SIGNED STAR \((j, k)\)-DOMATIC NUMBER OF A GRAPH

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Abstract. Let \(G\) be a simple graph without isolated vertices with edge set \(E(G)\), and let \(j\) and \(k\) be two positive integers. A function \(f: E(G) \to \{-1, 1\}\) is said to be a signed star \(j\)-dominating function on \(G\) if \(\sum_{e \in E(v)} f(e) \geq j\) for every vertex \(v\) of \(G\), where \(E(v) = \{uv \in E(G) \mid u \in N(v)\}\). A set \(\{f_1, f_2, \ldots, f_d\}\) of distinct signed star \(j\)-dominating functions on \(G\) with the property that \(\sum_{i=1}^{d} f_i(e) \leq k\) for each \(e \in E(G)\), is called a signed star \((j, k)\)-dominating family (of functions) on \(G\). The maximum number of functions in a signed star \((j, k)\)-dominating family on \(G\) is the signed star \((j, k)\)-domatic number of \(G\) denoted by \(d_{SS}^{(j, k)}(G)\).

In this paper we study properties of the signed star \((j, k)\)-domatic number of a graph \(G\). In particular, we determine bounds on \(d_{SS}^{(j, k)}(G)\). Some of our results extend those ones given by Atapour, Sheikholeslami, Ghameslou and Volkmann [1] for the signed star domatic number, Sheikholeslami and Volkmann [4] for the signed star \(k\)-domatic number and the signed star \(k\)-domination number.

1. Introduction

Let \(G\) be a graph with vertex set \(V(G)\) and edge set \(E(G)\). We use [2] for terminology and notation which are not defined here and consider simple graphs without isolated vertices only. The integers \(n = |V(G)|\) and \(m = |E(G)|\) are the order and the size of the graph \(G\), respectively. For every vertex \(v \in V(G)\), the open neighborhood \(N(v)\) of \(v\) is the set \(\{u \in V(G) \mid uv \in E(G)\}\), and the

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closed neighborhood of $v$ is the set $N[v] = N(v) \cup \{v\}$. The degree of a vertex $v$ is $d(v) = |N(v)|$. The minimum and maximum degree of a graph $G$ are denoted by $\delta(G)$ and $\Delta(G)$, respectively. The complement $\overline{G}$ of a graph $G$ is the graph with vertex set $V(G)$ such that two vertices are adjacent in $\overline{G}$ if and only if these vertices are not adjacent in $G$.

The open neighborhood $N_G(e)$ of an edge $e \in E(G)$ is the set of all edges adjacent to $e$. Its closed neighborhood is $N_G[e] = N_G(e) \cup \{e\}$. For a function $f : E(G) \to \{-1, 1\}$ and a subset $S$ of $E(G)$, we define $f(S) = \sum_{e \in S} f(e)$. The edge-neighborhood $E_G(v) = E(v)$ of a vertex $v \in V(G)$ is the set of all edges incident with the vertex $v$. For each vertex $v \in V(G)$, we also define $f(v) = \sum_{e \in E_G(v)} f(e)$.

Let $j$ be a positive integer. A function $f : E(G) \to \{-1, 1\}$ is called a signed star $j$-dominating function (SSjDF) on $G$ if $f(v) \geq j$ for every vertex $v$ of $G$. The signed star $j$-domination number of a graph $G$ is $\gamma_{jSS}(G) = \min\{\sum_{e \in E(G)} f(e) \mid f \text{ is a SSjDF on } G\}$. The signed star $j$-dominating function $f$ on $G$ with $f(E(G)) = \gamma_{jSS}(G)$ is called a $\gamma_{jSS}(G)$-function. As the assumption $\delta(G) \geq j$ is clearly necessary, we will always assume that satisfy $\delta(G) \geq j$ while discussing $\gamma_{jSS}(G)$ all graphs involved. The signed star $j$-domination number was introduced by Xu and Li [10] in 2009 and has been studied by several authors (see for instance, [3, 4, 7]). The signed star 1-domination number is the usual signed star domination number, introduced in 2005 by Xu [8]. The signed star domination number was investigated for example, by [3, 6, 9].

