Moment transport equations for non-Gaussianity

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Abstract. We present a novel method for calculating the primordial non-Gaussianity produced by super-horizon evolution during inflation. Our method evolves the distribution of coarse-grained inflationary field values using a transport equation. We present simple evolution equations for the moments of this distribution, such as the variance and skewness. This method possesses some conceptual advantages over existing techniques. Among them, it cleanly separates multiple sources of primordial non-Gaussianity and may diagnose "tuned" inflaton trajectories. It has computational advantages when compared with popular alternatives, such as the δN framework. We adduce numerical calculations demonstrating that our new method offers good agreement with those already in the literature. We focus on two fields and the f_{NL} parameter, but we expect our method will generalize to multiple scalar fields and to moments of arbitrarily high order. We present our expressions in a field-space covariant form which we postulate to be valid for any number of fields.

1. Introduction

Inflation generically predicts a primordial spectrum of density perturbations which is almost precisely Gaussian [1, 2, 3, 4, 5]. In recent years the small non-Gaussian component [6, 7, 8, 9, 10, 11, 12] has emerged as an important observable [13], and will be measured with good precision by the *Planck Surveyor* satellite [14]. In the near future, as observational data become more plentiful, it will be important to understand the non-Gaussian signal expected in a wide variety of models, and to anticipate what conclusions can be drawn about early-universe physics from a prospective detection of primordial non-Gaussianity.

In this paper, we present a novel method for calculating the primordial non-Gaussianity produced by super-horizon evolution in two-field models of inflation. Our method is based on the real-space distribution of inflationary field values on a flat hypersurface, which can be thought of as a probability density function whose evolution is determined by a form of the collisionless Boltzmann equation. Using a cumulant representation [15, 16, 17, 18] to expand our density function around an exact Gaussian, we derive ordinary differential equations which evolve the moments of this distribution. Further, we show how these moments are related to observable quantities, such as the dimensionless bispectrum measured by f_{NL} [14, 10]. We present numerical results which show that this method gives good agreement with other techniques. It is not necessary to make any assumptions about the inflationary model beyond requiring a canonical kinetic term and applying the slow-roll approximation. While there are already numerous methods for computing the super-horizon contribution to f_{NL}, including the widely used δN formalism, we believe the one reported here has a number of advantages.

First, this new technique is ideally suited to unraveling the various contributions to f_{NL}. This is because we follow the moments of the inflaton distribution directly, which makes it straightforward to identify large contributions to the skewness or other moments. The evolution equation for each moment is simple and possesses clearly identifiable source terms, which are related to the properties of the inflationary flow on field space. This offers a clear separation between two key sources of primordial non-Gaussianity. One of these is the intrinsic non-linearity associated with evolution of the probability density function between successive flat hypersurfaces; the other is a gauge transformation from field fluctuations to the curvature peturbation, ζ. It would be difficult or impossible to observe this split within the context of other calculational schemes, such as the conventional δN formalism.

As a second advantage of our method, it employs the conventional definition of statistical moments. When applied to the primordial density perturbation, the moments of order two, three and four correspond to the power spectrum, bispectrum and trispectrum, respectively. As will be described in more detail in §2, many existing formalisms introduce a fiducial trajectory in field space, with respect to which fluctuations are measured. This trajectory is determined by solving the classical field equations. To obtain a meaningful answer, the fiducial trajectory must provide a baseline

which is very close to the ensemble average. This need not be true; in certain models it may happen that small subsets of trajectories in field space are quite unrepresentative of the ensemble average. In such cases, formalisms based on the use of a fiducial trajectory are at risk of yielding misleading estimates.

A third advantage of our method is connected with the computational cost of numerical implementation. Analytic formulas for $f_{\rm NL}$ are known in certain cases, mostly in the context of the δN framework, but only for very specific choices of the potential [19, 20, 21, 22] or Hubble rate [23]. These formulas become increasingly cumbersome as the number of fields increases, or if one studies higher moments [24, 25]. In the future, it seems clear that studies of complex models with many fields will increasingly rely on numerical methods. The numerical δN framework requires the solution to a number of ordinary differential equations which scales exponentially with the number of fields. Since some models include hundreds of fields, this may present a significant obstacle [26]. Moreover, the δN formalism depends crucially on a numerical integration algorithm with low noise properties, since finite differences must be extracted between perturbatively different initial conditions after ~ 60 e-folds of evolution. Thus, the background equations must be solved to great accuracy, since any small error has considerable scope to propagate.

In this paper we ultimately solve our equations numerically to determine the evolution of moments in specific models. Our method requires the solution to a number of differential equations which scales at most polynomially (or in certain cases perhaps even linearly) with the number of fields. It does not rely on extracting finite differences, and therefore is much less susceptible to numerical noise. These advantages may be shared with other schemes, such as the numerical method recently employed by Lehners & Renaux-Petel [27], but we are not aware of any other method which also shares the first two advantages enumerated above.

This paper is organized as follows. In §2, we show how the non-Gaussian parameter $f_{\rm NL}$ can be computed in our framework. The calculation remains in real space throughout (as opposed to Fourier space), which modifies the relationship between $f_{\rm NL}$ and the multi-point functions of the inflaton field. Our expression for $f_{\rm NL}$ shows a clean separation between different contributions to non-Gaussianity, especially between the intrinsic nonlinearity of the field evolution and the gauge transformation between comoving and flat hypersurfaces. In §3, we introduce our model for the distribution of inflaton field values, which is a "moment expansion" around a purely Gaussian distribution. We derive the equations which govern the evolution of the moments of this distribution in the one- and two-field cases. In §4, we present a comparison of our new technique and those already in the literature. We compute $f_{\rm NL}$ numerically in several two-field models, and find excellent agreement between techniques. We conclude in §5.

Throughout this paper, we use units in which $c = \hbar = 1$, and the reduced Planck mass $M_{\rm P}^{-2} \equiv 8\pi G$ is set to unity.

2. Frameworks for computing f_{NL}

In this section, we introduce our new method for computing the non-Gaussianity parameter f_{NL}. This method requires three main ingredients: a formula for the curvature perturbation, ζ, in terms of the field values on a spatially flat hypersurface; expressions for the derivatives of the number of e-foldings, N, as a function of field values at horizon exit; and a prescription for evolving the field distribution from horizon exit to the time when we require the statistical properties of ζ. The first two ingredients are given in Eqs. (10)–(11) and (26)–(28), found at the end of §2.2 and §2.3 respectively. The final ingredient is discussed in §3.

2.1. Calculations beyond linear order

Once it became clear that non-linearities of the microwave background anisotropies could be detected by the WMAP and *Planck* survey satellites [14], many authors studied higher-order correlations of the curvature perturbation. In early work, direct calculations of a correlation function were matched to the known limit of local non-gaussianity [9, 10, 28, 29, 30, 31]. This method works well if isocurvature modes are absent, so that the curvature perturbation is constant after horizon exit. In the more realistic situation that isocurvature modes cause evolution on superhorizon scales, all correlation functions become time dependent. Various formalisms have been employed to describe this evolution. Lyth & Rodríguez [11] extended the δN method [32, 33] beyond linear order. This method is simple and well-suited to analytical calculation. Rigopoulos, Shellard and van Tent [34, 35] worked with a gradient expansion, rewriting the field equations in Langevin form. The noise term was used as a proxy for setting initial conditions at horizon crossing. A similar 'exact' gradient formalism was written down by Langlois & Vernizzi [36, 37, 38]. In its perturbative form, this approach has been used by Lehners & Renaux-Petel to obtain numerical results [27]. Another numerical scheme has been introduced by Huston & Malik [39].

What properties do we require of a successful prediction? Consider a typical observer, drawn at random from an ensemble of realizations of inflation. In any of the formalisms discussed above, we aim to estimate the statistical properties of the curvature perturbation which would be measured by such an observer. It may happen that some realizations yield statistical properties which are quite different from the ensemble average, but such large excursions are uninteresting unless anthropic arguments are in play.

