Abstract—In this paper robust stability of continuous linear time-varying systems is addressed based on Lyapunov functions which are constructed by max-composition of continuously differentiable functions. The resulting Lyapunov functions are continuous but not necessarily differentiable and no individual component needs to be positive definite. When the components are quadratic functions it will be possible to prove robust stability of systems which fail the classic quadratic stability test. The resulting conditions are matrix inequalities which are linear after choosing a set of tuning parameters. The robust stability condition is also extended to provide upper-bounds on integral performance measures.

I. INTRODUCTION

Despite many recent developments in the study of robust stability analysis for continuous linear time-invariant systems (e.g. [1]–[4]) the standard approach in the literature for robust stability analysis of continuous linear time-varying (LTV) systems continues to be the classic quadratic stability criterion [5]. Difficulties with continuous linear time-varying systems arise when the uncertainty may be allowed to have arbitrary rates of variation, forcing the underlying Lyapunov functions to be parameter-independent. Uncertain models with arbitrary rates of variation appear often in the context of hybrid and switched systems [6], [7]. Interestingly, this difficulty is not present in discrete-time linear time-varying systems, where boundedness of the uncertainty domain naturally implies boundedness of the rate of uncertainty variation by finite sampling times. See for example [8], [9] which successfully make use of parameter-dependent Lyapunov functions to study the stability of discrete-time linear time-varying uncertain systems.

In this paper robust stability of uncertain continuous linear time-varying systems is addressed by developing parameter-independent Lyapunov functions which are obtained by max-type compositions of general continuously differentiable functions. The resulting Lyapunov functions are continuous but not necessarily differentiable everywhere. The stability conditions are therefore formulated in terms of directional derivatives which are continuous almost everywhere as discussed for example in [10]. Of note is the fact that no component needs to be a positive definite function but that the resulting composition is indeed positive definite. A similar idea has been successfully applied to design switching rules for affine switched linear systems in [7].

We first derive a general sufficient condition for stability of uncertain continuous LTV systems which do not impose restrictions on the components of the max-composition. We then specialize this result to the case in which the components are quadratic forms, obtaining matrix inequalities which must be verified for a set of user controlled tuning parameters. In the case of quadratic forms, the resulting condition is never more conservative than quadratic stability, which in fact is a special case of the condition outlined in this paper. As is well known in the literature, it will be possible to prove robust stability of a continuous-time LTV system with a max-composition of quadratic forms for which no single quadratic Lyapunov function exists. This will be illustrated by Example 1 in Section V. Finally we extend the robust stability conditions to provide computable upper-bounds on certain integral performance measures. In the quadratic case, performance can be assessed based on parametrized matrix inequalities.

The paper is organized as follows: Section II defines the uncertain system dynamics at hand and reviews the notion of quadratic stability. The main robust stability theorem is developed in Section III and extended to cover performance bounds in Section IV. In Section V, the results are particularized for compositions of quadratic functions with quadratic performance measures and illustrated by numerical examples.

II. PRELIMINARIES

Consider the uncertain LTV system

\[ \dot{x}(t) = A(\xi(t)) x(t), \]

where \( x(t) \in \mathbb{R}^n \) and the uncertain system matrix \( A(\xi) \) belongs to the polytope

\[ A := \left\{ A(\xi) : A(\xi) = \sum_{i=1}^{m} \xi_i A_i, \quad \xi \in \Xi_m \right\} \]

for some matrices \( A_i \in \mathbb{R}^{n \times n}, \quad i = 1, \ldots, m \) and \( \Xi_m \) denotes the \( m \)-dimensional unit simplex

\[ \Xi_m := \left\{ \xi \in \mathbb{R}_+^m : \sum_{i=1}^{m} \xi_i = 1 \right\}. \]

We are particularly interested in the case when \( \xi \) is time-varying with potentially unbounded rate of variation, that is, when \( \xi : \mathbb{R}_+ \mapsto \Xi_m \) is a function of time \( \xi(t) \) that is potentially discontinuous. This encompasses the important class of switched systems as a particular case [6]. When the parameter \( \xi(t) \) is discontinuous, continuous solutions to the differential equation (1) are assumed to exist in the sense of Filippov [10]. A well-known condition for stability of uncertain linear time-varying systems is quadratic stability [5]:

\[ \text{Stability Criteria for Uncertain Linear Time-Varying Systems} \]

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\textbf{Definition 1}: The linear time-varying uncertain system (1) is said to be quadratically stable if there exists a symmetric positive-definite matrix \( P > 0 \) such that
\[
A_i^T P + PA_i < 0
\] (3)
for all \( i = 1, \ldots, m \).