Let $k$ be a further positive integer. A set $\{f_1, f_2, \ldots, f_d\}$ of distinct signed star $j$-dominating functions on $G$ with $\sum_{i=1}^{d} f_i(e) \leq k$ for each $e \in E(G)$, is called a signed star $(j,k)$-dominating family (SS$(j,k)$D family) (of functions) on $G$. The maximum number of functions in a signed star $(j,k)$-dominating family on $G$ is the signed star $(j,k)$-domatic number of $G$ denoted by $d_{SS}^{(j,k)}(G)$. The signed star $(j,k)$-domatic number is well-defined and

\[
(1) \quad d_{SS}^{(j,k)}(G) \geq 1
\]
for all graphs $G$ with $\delta(G) \geq j$, since the set consisting of any signed star $j$-dominating function forms a SS(j,k)D family on $G$. A $d_{SS}^{(j,k)}$-family of a graph $G$ is a SS(j,k)D family containing exactly $d_{SS}^{(j,k)}(D)$ signed star $j$-dominating functions. The signed star $(1,1)$-domatic number $d_{SS}^{(1,1)}(G)$ is the usual signed star domatic number $d_{SS}(G)$ which was introduced by Atapour, Sheikholeslami, Ghameslou and Volkmann [1] in 2010.

Our purpose in this paper is to initiate the study of the signed star $(j,k)$-domatic number in graphs. We study basic properties and bounds for the signed star $(j,k)$-domatic number $d_{SS}^{(j,k)}(G)$ of a graph $G$. In addition, we derive Nordhaus-Gaddum type results and bounds of the product and the sum of $\gamma_{jSS}(G)$ and $d_{SS}^{(j,k)}(G)$. Many of our results extend those given by Atapour, Sheikholeslami, Ghameslou and Volkmann [1] for the signed star domatic number, Sheikholeslami and Volkmann [5] for the signed star $(k,k)$-domatic number and Sheikholeslami and Volkmann [4] for the signed star $k$-domatic number.

**Observation 1 ([4]).** Let $G$ be a graph of size $m$ with $\delta(G) \geq j$. Then $\gamma_{jSS}(G) = m$ if and only if each edge $e \in E(G)$ has an endpoint $u$ such that $d(u) = j$ or $d(u) = j + 1$.

2. Properties of the signed star $(j,k)$-domatic number

**Theorem 2.** Let $j, k \geq 1$ be two integers. If $G$ is a graph of minimum degree $\delta(G) \geq j$, then

$$d_{SS}^{(j,k)}(G) \leq \frac{k\delta(G)}{j}.$$ 

Moreover, if $d_{SS}^{(j,k)}(G) = k\delta(G)/j$, then for each function of any signed star $(j,k)$-dominating family $\{f_1, f_2, \ldots, f_d\}$ with $d = d_{SS}^{(j,k)}(G)$ and for all vertices $v$ of degree $\delta(G)$, $\sum_{e \in E_G(v)} f_i(e) = j$ and $\sum_{i=1}^{d} f_i(e) = k$ for every $e \in E_G(v)$. 


Proof. Let \( \{f_1, f_2, \ldots, f_d\} \) be a signed star \((j, k)\)-dominating family on \( G \) such that \( d = d^{(j, k)}_{SS}(G) \). If \( v \in V(G) \) is a vertex of minimum degree \( \delta(G) \), then it follows that

\[
d \cdot j = \sum_{i=1}^{d} j \cdot f_i(e) \leq \sum_{i=1}^{d} \sum_{e \in E_G(v)} f_i(e)
\]

\[
= \sum_{e \in E_G(v)} \sum_{i=1}^{d} f_i(e)
\]

\[
\leq \sum_{e \in E_G(v)} k = k \cdot \delta(G),
\]

and this implies the desired upper bound on the signed star \((j, k)\)-domatic number.

If \( d^{(j, k)}_{SS}(G) = k \delta(G)/j \), then the two inequalities occurring in the proof become equalities, which leads to the two properties given in the statement. \( \square \)

The special cases \( j = k = 1, j = 1 \) and \( j = k \) in Theorem 2 can be found in \([1], [4] \) and \([5] \), respectively. As an application of Theorem 2, we will prove the following Nordhaus-Gaddum type result.