Next we introduce a collection of comparably-sized spacetime volumes whose mutual scatter is destined to dominate the microwave background anisotropy on a given scale. Neglecting spatial gradients, each spacetime volume will follow a trajectory in field space which is slightly displaced from its neighbors. The scatter between trajectories is determined by initial conditions set at horizon exit, which are determined in inflationary

models by promoting the vacuum fluctuation to a classical perturbation.† A correct prediction is a function of the trajectories followed by every volume in the collection, taken as a whole. One never makes a prediction for a single trajectory.

Each spacetime volume follows a trajectory, which we label with its position $\varphi^\star$ at some fixed time, to be made precise below. Throughout this paper, superscript '$\star$' denotes evaluation on a spatially flat hypersurface. Consider the evolution of some quantity of interest, F, which is a function of trajectory. If we know the distribution $P(\varphi^\star)$ we can study statistical properties of F such as the mth moment κ_m,

$$\kappa_m \equiv \int \mathrm{d}\varphi^\star \, P(\varphi^\star) \left[F(\varphi^\star) - \langle F \rangle \right]^m, \tag{1}$$

where we have introduced the ensemble average of F,

$$\langle F \rangle \equiv \int \mathrm{d}\varphi^\star \, P(\varphi^\star) F(\varphi^\star). \tag{2}$$

In Eqs. (1)–(2), and the remainder of this section, $\varphi^\star$ stands for a collection of any number of fields.

Eq. (1) defines what we will call the exact separate universe picture. The κ_m are the relevant observable quantities, but many existing formalisms do not compute Eq. (1) precisely. Instead, we pick a fiducial trajectory $\varphi^\star_{\text{fid}}$, and write F evaluated on this trajectory as F_{fid}. Formalisms of this type compute a subtly different moment, which we denote $\hat{\kappa}_m$,

$$\hat{\kappa}_m \equiv \int \mathrm{d}\varphi^\star \, P(\varphi^\star) \left[F(\varphi^\star) - F_{\text{fid}} \right]^m. \tag{3}$$

If F_{fid} is a good estimate of $\langle F \rangle$, then $\hat{\kappa}_m$ will be an accurate predictor of κ_m. In certain models, however, trajectories may exist for which F_{fid} exhibits significant departures from $\langle F \rangle$. For these trajectories, $\hat{\kappa}_m$ yields distorted estimates which over-emphasize uninteresting outlying values. We describe Eq. (3), and formalisms which implement it, as the 'fiducial' separate universe picture.

When the fiducial picture is in use, it is often convenient to expand $F(\varphi^\star)$ as a power series in the field values centered on the fiducial trajectory,

$$F(\varphi^\star) - F(\varphi^\star_{\text{fid}}) = \sum_{n=1}^{\infty} \frac{1}{n!} (\varphi^\star - \varphi^\star_{\text{fid}})^n \left. \frac{\partial^n F}{\partial (\varphi^\star)^n} \right|_{\varphi^\star = \varphi^\star_{\text{fid}}}. \tag{4}$$

We call Eq. (4) the 'perturbative' separate universe picture. (A similar construction can be performed for Eq. (1), but in that case there is no natural distinguished trajectory around which to expand F.) If all terms in the power series are retained,

† The scatter among trajectories may grow with the overall volume, owing to back-reaction effects in de Sitter space [40, 41, 42, 43]. If inflation can end in more than one vacuum, leading to different predictions for observable quantities, it may be helpful to evaluate this enhanced dispersion for the purpose of deciding into which vacuum the field will fall. Once this minimum has been decided, however, large-volume effects needlessly complicate the calculation. In this paper, we will always assume that predictions are being made by following a sufficient number of trajectories to determine the statistical properties with reasonable precision, but no more.

these two versions of the calculation are formally equivalent. In unfavorable cases, however, convergence may occur slowly or not at all. This possibility was discussed in Refs. [44, 45].

Before discussing the details of our method, we would like to make two observations. First, the formalism we propose in this paper is designed to compute κ_m, rather than the fiducial estimate $\hat{\kappa}_m$. For this reason we anticipate that, in models where tuned trajectories traverse outlying regions in field space, our method may be less sensitive to the issue of distorted moments. Second, our calculation is formally perturbative, but in a way which is not equivalent to Eq. (4). We believe this may make it less susceptible to problems associated with slow convergence. We briefly discuss the relation of our calculation to conventional perturbation theory in §5.

2.2. Calculating f_{NL} in the separate universe picture

By definition, the curvature perturbation ζ measures local fluctuations in expansion history (expressed in e-folds N), calculated on a comoving hypersurface. In many models, the curvature perturbation is synthesized by superhorizon physics, which reprocesses a set of Gaussian fluctuations generated at horizon exit. In a single-field model, only one Gaussian fluctuation can be present, which we label ζ_g. Neglecting spatial gradients, the total curvature perturbation must then be a function of ζ_g alone. For $|\zeta_g| \ll 1$, this can be well-approximated by

$$\zeta \simeq \zeta_g + \frac{3}{5} f_{\mathrm{NL}} \zeta_g^2 \tag{5}$$

where f_{NL} is independent of spatial position. Eq. (5) defines the so-called "local" form of non-gaussianity. It applies only when quantum interference effects can be neglected, making ζ a well-defined object rather than a superposition of operators [46]. If this condition is satisfied, spatial correlations of ζ may be extracted and it follows that f_{NL} can be estimated using the rule

$$f_{\mathrm{NL}} \simeq \frac{5}{27} \frac{\langle \zeta\zeta\zeta \rangle}{\langle \zeta\zeta \rangle^2}, \tag{6}$$

where we have recalled that ζ is nearly Gaussian, or equivalently that $|f_{\mathrm{NL}}| \ll |\zeta_g|^{-1}$.

Eq. (5) strictly applies only in single-field inflation. In this case one can accurately determine f_{NL} by applying Eq. (5) to a single trajectory, as in the method of Lehners & Renaux-Petel [27]. Where more than one field is present, f_{NL} may vary in space because it depends on the isocurvature modes. In this case one must determine f_{NL} statistically on a bundle of adjacent trajectories which sample the local distribution of isocurvature modes. Eq. (6) is then indispensible. Following Maldacena [10], and later Lyth & Rodríguez [11], we adopt Eq. (6) as our definition of f_{NL}, whatever its origin.

To proceed, we require estimates of the correlation functions $\langle \zeta\zeta \rangle$ and $\langle \zeta\zeta\zeta \rangle$. We first describe the conventional approach, in which '$\star$' denotes a flat hypersurface at a fixed initial time. The quantity $N(\varphi_i^\star)$ denotes the number of e-foldings between this

initial slice and a final comoving hypersurface, where i indexes the species of light scalar fields. The local variation in expansion can be written in the fiducial picture as

$$\zeta(\mathbf{x}) \equiv \delta N(\mathbf{x}) = \sum_{n=1}^{\infty} \frac{1}{n!} \left. \frac{\partial^n N(\varphi_i^\star)}{\partial \varphi_{j_1}^\star \cdots \partial \varphi_{j_n}^\star} \right|_{\varphi_i^\star = \varphi_{i,\mathrm{fid}}^\star} \delta\varphi_{j_1}^\star(\mathbf{x}) \cdots \delta\varphi_{j_n}^\star(\mathbf{x}), \qquad (7)$$

where $\delta\varphi_j^\star \equiv \varphi_j^\star - \varphi_{j,\mathrm{fid}}^\star$.