Quadratic stability works by enforcing the existence of a parameter-independent quadratic Lyapunov function
\[
V(x) = x^T P x > 0, \quad \text{for all } x \neq 0
\]
with the LMI (3) implying that
\[
\dot{V}(x) = 2 x(t)^T P \dot{x}(t) = x(t)^T [A(\xi(t))^T P + PA(\xi(t))] x(t) < 0
\]
for all \( x(t) \neq 0 \) and \( \xi(t) \in \Xi_m \) for all \( t \in \mathbb{R}_+ \).

In other words, quadratic stability implies that the origin is a globally asymptotically stable equilibrium point of the uncertain linear time-varying system (1) for all \( \xi : \mathbb{R}_+ \rightarrow \Xi_m \).

Note that for all \( x(t) \neq 0 \) and \( \xi(t) \in \Xi_m \) for all \( t \in \mathbb{R}_+ \).

\section{III. Stability Analysis with Max-Compositions}

In this section, we introduce a sufficient condition for stability of the uncertain time-varying system (1). This condition is based on parameter-independent Lyapunov functions obtained via pointwise maximization (max-composition) over a set of continuously differentiable functions. These Lyapunov functions are of the form
\[
V(x) = \max_{1 \leq j \leq p} V_j(x),
\]
where each \( V_j \) is a continuously differentiable function. Note that the number of functions \( p \) does not have to equal the number of vertices \( m \) and that when \( p = 1 \) and \( V_1(x) \) is quadratic, we recover the standard quadratic function used for quadratic stability. Our first result is a condition on the functions \( V_j(x) \) such that \( V(x) > 0 \) for all \( x \neq 0 \), which can be seen as a form of weak duality [11]:

\textbf{Lemma 1}: Let \( V_j : \mathbb{R}^n \rightarrow \mathbb{R}, \; j = 1, \ldots, p \) be given continuously differentiable functions. If there exists \( \xi \in \Xi_p \) such that \( \sum_{j=1}^p \bar{\xi}_j V_j(x) > 0 \) for all \( x \neq 0 \), then \( V(x) = \max_j V_j(x) > 0 \) for all \( x \neq 0 \).

\textbf{Proof}: Note that
\[
V(x) = \max_{1 \leq j \leq p} V_j(x) = \max_{\xi(x) \in \Xi_p} \sum_{j=1}^p \bar{\xi}_j V_j(x).
\]
Hence, if there exists \( \bar{\xi} \in \Xi_p \) such that \( \sum_{j=1}^p \bar{\xi}_j V_j(x) > 0 \) for all \( x \neq 0 \), then
\[
V(x) = \max_{\xi(x) \in \Xi_p} \sum_{j=1}^p \bar{\xi}_j V_j(x) \geq \sum_{j=1}^p \bar{\xi}_j V_j(x) > 0
\]
for all \( x \neq 0 \).

One might enforce \( V(x) \geq 0 \) by selecting \( V_j(x) \equiv 0 \) for some \( j \). However, Lemma 1 goes further and guarantees that this inequality holds strictly for any \( x \neq 0 \). Consequently, \( V(x) \) can be used as a Lyapunov function candidate. The following theorem is our main stability result:

\textbf{Theorem 1}: Let \( V_j : \mathbb{R}^n \rightarrow \mathbb{R}, \; j = 1, \ldots, p \) be given continuously differentiable functions and
\[
V(x) = \max_{1 \leq j \leq p} V_j(x).
\]
Assume that \( V(0) = 0 \) and \( V(x) \rightarrow \infty \) as \( \|x\| \rightarrow \infty \). If there exists \( \bar{\xi} \in \Xi_p \) such that \( \sum_{k=1}^p \bar{\xi}_k V_k(x) > 0 \) for all \( x \neq 0 \) and \( \mu_{ij} \in \mathbb{R}_+ \) such that
\[
\langle \nabla V_j(x), A(\xi) x \rangle + \mu_{ij} \left( V_j(x) - \sum_{k=1}^p \bar{\xi}_k V_k(x) \right) < 0
\]
for all \( i = 1, \ldots, m \), \( j = 1, \ldots, p \) and \( x \neq 0 \), then the origin of the uncertain linear time-varying system (1) is a globally asymptotically stable equilibrium point for all \( \xi : \mathbb{R}_+ \rightarrow \Xi_m \).

\textbf{Proof}: Use Lemma 1 to prove that \( V(x) > 0 \) for all \( x \neq 0 \). Convex combinations of the inequalities (4) imply that
\[
\langle \nabla V_j(x), A(\xi) x \rangle + \mu_j(\xi) \left( V_j(x) - \sum_{k=1}^p \bar{\xi}_k V_k(x) \right) < 0
\]
for all \( j \in \mathbb{J}(x) \), \( \xi \in \Xi_m \), \( x \neq 0 \). This means that \( \mu_j(\xi) := \sum_{i=1}^m \bar{\xi}_i \mu_{ij} \geq 0 \) for all \( \xi \in \Xi_m \) and \( x \neq 0 \).

Now let \( \mathbb{J}(x) \equiv \mathbb{J} = \{ 1, \ldots, p \} \) denote the subset of indices for which \( V_j(x) = V(x) \) at a given \( x \). Because \( \mu_j(\xi) \geq 0 \) and
\[
V_j(x) \geq \sum_{k=1}^p \bar{\xi}_k V_k(x)
\]
for all \( j \in \mathbb{J}(x) \) and \( \xi \in \Xi_m \), (5) implies that
\[
\langle \nabla V_j(x), A(\xi) x \rangle < 0
\]
for all \( \xi \in \Xi_m \), \( j \in \mathbb{J}(x) \) and \( x \neq 0 \). This means that
\[
\max_{j \in \Xi(x)} \langle \nabla V_j(x), A(\xi) x \rangle < 0
\]
for all \( \xi \in \Xi_m \) and \( x \neq 0 \). Recalling that the directional derivative of the function \( V(x) \) in the direction \( h \), denoted \( DV(x)[h] \), is computed as (see for example [12, p.28])
\[
DV(x)[h] = \max_{j \in \mathbb{J}(x)} (\nabla V_j(x), h)
\]
we conclude from (6) that \( DV(x)[\dot{x}(t)] < 0 \) for all \( (x(t), \dot{x}(t)) \) satisfying the uncertain system (1). Since \( V(0) = 0 \), \( V(x) > 0 \) for all \( x \neq 0 \) and \( V(x) \) is radially unbounded, the origin is a globally asymptotically stable equilibrium point [10, Theorem 1, p. 153]. As \( V(x) \) does not depend on \( \xi \), the proof remains the same if \( \xi(t) \) is time-varying or discontinuous.

Note that Filipov’s Theorem 1 in [10] is a local result, but one can use standard methods to extend the result to hold globally when \( V \) is radially unbounded [13]. Moreover,
Theorem 1 allows that some or all $V_j$'s are such that $V_j(x) \neq 0$ for $x \neq 0$. This property yields a major gain in flexibility when it comes to the construction of Lyapunov functions for large scale time-varying systems. In particular, the conditions in the theorem will enable the construction of quadratic max-composition for systems which are not quadratically stable.