**Corollary 3.** Let \( j, k \geq 1 \) be integers. If \( G \) is a graph of order \( n \) such that \( \delta(G) \geq j \) and \( \delta(G) \geq j \), then

\[
d^{(j, k)}_{SS}(G) + d^{(j, k)}_{SS}(\overline{G}) \leq \frac{k}{j}(n - 1).
\]

If \( d^{(j, k)}_{SS}(G) + d^{(j, k)}_{SS}(\overline{G}) = k(n - 1)/j \), then \( G \) is regular.
Proof. Since $\delta(G) \geq j$ and $\delta(\overline{G}) \geq j$, it follows from Theorem 2 that

$$d_{SS}^{(j,k)}(G) + d_{SS}^{(j,k)}(\overline{G}) \leq \frac{k\delta(G)}{j} + \frac{k\delta(\overline{G})}{j}$$

$$= \frac{k}{j}(\delta(G) + (n - \Delta(G) - 1)) \leq \frac{k}{j}(n - 1),$$

and this is the desired Nordhaus-Gaddum inequality. If $G$ is not regular, then $\Delta(G) - \delta(G) \geq 1$, and the above inequality chain leads to the better bound $d_{SS}^{(j,k)}(G) + d_{SS}^{(j,k)}(\overline{G}) \leq \frac{k}{j}(n - 2)$. This completes the proof.

**Theorem 4.** Let $j, k \geq 1$ be integers. If $v$ is a vertex of a graph $G$ such that $d(v)$ is odd and $j$ is even or $d(v)$ is even and $j$ is odd, then

$$d_{SS}^{(j,k)}(G) \leq \frac{k}{j+1} \cdot d(v).$$

Proof. Let $\{f_1, f_2, \ldots, f_d\}$ be a signed star $(j, k)$-dominating family on $G$ such that $d = d_{SS}^{(j,k)}(G)$. Assume first that $d(v)$ is odd and $j$ is even. The definition yields to $\sum_{e \in E_G(v)} f_i(e) \geq j$ for each $i \in \{1, 2, \ldots, d\}$. On the left-hand side of this inequality a sum of an odd number of odd summands occurs. Therefore it is an odd number, and as $j$ is even, we obtain $\sum_{e \in E_G(v)} f_i(e) \geq j + 1$ for each $i \in \{1, 2, \ldots, d\}$. It follows that

$$k \cdot d(v) = \sum_{e \in E_G(v)} k \geq \sum_{e \in E_G(v)} \sum_{i=1}^{d} f_i(e) = \sum_{i=1}^{d} \sum_{e \in E_G(v)} f_i(e) \geq \sum_{i=1}^{d} (j + 1) = d(j + 1),$$
and this leads to the desired bound. Assume next that \( d(v) \) is even and \( j \) is odd. Note that 
\[
\sum_{e \in E_G(v)} f_i(e) \geq j \quad \text{for each} \quad i \in \{1, 2, \ldots, d\}.
\]
On the left-hand side of this inequality a sum of an even number of odd summands occurs. Therefore it is an even number, and as \( j \) is odd, we obtain 
\[
\sum_{e \in E_G(v)} f_i(e) \geq j + 1 \quad \text{for each} \quad i \in \{1, 2, \ldots, d\}.
\]
Now the desired bound follows as above, and the proof is complete. \( \square \)

The next result is an immediate consequence of Theorem 4.

**Corollary 5.** Let \( j, k \geq 1 \) be integers. If \( G \) is a graph such that \( \delta(G) \) is odd and \( j \) is even or \( \delta(G) \) is even and \( j \) is odd, then

\[
d_{SS}^{(j,k)}(G) \leq \frac{k}{j + 1} \cdot \delta(G).
\]

As an application of Corollary 5, we will improve the Nordhaus-Gaddum bound in Corollary 3 for many cases.

**Theorem 6.** Let \( j, k \geq 1 \) be two integers and let \( G \) be a graph of order \( n \) such that \( \delta(G) \geq j \) and \( \delta(G) \geq j \). If \( \Delta(G) - \delta(G) \geq 1 \) or \( j \) is odd or \( j \) is even and \( \delta(G) \) is odd or \( j \), \( \delta(G) \) and \( n \) are even, then

\[
d_{SS}^{(j,k)}(G) + d_{SS}^{(j,k)}(\overline{G}) < \frac{k}{j}(n - 1).
\]