Subject to the condition that the relevant scales are all outside the horizon, we are free to choose the initial time—set by the hypersurface '$\star$'—at our convenience. In the conventional approach, '$\star$' is taken to lie a few e-folds after our collection of spacetime volumes passes outside the causal horizon [11, 12]. This choice has many virtues. First, we need to know statistical properties of the field fluctuations $\delta\varphi_i^\star$ only around the time of horizon crossing, where they can be computed without the appearance of large logarithms [43, 47]. Second, the $\delta\varphi_i^\star$ are uncorrelated at this time, leading to algebraic simplifications. Finally, the δN formula subsumes a gauge transformation from the field variables $\delta\varphi_i^\star$ to the observational variable ζ. With this choice, and combining Eqs. (6) and (7), one finds that f_{NL} can be written to a good approximation [11]

$$f_{\mathrm{NL}} \approx \frac{5}{6} \frac{N_{,i} N_{,j} N_{,ij}}{(N_{,k} N_{,k})^2}, \qquad \text{'$\star$' at horizon crossing} \qquad (8)$$

where $N_{,i} \equiv \partial N / \partial \varphi_i^\star$ and for simplicity we have dropped the '$\star$' which indicates time of evaluation. A similar definition applies for $N_{,ij}$.

Eq. (8) is accurate up to small intrinsic non-Gaussianities present in the field fluctuations at horizon exit. As a means of predicting f_{NL} it is pleasingly compact, and straightforward to evaluate in many models. Unfortunately, it also obscures the physics which determines the magnitude of $|f_{\mathrm{NL}}|$. For this reason, it is hard to obtain guidance from Eq. (8) which would help us identify classes of models in which $|f_{\mathrm{NL}}|$ is always large or small.‡

Our strategy is quite different. We choose '$\star$' to lie around the time when we require the statistical properties of ζ. The role of the δN formula, Eq. (7), is then to encode *only* the gauge transformation between the $\delta\varphi_i^\star$ and ζ. In §2.3 below, we show how the appropriate gauge transformation is computed using the δN formula. In the present section we restrict our attention to determining a formula for f_{NL} under the assumption that the distribution of field values on '$\star$' is known. In §3, we will supply the required prescription to evolve the distribution of field values between horizon exit and '$\star$'.

Although the $\delta\varphi_i^\star$ are independent random variables at horizon exit, correlations can be induced by subsequent evolution. One must therefore allow for off-diagonal terms in the two-point function. Remembering that we are working with a collection of spacetime volumes in real space, smoothed on some characteristic scale, we write

$$\langle \delta\varphi_i^\star \delta\varphi_j^\star \rangle \equiv \Sigma_{ij}. \qquad (9)$$

‡ Even in simple models it can be quite subtle to determine what range of $|f_{\mathrm{NL}}|$ is dynamically allowed. See, for example, Refs. [21, 22].

Σ_{ij} does not vary in space, but it may be a function of the scale which characterizes our ensemble of spacetime volumes. In all but the simplest models it will vary in time. It is also necessary to account for intrinsic non-linearities among the $\delta\varphi_i^\star$, which are small at horizon crossing but may grow. We define

$$\langle \delta\varphi_i^\star \delta\varphi_j^\star \delta\varphi_k^\star \rangle \equiv \alpha_{ijk}. \tag{10}$$

Likewise, α_{ijk} should be regarded as a function of time and scale. The permutation symmetries of an expectation value such as (10) guarantee that, for example, $\alpha_{122} = \alpha_{212} = \alpha_{221}$. When written explicitly, we place the indices of symbols such as α in numerical order. Neglecting a small ($\lesssim \mathrm{O}(\Sigma^3)$) intrinsic four-point correlation, it follows that

$$\langle \delta\varphi_i^\star \delta\varphi_j^\star \delta\varphi_k^\star \delta\varphi_m^\star \rangle = \Sigma_{ij}\Sigma_{km} + \Sigma_{ik}\Sigma_{jm} + \Sigma_{im}\Sigma_{jk} \equiv \beta_{ijkm}. \tag{11}$$

Now we specialize to a two-field model, parametrized by fields φ_1 and φ_2. In this model, the two-point function of ζ satisfies

$$\langle \zeta\zeta \rangle = N_{,1}^2 \Sigma_{11} + N_{,2}^2 \Sigma_{22} + 2N_{,1}N_{,2}\Sigma_{12} \qquad \text{`$\star$' arbitrary} \tag{12}$$

Moreover the three-point function can be written

$$
\begin{aligned}
\langle \zeta\zeta\zeta \rangle = {}& N_{,1}^3 \alpha_{111} + N_{,2}^3 \alpha_{222} + 3N_{,1}^2 N_{,2} \alpha_{112} + 3N_{,1}N_{,2}^2 \alpha_{122} \qquad \text{`$\star$' arbitrary} \\
&+ \frac{3}{2}N_{,1}^2 N_{,11} \beta_{1111} + \frac{3}{2}N_{,2}^2 N_{,22} \beta_{2222} \\
&+ 3\left(N_{,1}N_{,2}N_{,11} + N_{,1}^2 N_{,12}\right)\beta_{1112} \\
&+ 3\left(N_{,1}N_{,2}N_{,22} + N_{,2}^2 N_{,12}\right)\beta_{1222} \\
&+ \left(\frac{3}{2}N_{,2}^2 N_{,11} + \frac{3}{2}N_{,1}^2 N_{,22} + 6N_{,1}N_{,2}N_{,12}\right)\beta_{1122},
\end{aligned}
\tag{13}
$$

from which f_{NL} can be obtained after application of Eq. (6).

2.3. The derivatives of N

To compute f_{NL} in concrete models, we require expressions for the derivatives $N_{,i}$ and $N_{,ij}$. For generic initial and final times, these are difficult to obtain. Lyth & Rodríguez [11] used direct integration, which is effective for quadratic potentials and constant slow-roll parameters. Vernizzi & Wands [19] obtained expressions in a two-field model with an arbitrary sum-separable potential by introducing Gaussian normal coordinates on the space of trajectories. Their approach was generalized to many fields by Battefeld & Easther [20]. Product-separable potentials can be accommodated using the same technique [48].

A considerable simplification occurs in the present case, because we only require the derivative evaluated between flat and comoving hypersurfaces which coincide in the unperturbed universe. For any species i, and to leading order in the slow-roll approximation, the number of e-folds N between the flat hypersurface '$\star$' and a comoving hypersurface 'c' satisfies

$$N \equiv -\int_{\varphi_i^\star}^{\varphi_i^c} \frac{V}{V_{,i}}\,\mathrm{d}\varphi_i \qquad \text{no sum on } i, \tag{14}$$

where $V_{,i} \equiv \partial V/\partial \varphi_i$ and $\{\varphi_i^\star, \varphi_i^c\}$ are the field values evaluated on '$\star$' and 'c,' respectively. Under an infinitesimal shift of $\varphi_i^\star$, we deduce that $N_{,i}$ obeys

$$N_{,i} = \left(\frac{V}{V_{,i}}\right)^\star - \left(\frac{V}{V_{,i}}\right)^c \frac{\partial \varphi_i^c}{\partial \varphi_i^\star} \qquad \text{no sum on } i. \qquad (15)$$

Note that this applies for an arbitrary V, which need not factorize into a sum or product of potentials for the individual species i. In principle a contribution from variation of the integrand is present, which spoils a naïve attempt to generalize the method of Refs. [19, 20, 48] to an arbitrary potential. This contribution vanishes in virtue of our supposition that '$\star$' and 'c' are infinitesimally separated.

To compute $\partial \varphi_i^c/\partial \varphi_j^\star$ it is helpful to introduce a quantity C, which in the sum-separable case coincides with the conserved quantity of Vernizzi & Wands [19, 49]. For our specific choice of a two-field model, this takes the form

$$C(\varphi_1, \varphi_2) \equiv \int^{\varphi_1} \frac{H^2}{V_{,1}} \, \mathrm{d}\varphi_1' - \int^{\varphi_2} \frac{H^2}{V_{,2}} \, \mathrm{d}\varphi_2', \qquad (16)$$

where the integrals are evaluated on a single spatial hypersurface. In an M-field model, one would obtain $M-1$ conserved quantities which label the isocurvature fields. The construction of these quantities is discussed in Refs. [20, 24]. For sum-separable potentials one can show using the equations of motion that C is conserved under time evolution to leading order in slow-roll. It is not conserved for general potentials, but the variation can be neglected for infinitesimally separated hypersurfaces.

