_i| ≪ 1 generally holds true for actual rotors. The separation property of \( v_{all} \) into \( v_F \) and \( v_M \) enables us to handle the translational and rotational manipulability optimization problems independently, which greatly helps us solve the optimization problem.

On the other hand, the maximum translational acceleration \( L_F \) defined in (2) can be reduced to

\[
L_F = 2\max|f_i|\|g_1 s_1 + g_2 s_2 + g_3 s_3\|
\]

\[
= 2\max|f_i|\|s_1 + g_2 s_2 + g_3 s_3\| \text{.}
\]

In this work, we consider that \(-s_i\) means the different structure from \( s_i \). Then, the maximization of \( L_F \) implies maximizing the following \( l_F \geq 0 \).

\[
l_F = \max\|s_1 + s_2 + s_3\| \text{.}
\]

We thus consider \( l_F \) for translational acceleration optimization in the subsequent discussion.

Then, the objective of this work is to propose an optimization method of the triplet \( (v_F, v_M, l_F) \) appropriate for some control purposes while guaranteeing 6-DOF control.

III. DEFINITION OF STRUCTURE VARIABLES

From the definitions of \( (v_F, v_M, l_F) \), the variables to be designed are \( s_i \) and \( m_i \). Therefore, we next derive the key parameters to determine these variables in this section. We first note, without loss of generality, that the optimized structure satisfies

\[
\xi_i \perp s_i \forall i \text{.} \quad (3)
\]

This is because that unless (3) is satisfied, \( \|m_i\| \) becomes small (i.e. \( v_M \) is small). We next impose the following assumption for simplicity.
Assumption 1: \( \| \xi_1 \| = \| \xi_2 \| = \| \xi_3 \| \) hold.

We now set \( \| \xi_1 \| = 1 \) without loss of generality under Assumption 1. Then, we obtain \( \| \mathbf{m}_i \| = 1 \) \( \forall i \) from the definition.

Under this setting, the pair \((s_i, \mathbf{m}_i)\) uniquely determines the position and attitude of the \(i\)-th rotor. It is because that we neglect the orientation around the rotor rotation axis in the optimization from the fact that it does not make any sense in optimization.

Remark 1: Under Assumption 1, we can have some different structure a optimal triplet \((v_F, v_M, l_F)\) by changing \(\| \xi_i \|\) and the angle formed by \(\xi_i\) and \(s_i\). In this case, only the inertia can vary, and thus the assumption \(|\det(I)| = 1\) restricts all the structures obtained from a optimal triplet to the structures satisfying it.

A. Parameters for \(s_i\)

We now give parameters to define \(s_i\), where we distinguish \(s_i\) and \(-s_i\). Hereafter, we utilize the notations \(C_x := \cos x, \ S_x := \sin x\) for notational simplicity. From the symmetric property of the 3-twin rotors structure and \(\| s_i \| = 1\), we give the following definition.

Definition 2: \(s_i, \ i \in \{1,2,3\}\) are defined as

\[
\begin{align*}
 s_1 & = \begin{bmatrix} 1 \\
 0 \\
 0 
\end{bmatrix},
 s_2 & = \begin{bmatrix} C_{\alpha_1} \\
 S_{\alpha_1} \\
 0 
\end{bmatrix},
 s_3 & = \begin{bmatrix} S_{\alpha_2} C_{\alpha_3} \\
 S_{\alpha_2} S_{\alpha_3} \\
 C_{\alpha_2} 
\end{bmatrix},
\end{align*}
\]

\[
0 \leq \alpha_1 \leq \pi, \ 0 \leq \alpha_2 \leq \frac{\pi}{2}, \ 0 \leq \alpha_3 \leq 2\pi, \quad s_1 \cdot s_2 \geq s_2 \cdot s_3, \ s_1 \cdot s_2 \geq s_3 \cdot s_1,
\]

\[
\frac{\alpha_1}{2} \leq \alpha_3 \leq \pi + \frac{\alpha_1}{2}.
\]