**Proof.** If \( \Delta(G) - \delta(G) \geq 1 \), then Corollary 3 implies the desired bound. Thus assume now that \( G \) is \( \delta(G) \)-regular.
Case 1. Assume that $j$ is odd. If $\delta(G)$ is even, then from Theorem 2 and Corollary 5 it follows that

$$d_{SS}^{(j,k)}(G) + d_{SS}^{(j,k)}(\overline{G}) \leq \frac{k}{j+1}\delta(G) + \frac{k}{j}\delta(\overline{G})$$

$$< \frac{k}{j}(\delta(G) + (n - \delta(G) - 1))$$

$$= \frac{k}{j}(n - 1).$$

If $\delta(G)$ is odd, then $n$ is even and thus $\delta(\overline{G}) = n - \delta(G) - 1$ is even. Combining Theorem 2 and Corollary 5, we find that

$$d_{SS}^{(j,k)}(G) + d_{SS}^{(j,k)}(\overline{G}) \leq \frac{k}{j}\delta(G) + \frac{k}{j+1}\delta(\overline{G})$$

$$< \frac{k}{j}(\delta(G) + (n - \delta(G) - 1))$$

$$= \frac{k}{j}(n - 1),$$

and this completes the proof of Case 1.

Case 2. Assume that $j$ is even. If $\delta(G)$ is odd, then from Theorem 2 and Corollary 5 it follows that

$$d_{SS}^{(j,k)}(G) + d_{SS}^{(j,k)}(\overline{G}) \leq \frac{k}{j+1}\delta(G) + \frac{k}{j}(n - \delta(G) - 1) < \frac{k}{j}(n - 1).$$

If $\delta(G)$ is even and $n$ is even, then $\delta(\overline{G}) = n - \delta(G) - 1$ is odd, and we obtain the desired bound as above. □
Theorem 7. Let $j, k \geq 1$ be integers. If $G$ is a graph such that $k$ is odd and $d_{SS}^{(j,k)}(G)$ is even or $k$ is even and $d_{SS}^{(j,k)}(G)$ is odd, then

$$d_{SS}^{(j,k)}(G) \leq \frac{k - 1}{j} \cdot \delta(G).$$

Proof. Let $\{f_1, f_2, \ldots, f_d\}$ be a signed star $(j, k)$-dominating family on $G$ such that $d = d_{SS}^{(j,k)}(G)$. Assume first that $k$ is odd and $d$ is even. If $e \in E(G)$ is an arbitrary edge, then $\sum_{i=1}^{d} f_i(e) \leq k$. On the left-hand side of this inequality a sum of an even number of odd summands occurs. Therefore, it is an even number, and as $k$ is odd, we obtain $\sum_{i=1}^{d} f_i(e) \leq k - 1$ for each $e \in E(G)$. If $v$ is a vertex of minimum degree, then it follows that

$$d \cdot j = \sum_{i=1}^{d} j \leq \sum_{i=1}^{d} \sum_{e \in E_G(v)} f_i(e)$$

$$= \sum_{e \in E_G(v)} \sum_{i=1}^{d} f_i(e) \leq \sum_{e \in E_G(v)} (k - 1) = \delta(G)(k - 1),$$

and this yields to the desired bound. Assume second that $k$ is even and $d$ is odd. If $e \in E(G)$ is an arbitrary edge, then $\sum_{i=1}^{d} f_i(e) \leq k$. On the left-hand side of this inequality a sum of an odd number of odd summands occurs. Therefore, it is an odd number and as $k$ is even, we obtain $\sum_{i=1}^{d} f_i(e) \leq k - 1$ for each $e \in E(G)$. Now the desired bound follows as above, and the proof is complete. $\square$

The special cases $j = k = 1$, $j = 1$ and $j = k$ of Theorem 4, Corollary 5 and Theorem 7 can be found in [1], [4] and [5], respectively. According to (1), $d_{SS}^{(j,k)}(G)$ is a positive integer. If we
suppose in the case $j = k = 1$ that $d_{SS}(G) = d_{SS}^{(1,1)}(G)$ is an even integer, then Theorem 7 leads to the contradiction $d_{SS}(G) \leq 0$. Consequently, we obtain the next known result.