Under a change of trajectory, C varies according to the rules

$$\frac{\partial C}{\partial \varphi_1^\star} = \frac{H^2}{V_{,1}} \qquad (17)$$

and

$$\frac{\partial C}{\partial \varphi_2^\star} = -\frac{H^2}{V_{,2}}. \qquad (18)$$

The comoving hypersurface 'c' is defined by

$$\frac{1}{2}\left(\dot{\varphi}_1^2 + \dot{\varphi}_2^2\right) + V = \text{constant}. \qquad (19)$$

We are assuming that the slow-roll approximation applies, so that the kinetic energy may be neglected in comparison with the potential V. Therefore on 'c' we have

$$\frac{\partial V}{\partial \varphi_1^c} \frac{\partial \varphi_1^c}{\partial C} + \frac{\partial V}{\partial \varphi_2^c} \frac{\partial \varphi_2^c}{\partial C} = 0. \qquad (20)$$

Combining Eqs. (17), (18) and (20) we obtain expressions for $\partial \varphi_i^c/\partial \varphi_j^\star$, namely

$$\frac{\partial \varphi_1^c}{\partial \varphi_1^\star} = \left(\frac{V_{,1}}{V}\right)^c \left(\frac{V}{V_{,1}}\right)^\star \sin^2 \theta, \qquad (21)$$

$$\frac{\partial \varphi_1^c}{\partial \varphi_2^\star} = -\left(\frac{V_{,1}}{V}\right)^c \left(\frac{V}{V_{,2}}\right)^\star \sin^2 \theta, \qquad (22)$$

$$\frac{\partial \varphi_2^c}{\partial \varphi_1^\star} = -\left(\frac{V_{,2}}{V}\right)^c \left(\frac{V}{V_{,1}}\right)^\star \cos^2 \theta, \qquad (23)$$

$$\frac{\partial \varphi_2^c}{\partial \varphi_2^\star} = \left(\frac{V_{,2}}{V}\right)^c \left(\frac{V}{V_{,2}}\right)^\star \cos^2 \theta, \qquad (24)$$

where we have defined

$$\tan^2\theta \equiv \frac{(V_{,2})^2}{(V_{,1})^2}.$$

(25)

Eqs. (21)–(24) can alternatively be derived without use of C by comparing Eq. (15) with the formulas of Ref. [50], which were derived using conventional perturbation theory. Applying (15), we obtain

$$N_{,1} = \left(\frac{V}{V_{,1}}\right)^\star \cos^2\theta; \qquad N_{,2} = \left(\frac{V}{V_{,2}}\right)^\star \sin^2\theta.$$

(26)

To proceed, we require the second derivatives of N. These can be obtained directly from (26), after use of Eqs. (21)–(24). We find

$$N_{,11} = \left[1 - \frac{VV_{,11}}{(V_{,1})^2}\right]^\star \cos^2\theta + 2\left(\frac{V}{V_{,1}}\right)^{\star 2}\cos^2\theta$$
$$\times \left[\frac{V_{,11}}{V}\sin^2\theta - \frac{V_{,1}V_{,12}}{VV_{,2}}\sin^4\theta - \left(\frac{V_{,11}}{V} - \frac{V_{,22}}{V} + \frac{V_{,2}V_{,12}}{VV_{,1}}\right)\cos^2\theta\sin^2\theta\right]^c.$$

(27)

An analogous expression for $N_{,22}$ can be obtained after the simultaneous exchange $\{1\leftrightarrow 2, \sin\leftrightarrow\cos\}$. The mixed derivative satisfies

$$N_{,12} = 2\left(\frac{V}{V_{,1}}\right)^\star\left(\frac{V}{V_{,2}}\right)^\star\cos^2\theta$$
$$\times\left[-\frac{V_{,11}}{V}\sin^2\theta + \frac{V_{,1}V_{,12}}{VV_{,2}}\sin^4\theta + \left(\frac{V_{,11}}{V} - \frac{V_{,22}}{V} + \frac{V_{,2}V_{,12}}{VV_{,1}}\right)\cos^2\theta\sin^2\theta\right]^c$$
$$+ \cos^2\theta\left(\frac{V_{,2}}{V_{,1}} - \frac{VV_{,12}}{V_{,1}^2}\right)^c.$$

(28)

Now that the calculation is complete, we can drop the superscripts '$\star$' and 'c,' since any background quantity is the same on either hypersurface. Once this is done it can be verified that (despite appearances) Eq. (28) is invariant under the exchange $1\leftrightarrow 2$.

3. Transport equations

In this section we return to the problem of evolution between horizon exit and the time of observation, and supply the prescription which connects the distribution of field values at these two times.

3.1. Non-gaussian distribution in the single-field case

We begin by discussing the single-field system, which lacks the technical complexity of the two-field case, yet still exhibits certain interesting features which recur there. Among these features are the subtle difference between motion of the statistical mean and the background field value, and the hierarchy of moment evolution equations. Moreover, the structure of the moment mixing equations is similar to that which obtains in the two-field case. For this reason, the one-field scenario provides an instructive example of the techniques we wish to employ.

Recall that we work in real space with a collection of comparably sized spacetime volumes, each with a slightly different expansion history, and the scatter in these histories determines the microwave background anisotropy on a given angular scale. Within each volume the smoothed background field φ takes a uniform value described by a density function $P(\varphi)$, where in this section we are dropping the superscript '$\star$' denoting evaluation of spatially flat hypersurfaces. Our ultimate goal is to calculate the reduced bispectrum, f_{NL}, which describes the third moment of $P(\varphi)$. In the language of probability this is the skewness, which we denote α. A Gaussian distribution has skewness zero, and inflation usually predicts that the skew is small. For this reason, rather than seek a distribution with non-zero third moment, as proposed in Ref. [18], we will introduce higher moments as perturbative corrections to the Gaussian. Such a procedure is known as a *cumulant expansion*.

The construction of cumulant expansions is a classical problem in probability theory. We seek a distribution with centroid φ_0, variance σ^2, and skew α, with all higher moments determined by σ and α alone. A distribution with suitable properties is

$$P(\varphi) = P_g(\varphi) \left[1 + \frac{\alpha}{6\sigma^3} H_3 \left(\frac{\varphi - \varphi_0}{\sigma} \right) \right], \tag{29}$$

where

$$P_g(\varphi) \equiv \frac{1}{\sqrt{2\pi}\sigma} \exp \left[-\frac{(\varphi - \varphi_0)^2}{2\sigma^2} \right] \tag{30}$$

is a pure Gaussian and H_n denotes the nth Hermite polynomial, for which there are multiple normalization conventions. We choose to normalize so that

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \, H_n(x) H_m(x) \, \mathrm{d}x = n! \, \delta_{mn}, \tag{31}$$

which implies that the leading term of $H_n(x)$ is x^n. This is sometimes called the "Probabilist's convention." We define expectation values $\langle \cdots \rangle$ by the usual rule,

$$\langle F \rangle \equiv \int_{-\infty}^{\infty} P(\varphi) F \, \mathrm{d}x. \tag{32}$$

The probability density function in Eq. (29) has the properties†

$$\langle 1 \rangle = 1, \quad \langle \varphi \rangle = \varphi_0, \quad \langle (\varphi - \varphi_0)^2 \rangle = \sigma^2, \quad \text{and} \quad \langle (\varphi - \varphi_0)^3 \rangle = \alpha. \tag{33}$$

The moments φ_0, σ, and α may be time-dependent, so evolution of the probability density in time can be accommodated by finding evolution equations for these quantities.