Fig. 3(a) illustrates the meanings of \(\alpha_1, \alpha_2, \alpha_3\). The constraints (5) are imposed to avoid duplication: the same structure with different angles \((\alpha_1, \alpha_2, \alpha_3)\). Also, the constraint (6) is for excluding the structure symmetric with respect to the plane including \(z\)-axis and the bisector between \(s_1\) and \(s_2\) since they have the same evaluation values (see Fig. 3(b)). Then, the constraints (5) and (6) reduce the domain (4) to

\[
0 \leq \alpha_1 \leq \frac{2\pi}{3}, \ 0 \leq \alpha_2 \leq \frac{\pi}{2}, \ 0 \leq \alpha_3 \leq \frac{4\pi}{3}.
\]

In the optimization, we determine \((\alpha_1, \alpha_2, \alpha_3)\) within the domain (7).

B. Parameters for \(m_i\)

We next provide parameters to decide the moment \(m_i\). It should be noted that \(m_i\) is constrained to be on the plane \(\Pi_i\) vertical to \(s_i\) and including the origin as shown in Fig. 4(a). Let the angle of \(m_i\) be the angle between \(m_i\) and the intersection of \(\Pi_i\) and the plane including \(s_i\) and \(s_{i+1}\) as shown in Fig. 4(b). Then, from the symmetric property of the 3-twin rotors structure and \(\| m_i \| = 1\), we define \(m_i, \ i \in \{1,2,3\}\) as follows.

Definition 3: \(m_i, \ i \in \{1,2,3\}\) are defined as

\[
\begin{align*}
 m_1 & = C_{\beta_1} \begin{bmatrix} 0 \\
 1 \\
 0 
\end{bmatrix} + S_{\beta_1} \begin{bmatrix} 0 \\
 0 \\
 1 
\end{bmatrix},
 m_2 & = C_{\frac{\pi}{2} + \beta_2} \begin{bmatrix} s_2 \times s_3 \\
 s_3 \times s_2 \\
 s_1 \times s_3 
\end{bmatrix} + S_{\frac{\pi}{2} + \beta_2} s_2 \times s_3 \times s_2 \times s_1 \times s_3, \quad (8)
 m_3 & = C_{\frac{\pi}{2} + \beta_3} \begin{bmatrix} s_1 \times s_3 \\
 s_2 \times s_3 \\
 s_1 \times s_3 
\end{bmatrix} + S_{\frac{\pi}{2} + \beta_3} s_1 \times s_3 \times s_1 \times s_3, \quad (9)
\end{align*}
\]

\[
-\frac{\pi}{2} \leq \beta_1 \leq \frac{\pi}{2}, \ -\frac{\pi}{2} \leq \beta_2 \leq \frac{\pi}{2}, \ -\frac{\pi}{2} \leq \beta_3 \leq \frac{\pi}{2}, \quad (10)
\]

Fig. 5 illustrates the meanings of \(\beta_1, \beta_2, \beta_3\). We note from (8) and (9) that \(m_2 (m_3)\) cannot be defined in the case that \(s_3 \times s_2 = 0 (s_1 \times s_3 = 0)\), but we have no interest in this case since the hexrotor loses at least one DOF (namely the original motivation for full 6-DOF control is lost). The constraint (10) is imposed to avoid duplication.

In summary, the angle variables \((\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3)\) are designed for the structural optimization with the evaluation indices \((v_F, v_M, l_F)\) in the next section.

IV. STRUCTURAL OPTIMIZATION

We now design \(s_i\) and \(m_i\), i.e. \((\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3)\), by solving nonlinear optimization problems with constraints for \((v_F, v_M, l_F)\). In this work, we apply the interior point method [12] to the problem as a solver because the objective and constraint functions are continuously differentiable twice. In the method, we give a large enough number of initial points so that local optima are avoided.