Of interest is the role of the function

$$V_\zeta(x) := \sum_{k=1}^{p} \zeta_k V_k(x), \quad \zeta \in \Xi_p,$$

which serves as a global lower bound to $V(x)$. Indeed, condition (4) is equivalent to

$$\langle \nabla V_j(x), A_i x \rangle + \mu_{ij} (V_j(x) - V_\zeta(x)) < 0. \quad (7)$$

Compare the above formula with the variant

$$\langle \nabla V_j(x), A_i x \rangle + \sum_{k=1}^{p} \lambda_{ijk} [V_j(x) - V_k(x)] < 0, \quad (8)$$

where $\lambda_{ijk} \in \mathbb{R}_+$ for all $i = 1, \ldots, m$, $j = 1, \ldots, p$ and $x \neq 0$. This variant is a particular case of (7) where

$$\mu_{ij} = \sum_{k=1}^{p} \lambda_{ijk} > 0, \quad \zeta_k = \frac{\lambda_{ijk}}{\mu_{ij}} \in \Xi_p$$

when at least one $\lambda_{ijk}$ is not zero. Indeed, the $\lambda$'s in (8) work by determining an implicit lower bound to $V(x)$. Since Theorem 1 uses this lower bound, $V_\zeta(x)$, which is needed to ensure positiveness of $V(x)$, we reapply it in condition (10).

IV. ROBUST PERFORMANCE WITH MAX-COMPOSITIONS

The conditions in Theorem 1 can be modified to calculate upper bounds on integral performance measures. For instance, consider the general cost function

$$J = \int_{0}^{\infty} F(x(t)) \, dt, \quad (9)$$

where $F : \mathbb{R}^n \rightarrow \mathbb{R}_+$ is a continuously differentiable non-negative function. The following theorem extends Theorem 1 by minimizing an upper bound to the cost function (9):

**Theorem 2:** Let $F : \mathbb{R}^n \rightarrow \mathbb{R}_+$ and $V_j : \mathbb{R}^n \rightarrow \mathbb{R}$, $j = 1, \ldots, p$ be given continuously differentiable functions and

$$V(x) = \max_j V_j(x), \quad j = 1, \ldots, p.$$

Assume that $V(0) = 0$ and $V(x) \rightarrow \infty$ as $\|x\| \rightarrow \infty$. If there exists $\zeta \in \Xi_p$ such that $\sum_{k=1}^{p} \zeta_k V_k(x) > 0$ for all $x \neq 0$ and $\mu_{ij} \in \mathbb{R}_+$ such that

$$\langle \nabla V_j(x), A_i x \rangle + F(x) + \mu_{ij} \left( V_j(x) - \sum_{k=1}^{p} \zeta_k V_k(x) \right) < 0 \quad (10)$$

for all $i = 1, \ldots, m$, $j = 1, \ldots, p$ and $x \neq 0$, then the origin is a globally asymptotically stable equilibrium point of the uncertain linear time-varying system (1) for all $\xi : \mathbb{R}_+ \rightarrow \Xi_m$, and

$$V(x(t_0)) > \int_{0}^{\infty} F(x(t)) \, dt \quad (11)$$

for any trajectory $x(t)$ of the uncertain linear time-varying system (1).

**Proof:** A key point is that for any $\xi(t) \in \Xi_m$, one can construct a continuous Filippov solution $x(t)$ satisfying the system equation (1) (see [10]). Consider a time-interval $t \in [t_\ell, t_{\ell+1}]$, $t_{\ell+1} > t_\ell$, in which $x(t)$ is confined to a region where $V(x(t))$ is differentiable, e.g. when $\mathcal{J}(x(t))$ is a constant singleton, then $DV [V(x(t))] \bar{x}(t)$ coincides with the traditional time-derivative of $V(x)$ along $x(t)$ and

$$\int_{t_\ell}^{t_{\ell+1}} DV [V(x(t))] \bar{x}(t) \, dt = V(x(t_{\ell+1})) - V(x(t_\ell)). \quad (12)$$