The density function given in Eq. (29) is well-known and has been applied in many situations. It is a solution to the problem of approximating a nearly-Gaussian

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† These formulas apply for arbitrary values of α, and do not depend on the approximation that α is small. However, for large α the density function (29) may become negative for some values of φ. It then ceases to be a probability density in the strict sense. This does not present a problem in practice, since we are interested in distributions which are approximately Gaussian, and for which α will typically be small. Moreover, our principal use of Eq. (29) is as a formal tool to extract evolution equations for each moment. For this reason we will not worry whether $P(\varphi)$ defines an honest probability density function in the strict mathematical sense.

distribution whose moments are known. Eq. (29) is in fact the first two terms of the *Gram–Charlier 'A' series*, also sometimes called the *Gauss–Hermite series*.‡ In recent years it has found multiple applications to cosmology, of which our method is closest to that of Taylor & Watts [51]. Other applications are discussed in Refs. [16, 17, 15, 52, 18, 51, 53, 54, 55, 56, 57, 58]. For a review of the 'A' series and related nearly-Gaussian probability distributions from an astrophysical perspective, see [59]. In this paper, we will refer to Eq. (29) and its natural generalization to higher moments as the "moment expansion."

In the slow-roll approximation, the field in each spacetime volume obeys a simple equation of motion

$$\frac{\mathrm{d}\varphi}{\mathrm{d}N} = -\frac{\partial \ln V(\varphi)}{\partial \varphi} \equiv u(\varphi), \tag{34}$$

where N records the number of e-foldings of expansion. We refer to $u(\varphi)$ as the velocity field. Expanding u about the instantaneous centroid φ_0 gives

$$u(\varphi) = u_0 + u_\varphi(\varphi - \varphi_0) + \frac{1}{2}u_{\varphi\varphi}(\varphi - \varphi_0)^2 + \cdots, \tag{35}$$

where

$$u_0 \equiv u|_{\varphi_0}, \quad u_\varphi \equiv \left.\frac{\mathrm{d}u}{\mathrm{d}\varphi}\right|_{\varphi_0}, \quad u_{\varphi\varphi} \equiv \left.\frac{\mathrm{d}^2 u}{\mathrm{d}\varphi^2}\right|_{\varphi_0}. \tag{36}$$

The value of φ_0 evolves with time, so each expansion coefficient is time-dependent. Hence, we do not assume that the velocity field is *globally* well-described by a quadratic Taylor expansion, but merely that it is well-described as such in the neighborhood of the instantaneous centroid. We expand the velocity field to second order, although in principle this expansion could be carried to arbitrary order.

It remains to specify how the probability density evolves in time. Conservation of probability leads to the transport equation

$$\frac{\partial P}{\partial N} + \frac{\partial}{\partial \varphi}(uP) = 0. \tag{37}$$

Eq. (37) can also be understood as the limit of a Chapman–Kolmogorov process as the size of each hop goes to zero. It is well known—for example, from the study of Starobinsky's diffusion equation which forms the basis of the stochastic approach to inflation [60]—that the choice of time variable in this equation is significant, with different choices corresponding to the selection of a temporal gauge. We have chosen to use the e-folding time, N, which means that we are evolving the distribution on hypersurfaces of uniform expansion. These are the spatially flat hypersurfaces whose field perturbations enter the δN formulas described in §2.

In principle, Eq. (37) can be solved directly. In practice it is simpler to extract equations for the moments of P, giving evolution equations for φ_0, σ and α. To achieve this, one need only resolve Eq. (37) into a Hermite series of the form

$$P_g \sum_{n \geqslant 0} c_n H_n(\varphi - \varphi_0) = 0 \tag{38}$$

‡ In the physics literature, this series has sometimes erroneously been called the Edgeworth expansion.

The Hermite polynomials are linearly independent, and application of the orthogonality condition (31) shows that the c_n must all vanish. This leads to a hierarchy of equations $c_n = 0$, which we refer to as the moment hierarchy. At the top of the hierarchy, the equation $c_0 = 0$ is empty and expresses conservation of probability.

The first non-trivial equation requires $c_1 = 0$ and yields an evolution equation for the centroid φ_0,

$$\frac{\mathrm{d}\varphi_0}{\mathrm{d}N} = u_0 + \frac{1}{2}u_{\varphi\varphi}\sigma^2. \tag{39}$$

The first term on the right-hand side drives the centroid along the velocity field, as one would anticipate based on the background equation of motion, Eq. (34). However, the second term shows that the centroid is also influenced as the wings of the probability distribution probe the nearby velocity field. This influence is not captured by the background equation of motion. If we are in a situation with $u_{\varphi\varphi} > 0$, then the wings of the density function will be moving faster than the center. Hence, the velocity of the centroid will be larger than one might expect by restricting attention to φ_0. Accordingly, the mean fluctuation value is not following a solution to the background equations of motion. This is a reflection of the fact that our formalism computes the moment κ_m given in Eq. (1) rather than its fiducial proxy, Eq. (3).

Evolution equations for the variance σ^2 and skew α are obtained after enforcing $c_2 = c_3 = 0$, yielding

$$\frac{\mathrm{d}\sigma^2}{\mathrm{d}N} = 2u_\varphi\sigma^2 + u_{\varphi\varphi}\alpha \tag{40}$$

$$\frac{\mathrm{d}\alpha}{\mathrm{d}N} = 3u_\varphi\alpha + 3u_{\varphi\varphi}\sigma^4 \tag{41}$$

In both equations, the first term on the right-hand sides describes how σ and α scale as the density function expands or contracts in response to the velocity field. These terms force σ^2 and α to scale inversely with the velocity field. If we temporarily drop the second terms in each equation above, one finds that $\sigma^2 \sim u^2$ and $\alpha \sim u^3$. This precisely matches our expectation for the scaling of these quantities. Hence, these terms account for the Jacobians associated with infinitesimal transformations induced by the flow $u(\varphi)$.

For applications to inflationary non-Gaussianity, the second terms in (40) and (41) are more relevant. These terms describe how each moment is sourced by higher moments and the interaction of the density function with the velocity field. In the example above, if we are in a situation where $u_{\varphi\varphi} > 0$, the tails of the density function are moving faster than the core. This means that one tail is shrinking and the other is extending, skewing the probability density. The opposite occurs when $u_{\varphi\varphi} < 0$. These effects are measured by the second term in (41). Hence, by expanding our PDF to the third moment, and our velocity field to quadratic order, we are able to construct a set of evolution equations which include the leading-order source terms for each moment.

3.2. The two-field case

There is little conceptually new as we move from one field to two. The new features are mostly technical in nature. Our primary challenge is a generalization of the moment expansion to two fields, allowing for the possibility of correlation between the fields. With this done, we can write down evolution equations whose structure is very similar to those found in the single-field case.

The two-field system is described by a two-dimensional velocity field u_i, defined by

$$u_i = \frac{\mathrm{d}\varphi_i}{\mathrm{d}N},\tag{42}$$

where again we are using the number of e-folds N as the time variable. The index i takes values in $\{1,2\}$. While we think it is likely that our equations generalize to any number of fields, we have only explicitly constructed them for a two-field system. As will become clear below, certain steps in this construction apply only for two fields, and hence we make no claims at present concerning examples with three or more fields.

The two-dimensional transport equation is

$$\frac{\partial P(\varphi_i, N)}{\partial N} + \frac{\partial}{\partial \varphi_j}\left[u_j P(\varphi_i, N)\right] = 0.\tag{43}$$

Here and in the following we have returned to our convention that repeated species indices are summed. As in the single-field case, we construct a probability distribution which is nearly Gaussian, but has a small non-zero skewness. That gives

$$P(\varphi_i, N) \equiv P_g(\varphi_i, N) P_{ng}(\varphi_i, N)\tag{44}$$

where P_g is a pure Gaussian distribution, defined by

$$P_g(\varphi_i, N) = \frac{1}{2\pi\sqrt{\det \Sigma}} \exp\left[-\frac{1}{2}(\varphi_i - \Phi_i)(\Sigma^{-1})_{ij}(\varphi_j - \Phi_j)\right].\tag{45}$$

In this equation, Φ_i defines the center of the distribution and Σ describes the covariance between the fields. We adopt a conventional parametrization in terms of variances σ_i^2 and a correlation coefficient ρ,

$$\Sigma \equiv \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}.\tag{46}$$

The matrix σ defines two-point correlations of the fields,

$$\langle (\varphi_i - \Phi_i)(\varphi_j - \Phi_j) \rangle = \Sigma_{ij}.\tag{47}$$

All skewnesses are encoded in P_{ng}. Before defining this explicitly, it is helpful to pause and notice a complication inherent in Eqs. (45)–(46) which was not present in the single-field case. To extract a hierarchy of moment evolution equations from the transport equation, Eq. (37), we made the expansion given in (38) and argued that orthogonality of the Hermite polynomials implied the hierarchy $c_n = 0$. However, Hermite polynomials of the form $H_n[(\varphi_i - \Phi_i)/\sigma]$ are *not* orthogonal under the Gaussian measure of Eq. (45). Following an expansion analogous to Eq. (38) the moment hierarchy

would comprise linear combinations of the coefficients. The problem is essentially an algebraic question of Gram–Schmidt orthogonalization.