**Corollary 8 ([1]).** The signed star domatic number $d_{SS}(G)$ is an odd integer.

**Proposition 9.** Let $j, k$ be two integers such that $j \geq 1$ and $k \geq 2$, and let $G$ be a graph with minimum degree $\delta(G) \geq j$. Then $d_{SS}^{(j,k)}(G) = 1$ if and only if each edge $e \in E(G)$ has an endpoint $u$ such that $d(u) = j$ or $d(u) = j + 1$.

**Proof.** Assume that each edge $e \in E(G)$ has an endpoint $u$ such that $d(u) = j$ or $d(u) = j + 1$. It follows from Observation 1 that $\gamma_{jSS}(G) = m$ and thus $d_{SS}^{(j,k)}(G) = 1$.

Conversely, assume that $d_{SS}^{(j,k)}(G) = 1$. If $G$ contains an edge $e = uv$ such that $d(u) \geq j + 2$ and $d(v) \geq j + 2$, then the functions $f_1: E(G) \rightarrow \{-1, 1\}$ such that $f_1(x) = 1$ for each $x \in E(G)$ and $f_2(e) = -1$ and $f_2(x) = 1$ for each edge $x \in E(G) \setminus \{e\}$ are signed star $j$-dominating functions on $G$ such that $f_1(x) + f_2(x) \leq 2 \leq k$ for each edge $x \in E(G)$. Thus $\{f_1, f_2\}$ is a signed star $(j,k)$-dominating family on $G$, a contradiction to $d_{SS}^{(j,k)}(G) = 1$. 

The next result is an immediate consequence of Observation 1 and Proposition 9.

**Corollary 10.** Let $j, k$ be two integers such that $j \geq 1$ and $k \geq 2$, and let $G$ be a graph with minimum degree $\delta(G) \geq j$. Then $d_{SS}^{(j,k)}(G) = 1$ if and only if $\gamma_{jSS}(G) = m$.

Next we present a lower bound on the signed star $(j,k)$-domatic number.

**Proposition 11.** Let $j, k$ be two integers such that $k \geq j \geq 1$, and let $G$ be a graph with minimum degree $\delta(G) \geq j$. If $G$ contains a vertex $v \in V(G)$ such that all vertices of $N[N[v]]$ have degree at least $j + 2$, then $d_{SS}^{(j,k)}(G) \geq j$. 


Proof. Let \( \{u_1, u_2, \ldots, u_j\} \subset N(v) \). The hypothesis that all vertices of \( N[N[v]] \) have degree at least \( j + 2 \) implies that the functions \( f_i : E(G) \to \{-1, 1\} \) such that \( f_i(vu_i) = -1 \) and \( f_i(x) = 1 \) for each edge \( x \in E(G) \setminus \{vu_i\} \) are signed star \( j \)-dominating functions on \( G \) for \( i \in \{1, 2, \ldots, j\} \). Since \( f_1(x) + f_2(x) + \ldots + f_j(x) \leq j \leq k \) for each edge \( x \in E(G) \), we observe that \( \{f_1, f_2, \ldots, f_j\} \) is a signed star \((j,k)\)-dominating family on \( G \), and Proposition 11 is proved. \( \square \)

Corollary 12. Let \( j, k \) be two integers such that \( k \geq j \geq 1 \). If \( G \) is a graph of minimum degree \( \delta(G) \geq j + 2 \), then \( d_{SS}^{(j,k)}(G) \geq j \).

Corollary 13. Let \( j, k \geq 1 \) be integers, and let \( G \) be an \( r \)-regular graph with \( r \geq j \).

1. If \( j \leq r \leq j+1 \), then \( d_{SS}^{(j,k)}(G) = 1 \).
2. If \( r = j + 2p + 1 \) with an integer \( p \geq 1 \) and \( k \geq j \), then \( j \leq d_{SS}^{(j,k)}(G) \leq \frac{kr}{j+1} \).
3. If \( r = j + 2p \) with an integer \( p \geq 1 \) and \( k \geq j \), then \( j \leq d_{SS}^{(j,k)}(G) \leq \frac{kr}{j} \).

Proof. (1) Assume that \( j \leq r \leq j+1 \). According to Observation 1, \( \gamma_{jSS}(G) = m \) and thus \( d_{SS}^{(j,k)}(G) = 1 \).