More generally, in any finite interval $t \in [t_\ell, t_{\ell+1}]$, $t_{\ell+1} > t_\ell$, in which $x(t)$ is such that $\mathcal{J}(x(t))$ is a constant but not necessarily a singleton, e.g. on a sliding mode, $DV [V(x(t))] \bar{x}(t)$ coincides with the time-derivative of $V(x)$ along a trajectory in the differentiable manifold defined by

$$M(x) = \{ x : V_j(x) = V(x), \quad j \in \mathcal{J}(x) \ \text{constant} \},$$

on which (12) also holds. Because the functions $V_j(x)$ and $V(x)$ are continuous, a trajectory going through a point in which $\mathcal{J}(x(t)) \neq \lim_{t \to 0^+} \mathcal{J}(x(t + \epsilon))$ is such that $\lim_{t \to 0^+} V(x(t + \epsilon)) = V(x(t))$, and therefore

$$\int_{0}^{\infty} DV [V(x(t))] \bar{x}(t) \, dt =$$

$$\sum_{\ell=0}^{L-1} \left( V(x(t_{\ell+1})) - V(x(t_\ell)) \right) = V(x(t_L)) - V(x(t_0)),$$

where $t_0 = 0$ and $t_L \rightarrow \infty$, with $L$ itself possibly infinite. If (10) holds, then following steps similar to the ones used in the proof of Theorem 1, one obtains that

$$DV [V(x(t))] \bar{x}(t) < -F(x(t)) \leq 0$$

for all $x(t) \neq 0$. Making use of Theorem 1 and the fact that $F(x) \geq 0$, we can say that the origin of system (1) is a globally asymptotically stable equilibrium and therefore $x(t_L) \rightarrow 0$, which implies $V(x(t_L)) \rightarrow 0$ and

$$V(x(t_0)) = -\int_{0}^{\infty} DV [V(x(t))] \bar{x}(t) \, dt \geq \int_{0}^{\infty} F(x(t)) \, dt,$$

concluding this proof.

In the remainder of this paper we specialize Theorems 1 and 2 to the class of quadratic functions $V_j$, $j = 1, \ldots, p$.

V. MAX-COMPOSITION OF QUADRATIC FUNCTIONS

As noted above, consider

$$V_j(x) = x^T P_j x, \quad j = 1, \ldots, p.$$

Proofs for the Corollaries in this section can be obtained in a straightforward manner from Theorems 1 and 2 and are omitted. For quadratic functions $V_j(x)$, the stability conditions of Theorem 1 reduce to Linear Matrix Inequalities after choosing $\mu_{ij} \in \mathbb{R}_+$ and $\zeta \in \Xi_p$, as specified in the next corollary.
Corollary 1: If there exist symmetric matrices $P_j$, $j = 1, \ldots, p$, and $\bar{\zeta} \in \Xi_p$ such that
\[
\sum_{k=1}^{p} \bar{\zeta}_k P_k \succ 0, \tag{13}
\]
and $\mu_{ij} \in \mathbb{R}_+$ such that
\[
A_i^T P_j + P_j A_i + \mu_{ij} \left( P_j - \sum_{k=1}^{p} \bar{\zeta}_k P_k \right) < 0, \tag{14}
\]
for all $i = 1, \ldots, m$, $j = 1, \ldots, p$, then the origin of the uncertain linear time-varying system (1) is a globally asymptotically stable equilibrium point for all $\xi : \mathbb{R}_+ \to \Xi_m$.

Notice that we can choose $P_j \neq 0$ for some or all $j$ while still satisfying $V(x) > 0$ for all $x \neq 0$. As discussed earlier, feasibility of (13) is equivalent to the existence of a positive definite quadratic lower bound function
\[
V_\xi(x) = \sum_{k=1}^{p} \bar{\zeta}_k x^T P_k x > 0. \text{ } x \neq 0,
\]
Quadratic stability is a particular case of Corollary 1. Indeed, when $p = 1$, it follows trivially that $\bar{\zeta} = \tilde{\zeta}_1 = 1$ and the conditions in Corollary 1 reduce to those of Lemma 1. When $p > 1$, the conditions in Corollary 1 are also never more conservative than those of quadratic stability, because $P_j = P \succ 0$ for all $j = 1, \ldots, p$, arbitrary $\bar{\zeta} \in \Xi_p$ and $\mu_{ij} \in \mathbb{R}_+$ for all $i = 1, \ldots, m$, $j = 1, \ldots, p$ satisfy (14) if the system is quadratically stable. With $p > 1$, however, Theorem 1 can prove robust stability of uncertain systems which are not quadratically stable, as illustrated by the next example:

Example 1: Consider a spring-mass-damper system with uncertain, time-varying physical parameters. Based on mass $m$, spring coefficient $k$ and damping coefficient $d$, we obtain a second-order model:
\[
\dot{x} = A(t) x = \begin{bmatrix} 0 & 1 \\ -\xi_1(t) & -\xi_2(t) \end{bmatrix} x,
\]
where the time-varying parameters $\xi_1$ and $\xi_2$ can be used to represent variations on stiffness, damping and mass, e.g. $\xi_1 = k/m$ and $\xi_2 = d/m$. Say for instance that $m = 1$ is a constant. Given lower bounds $\underline{k} = \underline{d} = 0.5$ for the spring and damping coefficients we seek to determine the maximum possible upper bounds $\bar{k} = \bar{d} =: \gamma$ so that the corresponding uncertain time-varying system is still asymptotically stable. This problem can be directly cast as the stability of system (1) with 4 vertices, namely:
\[
A_1 = \begin{bmatrix} 0 & 1.0 \\ -0.5 & -0.5 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1.0 \\ -0.5 & -\gamma \end{bmatrix},
\]
\[
A_3 = \begin{bmatrix} 0 & 1.0 \\ -\gamma & -0.5 \end{bmatrix}, \quad A_4 = \begin{bmatrix} 0 & 1.0 \\ -\gamma & -\gamma \end{bmatrix}.
\]

We will attempt to construct max-compositions of quadratic functions using an increasing number of terms, parametrized by $p$, and compute their corresponding maximum possible value of the upper-bound $\gamma$, which we denote $\bar{\gamma}_p$.

$p = 1$: The case $p = 1$ is that of quadratic stability. Checking feasibility of the LMIs in Corollary 1, we were able to obtain a maximum possible value of $\bar{\gamma}_1 = 1.42$, associated with the positive-definite matrix
\[
P = \begin{bmatrix} 6.82 & 1.76 \\ 1.76 & 7.08 \end{bmatrix}.
\]

$p = 2$: With $p > 1$, feasibility depends also on choices of $\bar{\zeta} \in \Xi_2$ and $\mu$. Using brute force, we were able to obtain a maximum possible value of $\bar{\gamma}_2 = 1.66$ without losing feasibility of the LMIs in Corollary 1. Our choices for the parameters $\bar{\zeta}$ and $\mu$ were
\[
\bar{\zeta} = \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix}, \quad \mu = \begin{bmatrix} 4 & 0 \\ 0 & 1 \\ 0 & 8 \\ 1 & 6 \end{bmatrix},
\]
leading to
\[
P_1 = \begin{bmatrix} 72.1 & 12.6 \\ 12.6 & 63.8 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 67.7 & 24.0 \\ 24.0 & 67.9 \end{bmatrix}.
\]

The level sets of the corresponding function $V(x)$ are displayed in Figure 1. Note that these level sets are not ellipsoidal. Interestingly, both $P_1$ and $P_2$ calculated above are positive-definite matrices, but neither of them individually satisfies the quadratic stability inequalities (3) for $\bar{\gamma}_2 = 1.66$.

$p = 3$: In this case, the choice of parameters
\[
\bar{\zeta} = \begin{bmatrix} 0.2 \\ 0.4 \\ 0.4 \end{bmatrix}, \quad \mu = \begin{bmatrix} 4 & 0 & 5 \\ 2 & 0 & 1 \\ 0 & 6 & 0 \\ 4 & 1 & 1 \end{bmatrix},
\]

![Fig. 1. Level sets of $V(x)$ for $p = 2$, $\gamma = \bar{\gamma}_2 = 1.66$; bright background corresponds to $V_1(x)$ active, dark background to $V_2(x)$ active](image-url)

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produces the feasible solution
\[
P_1 = \begin{bmatrix} 75.7 & 11.6 \\ 11.6 & 65.8 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 71.1 & 24.8 \\ 24.8 & 71.3 \end{bmatrix}, \quad P_3 = \begin{bmatrix} 75.7 & 12.1 \\ 12.1 & 65.9 \end{bmatrix},
\]
associated with the maximum possible value of \( \bar{\gamma}_3 = 1.68 \). As before, all matrices are positive definite, but none can be used to prove quadratic stability. Figure 2 shows the level sets of \( V(x) \).