In this work, we put more emphasis on \(s_i\) related with translational motion than rotational \(m_i\). Namely, we first design \(s_i\) by regarding \(l_F\) and \(v_F\) as the objective function
and constraint function, respectively. Then, using the optimized $s_i^*$, we next optimize $m_i$ for the objective function $v_M$ ($^*$ means the optimized value). In both cases, optimization means maximization of the objective functions. If we put more emphasis on the rotational motion, we just have to design $m_i$ first.

**A. Optimization of Translational Motion**

We first optimize $x = [\alpha_1 \alpha_2 \alpha_3]^T \in \mathbb{R}^3$ defining $s_i$. The objective function $F_1(x) \geq 0$ and the constraint function $G_1(x \mid v_F) \in \mathbb{R}$ are respectively defined as follows.

$$F_1(x) = \frac{5}{2}, \quad G_1(x \mid v_F) = \frac{1}{2}.$$

Then, we constraint $G_1(x \mid v_F)$ for certain $v_F$ value as

$$G_1(x^* \mid v_F) = 0 \quad (11)$$

in optimization. Namely, $F_1$ and $G_1$ respectively correspond to $l_F$ and $v_F$, where we use the square values to easily handle differentiation. We now apply the interior point method with 20 initial points at each $v_F \in \{0, 0.1, \cdots, 1.0\}$.

The optimization results are shown in Figs. 6 and 7. We see from Fig. 6(a) that $l_F^*$ strictly decreases as $v_F$ increases. In fact, large $v_F$ implies, from the definition of $v_F$ based on the volume, that the angles made by two of $s_1$, $s_2$ and $s_3$ are large. Then, the resulting maximum translational acceleration becomes small because it is derived based on the summation of the vectors. As a special case, we get the optimization result $s_1 = s_2 = s_3 = [1 0 0]^T$ when $v_F = 0$. Also, when $v_F = 1$ is satisfied, all the angles formed by two of the three vectors become $\pi/2$, and we obtain $l_F = \sqrt{3}$.

We next investigate the structure of the results. Let us see Fig. 7 (three lines are overwritten) illustrating the inner products of the pairs of $s_1$, $s_2$ and $s_3$ which imply the angles formed by the pairs because of $\|s_i\| = 1 \forall i$. Then, we see that at all the sample points, the following equalities hold.

$$s_1 \cdot s_2 = s_2 \cdot s_3 = s_3 \cdot s_1. \quad (12)$$

Therefore, we impose the constraints (12) and $0 \leq \alpha_1 \leq \pi/2$ without loss of generality in the subsequent discussion.

Then, $x^*$ is determined by these constraints and the constraint equation (11) as

$$\begin{align*}
\alpha_1^* &= \arccos \left( \frac{1}{2} + \cos \left( \frac{\arg(2v_F^2 - 1 + 2v_F \sqrt{v_F^2 - 1}) - \frac{2}{3} \pi}{3} \right) \right), \\
\alpha_2^* &= \arccos \left( \frac{v_F}{S_{\alpha_1^*}} \right), \\
\alpha_3^* &= \frac{\alpha_1^*}{2}.
\end{align*}$$

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**Fig. 5.** Relation between $m_i$ and $\beta_i$

**Fig. 6.** Optimization Result of $s_i$

**Fig. 7.** Inner Product
Here, the function \( \text{arg} (\cdot) \) returns the argument of the complex number. These values equal to those in Fig. 6(b).

**B. Optimization of Rotational Motion**

We next optimize \( y = [\beta_1 \beta_2 \beta_3]^T \in \mathbb{R}^3 \) defining \( m_i \) by using \( x^* \). The objective function \( F_2(x^*(v_F), y) \geq 0 \) is defined as

\[
F_2(x^*(v_F), y) = \det(m_1 m_2 m_3)^2.
\]

We now also apply the interior point method with 80 initial points to maximization of \( F_2 \) at each \( v_F \in \{0.1, 0.2, \cdots, 1.0\} \).