(2) Assume that \( r = j + 2p + 1 \) with \( p \geq 1 \). The condition \( k \geq j \) and Corollary 12 imply that \( j \leq d_{SS}^{(j,k)}(G) \). If \( j \) is even, then \( r = j + 2p + 1 \) is odd, and if \( j \) is odd, then \( r = j + 2p + 1 \) is even. Therefore, Corollary 5 leads to the desired upper bound of \( d_{SS}^{(j,k)}(G) \).

(3) Assume that \( r = j + 2p \) with \( p \geq 1 \). The condition \( k \geq j \) and Corollary 12 imply that \( j \leq d_{SS}^{(j,k)}(G) \). In addition, Theorem 2 yields the desired upper bound of \( d_{SS}^{(j,k)}(G) \). \( \square \)
3. Bounds on the product and the sum of $\gamma_{jSS}(G)$ and $d_{SS}^{(j,k)}(G)$

Note that $\gamma_{jSS}(G) = m$ implies immediately $d_{SS}^{(j,k)}(G) = 1$, and so $\gamma_{jSS}(G) \cdot d_{SS}^{(j,k)}(G) = m$ and $\gamma_{jSS}(G) + d_{SS}^{(j,k)}(G) = m + 1$. In this section, we present general bounds of the product and the sum of $\gamma_{jSS}(G)$ and $d_{SS}^{(j,k)}(G)$.

**Theorem 14.** Let $j, k \geq 1$ be integers. If $G$ is a graph of size $m$ and minimum degree $\delta(G) \geq j$, then

$$\gamma_{jSS}(G) \cdot d_{SS}^{(j,k)}(G) \leq mk.$$ 

Moreover, if $\gamma_{jSS}(G) \cdot d_{SS}^{(j,k)}(G) = mk$, then for each $d_{SS}^{(j,k)}$-family $\{f_1, f_2, \ldots, f_d\}$ of $G$, each function $f_i$ is a $\gamma_{jSS}(G)$-function and $\sum_{i=1}^d f_i(e) = k$ for all $e \in E(G)$.

**Proof.** If $\{f_1, f_2, \ldots, f_d\}$ is a signed star $(j,k)$-dominating family on $G$ such that $d = d_{SS}^{(j,k)}(G)$, then the definitions imply

$$d \cdot \gamma_{jSS}(G) = \sum_{i=1}^d \gamma_{jSS}(G) \leq \sum_{i=1}^d \sum_{e \in E(G)} f_i(e)$$

$$= \sum_{e \in E(G)} \sum_{i=1}^d f_i(e) \leq \sum_{e \in E(G)} k = mk$$

as desired.

If $\gamma_{jSS}(G) \cdot d_{SS}^{(j,k)}(G) = mk$, then the two inequalities occurring in the proof become equalities. Hence for the $d_{SS}^{(j,k)}$-family $\{f_1, f_2, \ldots, f_d\}$ of $G$ and for each $i$, $\sum_{e \in E(G)} f_i(e) = \gamma_{jSS}(G)$, thus each function $f_i$ is a $\gamma_{jSS}(G)$-function and $\sum_{i=1}^d f_i(e) = k$ for all $e \in E(G)$. \qed
Theorem 15. Let $j, k \geq 1$ be integers. If $G$ is a graph of size $m$ and minimum degree $\delta(G) \geq j$, then

$$d_{SS}^{(j,k)}(G) + \gamma_{jSS}(G) \leq mk + 1.$$ 

Proof. According to Theorem 14, we have

$$d_{SS}^{(j,k)}(G) + \gamma_{jSS}(G) \leq d_{SS}^{(j,k)}(G) + \frac{km}{d_{SS}^{(j,k)}(G)}.$$ 

Using the fact that the function $g(x) = x + (km)/x$ is decreasing for $1 \leq x \leq \sqrt{km}$ and increasing for $\sqrt{km} \leq x \leq km$, we obtain

$$d_{SS}^{(j,k)}(G) + \gamma_{jSS}(G) \leq \max \left\{ 1 + mk, mk + \frac{km}{km} \right\} = mk + 1.$$

□

Next we improve Theorem 15 considerably.