\( p > 3 \): We were not able to improve beyond \( \bar{\gamma}_p = 1.68 \) using more than 3 terms in \( V(x) \).

As in Section IV, it is possible to compute an upper bound on the integral performance measure
\[
J = \int_0^\infty x(t)^T Q x(t) \, dt
\]
with \( Q \succeq 0 \), using the inequalities in the next corollary:

**Corollary 2:** If there exist symmetric matrices \( P_j, j = 1, \ldots, p \), and \( \bar{\zeta} \in \Xi_p \) such that
\[
\sum_{k=1}^p \bar{\zeta}_k P_k > 0,
\]
\( \mu_{ij} \in \mathbb{R}_+ \) such that
\[
A_i^T P_j + P_j A_i + Q + \mu_{ij} \left( P_j - \sum_{k=1}^p \bar{\zeta}_k P_k \right) < 0,
\]
for all \( i = 1, \ldots, m, j = 1, \ldots, p \), then the origin of the uncertain linear time-varying system (1) is a globally asymptotically stable equilibrium point for all \( \xi : \mathbb{R}_+ \to \Xi_m \), and
\[
\max_{1 \leq j \leq p} x(0)^T P_j x(0) > \int_0^\infty x(t)^T Q x(t) \, dt
\]
for any trajectory \( x(t) \) of the uncertain linear time-varying system (1).

Given \( x(0) = x_0 \neq 0 \), one can minimize the upper-bound to the performance measure (18) by mimizing an additional scalar variable \( \rho \) subject to the linear constraints
\[
x_0^T P_j x_0 \leq \rho, \quad j = 1, \ldots, p.
\]
The resulting problem for fixed \( \rho, \bar{\zeta} \) and \( \mu \) is convex. In fact, it is a Linear Matrix Inequality. Jointly solving for the \( P_j \)'s and the parameters \( \bar{\zeta} \) and \( \mu \) is a much more complicated task and one can devise several iterative algorithms with convergence to a local optimum. We will leave the analysis of possible algorithms to a future paper.

**Example continued 1:** We now minimize the upper bound on (15) using the inequalities from Corollary 2 for \( Q = I \) and \( x_0 = (1, 1)^T \). For each choice of \( p \) considered earlier we compute the least possible upper bound for all \( \gamma \in [0, \bar{\gamma}_p] \). The results are shown in Figure 3. Increasing \( p \) from 1 to 2 produced the most significant gain in feasibility range and least upper bound in this example.

VI. CONCLUSIONS

This paper introduced a new Lyapunov-based approach for assessing the robust stability of continuous linear time-varying systems whose system dynamics belong to convex polytopic sets at each time. Lyapunov functions based on the pointwise max over the state-space were introduced and used to generate sufficient conditions for robust stability and performance. These conditions were refined for the particular class of quadratic functions to generate max-compositions and measure performance. Examples were provided illustrating applicability and efficiency of the introduced methods in relation to standard quadratic stability tests.

The results presented in this paper can be extended by several means. For instance, the conditions presented can be used for design of robustly stabilizing feedback-controllers. Using the results presented for robust performance, it is expected that efficient optimization problems for control of uncertain time-varying systems can be cast. Further natural extensions are generalizations of the results for discrete-time and nonlinear system models as well as the examination of other functions to generate the composite Lyapunov functions. Moreover, it is expected that systematic algorithms for choosing the scaling parameters in the respective conditions will be of significant use. The authors are currently exploring linear functions to generate max-compositions.

**REFERENCES**

Fig. 3. Cost function bounds for $p = 1$ (blue), $p = 2$ (red) and $p = 3$ (black).