The optimization results are shown in Fig. 8. We see from Fig. 8(a) illustrating the optimal value of \( v_M \) that \( v_M \) strictly goes up to 1 as \( v_F \) increases and keeps 1 after that. This is because that the thrust force is dispersed as \( v_F \) increases and, as a result, the angles between planes \( \Pi_1, \Pi_2 \) and \( \Pi_3 \) are also become large. We next show the optimized angles in Fig. 8(b) (three lines are overwritten). Similarly to the case for the translational motion, it is seen from this figure that the following equalities hold at all the sample points.

\[
\beta_1^* = \beta_2^* = \beta_3^*.
\]

These results mean that symmetric structure is desirable when we consider dynamic manipulability and the maximum translational acceleration as the evaluation indices. Gaining this interesting knowledge is the main result of this work.

The resulting optimized structure is illustrated in Fig. 9 for each \( v_F \in \{0.4, 0.6, 0.8, 1.0\} \). In these figures, the frame with the 1st and 4th rotors is fixed. The three red arrows represent the thrust force direction vectors \( s_i \) and the purple arrows show the moment vectors \( m_i \). For each \( v_F \), we confirm that the arrows have symmetric properties as seen above (Figs. 7, and 8(b)). It is also confirmed that the red arrows become dispersed as \( v_F \) increases. Therefore, in order to achieve good payload performance, we should make \( v_F \) small.

**Remark 2:** In the present method, we regard \( v_F, l_F \) and \( v_M \) as the constraint, the first objective and the second purpose, respectively. Namely, we put the most emphasis on the translational dynamic manipulability, the second most emphasis on the maximum translational acceleration for payload performance. However, following some control purposes, we can choose what is the most important by changing the correspondence of the evaluation index to the objective or constraint function.

**V. Verification**

We finally demonstrate the validity of the optimization via simulations with simple feedback control.

In the simulations, we apply simple LQR controllers to two kinds of optimized hexrotors, where the LQR controllers are designed by taking the first order linear approximation for the hexrotor systems and the common weights for the quadratic cost functions (refer to [13], [14] for the details of the dynamic model of the hexrotor).

One hexrotor is optimized by the present method with \( v_F = 0.1 \) aiming at the better acceleration performance in the vertical direction. Another hexrotor is optimized with \( v_F = 1.0 \) to get larger dynamic manipulability. The goal is to drive the hexrotor state \( [\zeta^T \eta^T]^T \in \mathbb{R}^6 \) from the origin to \( [1 2 2 0 0 0]^T \), where \( \zeta = [x_y y_z]^T \in \mathbb{R}^3 \) and \( \eta = [\phi \theta \psi]^T \in \mathbb{R}^3 \) represent the hexrotor position and attitude (roll, pitch and yaw angles) defined in the inertial frame, respectively.
The simulation results are illustrated in Figs. 10 and 11. We see from these figures that as expected, the simulation with $v_F = 0.1$ has better and worse performance for the position control in the vertical and horizontal directions, respectively. On the other hand, it is seen that the simulation with $v_F = 1.0$ has almost the same performance for each direction, i.e. well-balanced control is achieved as expected. These mean that the present optimization method taking account of some control objectives works successfully.

VI. CONCLUSIONS

This paper has studied a structural optimization problem of hexrotors, where the 6-DOF dynamic manipulability and the maximum translational acceleration have been introduced as the structural evaluation indices. The special class of structure, 3-twin rotors structure, has been also introduced to easily solve the optimization problem. Then, we have considered the structural optimization problem and solved it by the interior point method. As a result, we have gained the interesting knowledge that the symmetric property is desirable for the present evaluation indices. Simulations with simple feedback control have finally verified the effectiveness of the present optimization scheme.

The future directions of this work include to consider inertia matrices whose determinant value is not fixed and to take mutual interference between the rotors into consideration.

REFERENCES