To avoid this problem it is convenient to diagonalize the covariance matrix Σ, introducing new variables X and Y for which Eq. (45) factorizes into the product of two measures under which the polynomials $H_n(X)$ and $H_n(Y)$ are separately orthogonal. The necessary redefinitions are

$$X \equiv \frac{1}{\sqrt{2(1+\rho)}} \left[\left(\frac{\varphi_1 - \Phi_1}{\sigma_1} \right) + \left(\frac{\varphi_2 - \Phi_2}{\sigma_2} \right) \right] \tag{48}$$

and

$$Y \equiv \frac{1}{\sqrt{2(1-\rho)}} \left[\left(\frac{\varphi_1 - \Phi_1}{\sigma_1} \right) - \left(\frac{\varphi_2 - \Phi_2}{\sigma_2} \right) \right]. \tag{49}$$

It is this step which prevents an immediate generalization of our results to an arbitrary number of fields. A simple expression for P_g can be given in terms of X and Y,

$$P_g = \frac{1}{2\pi} \exp\left(-\frac{X^2}{2} \right) \exp\left(-\frac{Y^2}{2} \right). \tag{50}$$

We now define the non-Gaussian factor, which encodes the skewnesses, to be

$$
\begin{aligned}
P_{ng} \equiv 1 &+ \frac{\alpha_{XXX}}{6} H_3(X) \\
&+ \frac{\alpha_{XXY}}{2} H_2(X) H_1(Y) + \frac{\alpha_{XYY}}{2} H_1(X) H_2(Y) + \frac{\alpha_{YYY}}{6} H_3(Y).
\end{aligned}
\tag{51}
$$

In these variables we find $\langle X^2 \rangle = \langle Y^2 \rangle = 1$, but $\langle XY \rangle = 0$. In addition, we have

$$\langle XXX \rangle = \alpha_{XXX}, \quad \langle XXY \rangle = \alpha_{XXY}, \quad \langle XYY \rangle = \alpha_{XYY}, \quad \text{and} \quad \langle YYY \rangle = \alpha_{YYY}. \tag{52}$$

In order for Eq. (51) to be useful, it is necessary to express the skewnesses associated with the physical variables φ_i in terms of X and Y. By definition, these satisfy

$$\langle (\varphi_i - \Phi_i)(\varphi_j - \Phi_j)(\varphi_k - \Phi_k) \rangle \equiv \alpha_{ijk}. \tag{53}$$

After substituting for the definition of these quantities inside the expectation values in Eq. (52) we arrive at the relations

$$\alpha_{XXX} = \frac{1}{2\sqrt{2}(1+\rho)^{3/2}} \left(\frac{\alpha_{111}}{\sigma_1^3} + 3\frac{\alpha_{112}}{\sigma_1^2 \sigma_2} + 3\frac{\alpha_{122}}{\sigma_1 \sigma_2^2} + \frac{\alpha_{222}}{\sigma_2^3} \right), \tag{54}$$

$$\alpha_{XXY} = \frac{1}{2\sqrt{2(1-\rho)}(1+\rho)} \left(\frac{\alpha_{111}}{\sigma_1^3} + \frac{\alpha_{112}}{\sigma_1^2 \sigma_2} - \frac{\alpha_{122}}{\sigma_1 \sigma_2^2} - \frac{\alpha_{222}}{\sigma_2^3} \right), \tag{55}$$

$$\alpha_{XYY} = \frac{1}{2\sqrt{2(1+\rho)}(1-\rho)} \left(\frac{\alpha_{111}}{\sigma_1^3} - \frac{\alpha_{112}}{\sigma_1^2 \sigma_2} - \frac{\alpha_{122}}{\sigma_1 \sigma_2^2} + \frac{\alpha_{222}}{\sigma_2^3} \right), \tag{56}$$

$$\alpha_{YYY} = \frac{1}{2\sqrt{2}(1-\rho)^{3/2}} \left(\frac{\alpha_{111}}{\sigma_1^3} - 3\frac{\alpha_{112}}{\sigma_1^2 \sigma_2} + 3\frac{\alpha_{122}}{\sigma_1 \sigma_2^2} - \frac{\alpha_{222}}{\sigma_2^3} \right). \tag{57}$$

The moments Φ_i, Σ_{ij} and α_{ijk} are time-dependent, but for clarity we will usually suppress this in our notation.

Next we must extract the moment hierarchy, which governs evolution of Φ_i, σ_i, ρ and α_{ijk}. We expand the velocity field in a neighborhood of the instantaneous centroid Φ_i according to

$$u_i(\varphi_j) = u_{i0} + u_{ij}(\varphi_j - \Phi_j) + \frac{1}{2}u_{ijk}(\varphi_j - \Phi_j)(\varphi_k - \Phi_k) + \cdots, \qquad (58)$$

where we have defined

$$u_{i0} \equiv u_i|_{\Phi_i}, \quad u_{ij} \equiv \left.\frac{\partial u_i}{\partial \varphi_j}\right|_{\Phi_i}, \quad \text{and} \quad u_{ijk} \equiv \left.\frac{\partial^2 u_i}{\partial \varphi_j \partial \varphi_k}\right|_{\Phi_i}. \qquad (59)$$

As in the single-field case, these coefficients are functions of time and vary with the motion of the centroid. The expansion can be pursued to higher order if desired.

Our construction of X and Y implies that the two-field transport equation can be arranged as a double Gauss–Hermite expansion,

$$\frac{\partial P(\varphi_i, N)}{\partial N} + \frac{\partial}{\partial \varphi_i}[u_i\,P(\varphi_i, N)] = P_g \sum_{m,n\geq 0} c_{mn} H_m(X) H_n(Y) = 0. \qquad (60)$$

Because the Hermite polynomials are orthogonal in the measure defined by P_g, we deduce the moment hierarchy

$$c_{mn} = 0. \qquad (61)$$

We define the "rank" r of each coefficient c_{mn} by $r \equiv m + n$. We terminated the velocity field expansion at quadratic order, and our probability distribution included only the first three moments. It follows that only c_{mn} with rank five or less are nonzero. If we followed the velocity field to higher order, or included higher terms in the moment expansion, we would obtain non-trivial higher-rank coefficients. Inclusion of additional coefficients requires no qualitative modification of our analysis and can be incorporated in the scheme we describe below.

A useful feature of the expansion in Eq. (60) is that the rank-r coefficients give evolution equations for the order-r moments. Written explicitly in components, the expressions that result from (60) are quite cumbersome. However, when written as field-space covariant expressions they can be expressed in a surprisingly compact form.

Rank 0 The rank-0 coefficient c_{00} is identically zero. This expresses the fact that the total probability is conserved as the distribution evolves.

Rank 1 The rank-1 coefficients c_{01} and c_{10} give evolution equations for the centroid Φ_i. These equations can be written in the form

$$\frac{\mathrm{d}\Phi_i}{\mathrm{d}N} = u_{i0} + \frac{1}{2}\Sigma_{jk}u_{ijk}. \qquad (62)$$

We remind the reader that here and below, terms like u_{i0}, u_{ij} and u_{ijk} represent the velocity field and its derivatives evaluated at the centroid Φ_i. The first term in (62) expresses the non-anomalous motion of the centroid, which coincides with the background velocity field of Eq. (42). The second term describes how the wings of the probability distribution sample the velocity field at nearby points. Narrow probability distributions have small components of Σ and hence are only sensitive

to the local value of $u_i(\varphi_j)$. Broad probability distributions have large components of Σ and are therefore more sensitive to the velocity field far from the centroid.