Theorem 16. Let $j, k \geq 1$ be two integers. If $G$ is a graph of size $m$ and minimum degree $\delta(G) \geq j$, then

$$\gamma_{jSS}(G) + d_{SS}^{(j,k)}(G) \leq \begin{cases} m + 1 & \text{if } k = 1, \\ \frac{mk}{2} + 2 & \text{if } k \geq 2. \end{cases}$$

Proof. If $k = 1$, then Theorem 15 leads to the desired bound. Therefore we assume next that $k \geq 2$. If the order $n = 2$, then $\gamma_{jSS}(G) = m = 1$ and $d_{SS}^{(j,k)}(G) = 1$ and hence the desired bound is valid. Now we assume that $n \geq 3$. Let $f$ be a SSjDF on $G$. Since $\sum_{e \in E_G(v)} f(e) \geq j$ for every
vertex \( v \) of \( G \), it follows that

\[
2 \sum_{e \in E(G)} f(e) = \sum_{v \in V(G)} \sum_{e \in E_G(v)} f(e) \geq \sum_{v \in V(G)} j = nj.
\]

This implies \( \gamma_{jSS}(G) \geq nj/2 \). As \( n \geq 3 \) and \( j \geq 1 \), we obtain \( \gamma_{jSS}(G) \geq 2 \). Theorem 14 implies that

\[
\gamma_{jSS}(G) + d_{SS}^{(j,k)}(G) \leq \gamma_{jSS}(G) + \frac{mk}{\gamma_{jSS}(G)}.
\]

If we define \( x = \gamma_{jSS}(G) \) and \( g(x) = x + (mk)/x \) for \( x > 0 \), then because \( 2 \leq \gamma_{jSS}(G) \leq m \), we have to determine the maximum of the function \( g \) in the interval \( I : 2 \leq x \leq m \). Using the condition \( k \geq 2 \) and the fact that \( m \geq 2 \), it is easy to see that

\[
\max_{x \in I} \{g(x)\} = \max\{g(2), g(m)\}
\]

\[
= \max \left\{ 2 + \frac{mk}{2}, m + \frac{mk}{m} \right\}
\]

\[
= \frac{mk}{2} + 2,
\]

and the proof is complete. \( \square \)

**Theorem 17.** Let \( j, k \geq 1 \) be two integers. If \( G \) is a graph of size \( m \), minimum degree \( \delta(G) \geq j \) and order \( n \geq 2p + 1 \) for an integer \( p \geq 1 \), then

\[
\gamma_{jSS}(G) + d_{SS}^{(j,k)}(G) \leq \begin{cases} 
  m + k & \text{if } 1 \leq k \leq p, \\
  \frac{mk}{p+1} + p + 1 & \text{if } k \geq p + 1.
\end{cases}
\]
Proof. We proceed by induction on $p$. Theorem 16 shows that the statement is valid for $p = 1$. Now let $p \geq 2$ and assume that the statement is true for all integers $1 \leq i \leq p - 1$. Then the induction hypothesis implies that $\gamma_{jSS}(G) + d^{(j,k)}_{SS}(G) \leq m + k$ for $1 \leq k \leq p - 1$. Thus assume next that $k \geq p$. The hypothesis $n \geq 2p + 1$ leads as in the proof of Theorem 16 to

$$\gamma_{jSS}(G) \geq \frac{n j}{2} \geq \frac{(2p + 1) j}{2} \geq \frac{2p + 1}{2}$$

and thus $p + 1 \leq \gamma_{jSS}(G) \leq m$. Therefore, it follows from Theorem 14 that

$$\gamma_{jSS}(G) + d^{(j,k)}_{SS}(G) \leq \gamma_{jSS}(G) + \frac{mk}{\gamma_{jSS}(G)} \leq \max\left\{p + 1 + \frac{mk}{p + 1}, m + k\right\}.$$ (2)

Note that the hypothesis $n \geq 2p + 1$ yields to $m \geq p + 1$.

If $k = p$, then we deduce from the inequality $m \geq p + 1$ that

$$\max\left\{p + 1 + \frac{mk}{p + 1}, m + k\right\} = \max\left\{p + 1 + \frac{mp}{p + 1}, m + p\right\} = m + p.$$ If $k \geq p + 1$, then

$$p + 1 + \frac{mk}{p + 1} \geq m + k$$

is equivalent with $m(k - p - 1) \geq (p + 1)(k - p - 1)$, and this inequality is valid since $k \geq p + 1$ and $m \geq p + 1$. Hence the desired result follows from (2), and the proof is complete. \(\square\)


5. ______, *Signed star (k,k)-domatic number of a graph*, submitted.


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