Rank 2 The rank-2 coefficients c_{02}, c_{11} and c_{20} give evolution equations for the variances σ_i^2 and the correlation ρ. These can conveniently be packaged as evolution equations for the matrix Σ

$$\frac{\mathrm{d}\Sigma_{ij}}{\mathrm{d}N} = u_{ik}\Sigma_{kj} + u_{jk}\Sigma_{ki} + \frac{1}{2}\left(\alpha_{imn}u_{jmn} + \alpha_{jmn}u_{imn}\right). \tag{63}$$

This equation describes the stretching and rotation of Σ as it is transported by the velocity field. It includes a sensitivity to the wings of the probability distribution, in a manner analogous to the similar term appearing in (40). Hence the skew α_{ijk} acts as a source for the correlation matrix.

Rank 3 The rank-3 coefficients c_{03}, c_{12}, c_{21} and c_{30} describe evolution of the moments α_{ijk}. These are

$$\frac{\mathrm{d}\alpha_{ijk}}{\mathrm{d}N} = u_{in}\alpha_{njk} + \Sigma_{jm}u_{imn}\Sigma_{nk}$$
$$+ \text{cyclic permutations } i \to j \to k. \tag{64}$$

The first term describes how the moments flow into each other as the velocity field rotates and shears the (X, Y) coordinate frame relative to the φ_i coordinate frame. The second term describes sourcing of non-Gaussianity from inhomogeneities in the velocity field and the overall spread of the probability distribution.

Some higher-rank coefficients—in our case, those of ranks four and five—are also nonzero, but do not give any new evolution equations. These coefficients measure the "error" introduced by truncating the moment expansion. If we had included higher cumulants, these higher-rank coefficients would have given evolution equations for the higher moments of the probability distribution. In general, all moments of the density function will mix so it is always necessary to terminate our expansion at a predetermined order—both in cumulants and powers of the field fluctuation. The order we have chosen is sufficient to generate evolution equations containing both the leading-order behavior of the moments—namely, the first terms in Eqs. (62), (63) and (64)—and the leading corrections, given by the latter terms in these equations.

4. Numerical results

At this point we put our new method into practice. We study two models for which the non-Gaussian signal is already known, using the standard δN approach. For each case we employ our method and compare it with results obtained using δN. To ensure a fair comparison, we solve the models numerically in both cases. Our new method employs the slow-roll approximation, as described above. Therefore, when using the δN approach we produce results both with and without slow-roll simplifications.
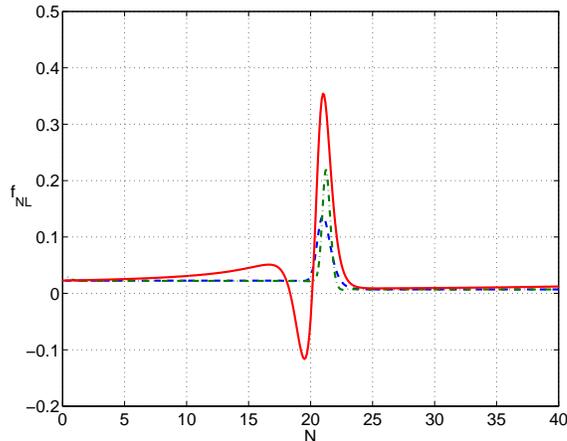
Figure 1. Evolution of f_{NL} in double quadratic inflation. The solid red line is obtained by numerically solving the moment transport equations obtained in §3. The blue dashed line and green dot-dashed line are the output of a numerical implementation of the standard δN approach, with and without slow roll respectively, using the fiducial picture.

First consider double quadratic inflation, which was studied by Rigopoulos, Shellard & van Tent [35, 61] and later by Vernizzi & Wands [19]. The potential is

$$V(\phi, \chi) = \frac{1}{2}m_\phi^2\phi^2 + \frac{1}{2}m_\chi^2\chi^2. \tag{65}$$

We use the initial conditions chosen in Ref. [35], where $m_\phi/m_\chi = 9$, and the fiducial trajectory has coordinates $\phi^\star = 8.2$ and $\chi^\star = 12.9$.

We plot the evolution of f_{NL} in Fig. 1, which also shows the prediction of the standard δN formula (with and without employing slow roll simplifications). We implement the δN algorithm using a finite difference method to calculate the derivatives of N. A similar technique was used in Ref. [19]. This model yields a very modest non-Gaussian signal, below unity even at its peak. If inflation ends away from the spike then f_{NL} is practically negligible. The method of moment transport allows us to separate contributions to f_{NL} from the intrinsic non-Gaussianity of the field fluctuations, and non-linearities of the gauge transformation to ζ. We denote the former $f_{\mathrm{NL}1}$ and the latter $f_{\mathrm{NL}2}$, and plot them separately in Fig. 2.

It has recently been shown by Byrnes *et al.* that a large non-Gaussian signal can be generated even within the slow-roll approximation [21, 22]. The conditions for this to occur are incompletely understood, but apparently require a specific choice of potential and strong tuning of initial conditions. In Figs. 3–4 we show the evolution of f_{NL} in a model with the potential

$$V = V_0\chi^2 e^{-\lambda\phi^2}, \tag{66}$$

which corresponds to Example A in §5 of Ref. [21] when we choose $\lambda = 0.05$ and initial conditions $\chi^\star = 16$, $\phi^\star = 0.001$.
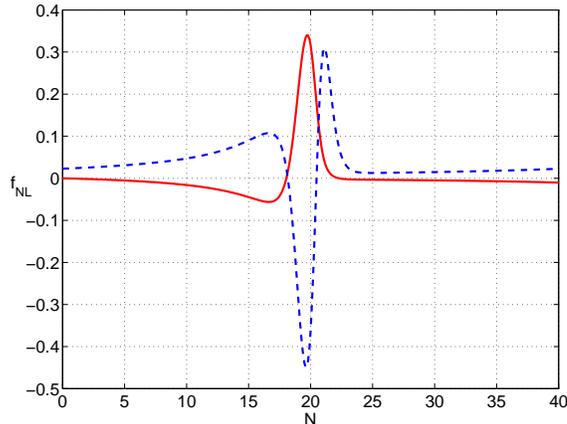
Figure 2. Evolution of $f_{\mathrm{NL}1}$ (solid red line), measuring the contribution of intrinsic non-linearities among the field fluctuations; and $f_{\mathrm{NL}2}$ (dashed blue line), measuring the contribution of the gauge transformation to ζ.
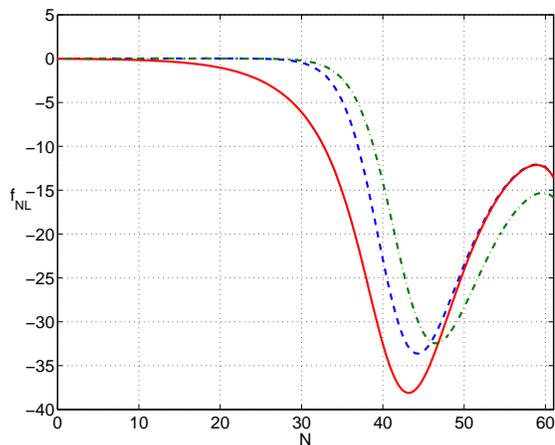

Figure 3. Evolution of f_{NL}, for the potential $V = V_0 \chi^2 e^{-\lambda \phi^2}$ (Example A from §5 of Ref. [21]). The solid red line represents the method of moment transport, whereas the blue dashed line and green dot-dashed line represents the output of conventional numerical δN with and without slow-roll respectively.

Figs. 1–4 show broad agreement between our new method and the outcome of the numerical δN formula. Above $|f_{\mathrm{NL}}| \gtrsim 1$, the absolute magnitude of f_{NL} is consistent within $\sim 10\%$. There is excellent agreement in the overall shape of f_{NL} as a function of N. Some small discrepancies can be observed. In Fig. 1, f_{NL} exhibits a small dip when computed using moment transport, before climbing to a positive peak. The trough has maximum depth $f_{\mathrm{NL}} \sim -0.1$, whereas the height of the peak is $f_{\mathrm{NL}} \sim 0.35$. Only the positive peak is visible by eye when f_{NL} is computed using the numerical δN formula, at a reduced height $f_{\mathrm{NL}} \sim 0.13$, but a small trough can be observed at large magnification. In this region f_{NL} is determined by a cancellation between two much larger components, as discussed below, and its final magnitude is exquisitely sensitive to their relative phase.
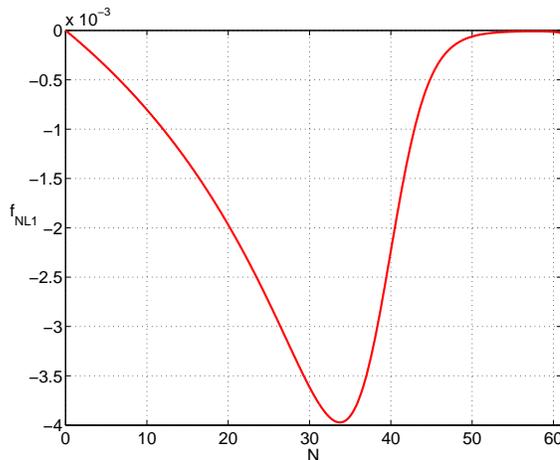
Figure 4. Evolution of f_{NL1}, for the potential $V = V_0\chi^2 e^{\lambda\phi^2}$. Comparison with Fig. 3 shows that f_{NL} is totally dominated by f_{NL2}.

Inspection of Fig. 2 clearly reveals the origin of this trough-to-peak structure. It arises from interference between f_{NL1} and f_{NL2}, which are of the same order of magnitude in this model. The intrinsic skewness, f_{NL1}, becomes very negative, inducing a trough. Later, the non-linearity of the second-order gauge transformation, f_{NL2}, becomes strongly positive, causing a peak. The exact location and depth of the trough depend on the relative amplitude and phase of f_{NL1} and f_{NL2}. We attribute the small difference in the evolution of f_{NL} to the difference between $\langle N \rangle$ and N_{fid}, as discussed in §2. Bearing in mind the approximations which are inherent in any estimate of f_{NL}, we do not believe these discrepancies of magnitude $\Delta|f_{NL}| \sim 0.1$ have any significance.

In Fig. 3, the estimate of f_{NL} from moment transport dips into a trough earlier than would be predicted in δN. Using moment transport, the trough is $\sim 10\%$ deeper. In the examples we have studied, $|f_{NL}|$ at maximum is at least as large as the δN result, and generally larger. A second discrepancy is that—when calculated using the method of moment transport for double quadratic inflation, Eq. (65)—f_{NL} grows slightly at late times. In comparison, using the δN formula, f_{NL} approaches a constant. It is possible this is due to small inaccuraces in our method: for example, its validity requires that growing secular terms, which may cancel between evolution of the field fluctuations and the gauge transformation to ζ, are correctly accounted for. Another possibility is that this difference reflects the actual behavior of κ_m (which our method computes) and $\hat{\kappa}_m$ (which is computed by δN).

Finally, we observe that discrepancies of a similar magnitude exist between the results of δN calculations which employ different approximation schemes. In Figs. 1 and 3 we compare the moment transport method and δN, with and without the slow-roll approximation. The scatter in f_{NL} between δN and the method of moment transport is comparable to the spread induced by employing the slow-roll approximation. This suggests that our new method agrees well with previous techniques, given the

approximations commonly employed in calculations of f_{NL}.

5. Discussion

Non-linearities are now routinely extracted from all-sky observations of the microwave background anisotropy. Our purpose in this paper has been to propose a new technique with which to predict the observable signal. Present data already give interesting constraints on the skewness parameter f_{NL}, and over the next several years we expect that the *Planck* survey satellite will make these constraints very stringent. It is even possible that higher-order moments, such as the kurtosis parameter g_{NL} [62] will become better constrained [63]. To meet the need of the observational community for comparison with theory, reliable estimates of these non-linear quantities will be necessary for various models of early-universe physics.

A survey of the literature suggests that the 'conventional' δN method, originally introduced by Lyth & Rodríguez, remains the method of choice for analytical study of non-Gaussianity. In comparison, our proposed moment transport method exhibits several clear differences. First, the conventional method functions best when we base the δN expansion on a flat hypersurface immediately after horizon exit. In our method, we make the opposite choice and move the flat hypersurface as close as possible to the time of observation. After this, the role of the δN formula is to provide no more than the non-linear gauge transformation between field fluctuations and the curvature perturbation.

Second, we substitute the method of moment transport to evolve the distribution of field fluctuations between horizon exit and observation. We believe this carries extra information which is neglected by the conventional δN formula. In particular, the transport equation is sensitive to an approximate distribution of spatial gradients because it is aware that nearby spacetime volumes may find themselves located at different values of the field.† The terms discussed in §3.2 which correct the background motion of Φ_i, Σ_{ij} and α_{ijk} can be thought of as a response to the distribution of spatial gradients in a typical inflating volume. This information is discarded in the simplest δN formula.

Third, in integrating the transport equation one uses an expansion of the velocity field such as the one given in Eqs. (58)–(59). This expansion is refreshed at each step of integration, so the result is related to conventional perturbative calculations in a very similar way to renormalization-group improved perturbation theory [64]. In this interpretation, derivatives of u_i play the role of couplings. At a given order, m, in the moment hierarchy, the equations for lower-order moments function as renormalization group equations for the couplings at level-m, resumming potentially large terms before they spoil perturbation theory. This property is shared with any formalism such as δN which is non-perturbative in time evolution, but may be an advantage in comparison

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† We emphasize, however, that there is no reason to suppose the distribution function will respond in the same way as it would, had gradient terms been retained in the equation of motion.

with perturbative methods. We also note that although δN is non-perturbative as a point of principle, practical implementations are frequently perturbative. For example, the method of Vernizzi & Wands [19] and Battefeld & Easther [20] depends on the existence of quantities which are conserved only to leading order in ϵN, and can lose accuracy after $N \sim \epsilon^{-1}$ e-foldings.

Numerical calculations confirm that our method gives results in good agreement with existing techniques. Small differences are apparent at a level of $\lesssim 10\%$. We believe that these discrepancies can be attributed to the fact that our method computes the conventional statistical moment, κ_m, rather than its fiducial proxy, $\hat{\kappa}_m$. In the examples we have studied, the centroid of our probability distribution closely tracks the fiducial trajectory. However, it does not always happen that this fiducial trajectory gives a good estimate of $\langle N \rangle$.

As a by-product of our analysis, we note that the large non-gaussianities which have recently been observed in sum- and product-separable potentials [21, 22] are dominated by non-linearities from the second-order part of the gauge transformation from $\delta\varphi_i$ to ζ. The contribution from intrinsic non-linearities of the field fluctuations, measured by the skewnesses α_{ijk}, is negligible. In such cases one can obtain a useful formula for $f_{\rm NL}$ by approximating the field distribution as an exact Gaussian. The non-Gaussianity produced in such cases arises from a distortion of comoving hypersurfaces with respect to adjacent spatially flat hypersurfaces.

Our new method joins many well-established techniques for estimating non-Gaussian properties of the curvature perturbation. In our experience, these techniques give comparable estimates of $f_{\rm NL}$, but they do not exactly agree. Each method invokes different assumptions, such as the neglect of gradients or the degree to which time dependence can be accommodated. The mutual scatter between different methods can be attributed to the theory error inherent in any estimate of $f_{\rm NL}$. The comparison presented in §4 shows that while all of these methods slightly disagree, the moment transport method gives good agreement with other established methods.

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