Comparing finite and biased infinite path length shooting random walk estimators for radiosity

Mateu Sbert*, Alex Brusi†

Abstract. In this paper we compare the best shooting random walk estimator with expected finite path length and the estimator resulting of biasing the infinite one. Heuristic formulae for the Mean Square Error of both estimators are given, and based on them a formula for the relative efficiency of both estimators is presented. The results are contrasted with different tests. The formulae for the MSE are also useful to know a priori the number of paths (or particles) needed to obtain a given error.

Keywords: Rendering, Radiosity, Monte Carlo, Random Walk.

1 Introduction

Shooting random walk radiosity proceeds by sending rays (or particles, or paths) from the sources which travel through the scene according to the Form Factors transition probabilities, either discrete patch-to-patch or continuous point-to-point Form Factors. On each bounce, the exit point can be the same impinging point, as in non-discrete methods [2], [4], [12] and Particle Tracing [5], or can be take at random on the patch [7]. The methods in [10], [3] can be considered as a breadth-first approach to random walk, which would be a depth-first approach. Also, shooting random walk estimators for radiosity can be classified according to the expected path length, finite or infinite. The purpose of this paper is to compare, for discrete methods such the ones in [7],[5], on the basis of the MSE, the best finite path length estimator to the *biased* infinite one, and also to give the number of paths (or particles) necessary to obtain a given error. This will be done, after a previous work section 2, in section 3. In section 4 we will present our results, and in section 5 alternative ways of biasing will be discussed. Finally in section 6 we present our conclusions.

2 Previous work

In [7] the variances for different shooting random walk estimators were given, and the best one (which we named Φ_T) was found. It was the one which updated each patch in its path, this obvious advantage overcoming the positive covariances. The way the estimator Φ_T proceeds is the following: Each path begins at a source and carries with it a given quantity of power. This power is the same through all the path. Each visited patch updates its received power by this quantity. On each visited patch a die-or-survive test is done according to the reflectivity of the patch. In [8] we presented the variances for the infinite path length estimators. The way the infinite estimator proceeds is the following: Each path begins at a source and carries with it a given quantity of power. This power, after updating a visited patch, is decreased by the reflectivity of the patch. The path never ends. Both finite and infinite path length estimators variances are given in table 1 (we suppose that the probability for a path to exit from a source is proportional to its power). We showed in [9] that the same formulae were valid for the Particle tracing estimators. In [7] we proved that the variance for the infinite path length estimator was inferior to the best finite path length one. However, it does not mean that it should be better, because of the cost. First, we must *bias* the infinite estimator, and there are many ways of doing that, leading to different *biased* estimators. Second, apart from knowing that it is lower, we don't know how much lower is the variance of the infinite estimator. This is because we do not know the exact

*Institut d'Informàtica i Aplicacions, Universitat de Girona, e-mail mateu@ima.udg.es
†Escola Politècnica Superior, Universitat de Girona

relation between b_i and β_i. This situation apparently drives us to work only on the experimental side to compare both estimators. However, it happens that heuristic formulae for the Mean Square Error (MSE) (not just bounds as were given in [7]) can be derived from the formulae in table 1, and thus a theoretical study can be undertaken. This error derivation will be done in the next section.

Estimator	**Variance**
Φ_T	$\frac{R_i b_i \Phi_T}{A_i}(1+2R_i\xi_i) - b_i^2$
infinite	$\frac{\beta_i \Phi_T}{A_i}(1+2R_i\xi_i) - b_i^2$

Table 1: Variances of Random Walk estimators. The meaning of the different quantities is in table 2.

E_i	Emissivity
b_i	Reflected radiosity $= B_i - E_i$
β_i	idem with each reflectivity substituted by its square
Φ_T	Total power in scene
A_i	Area
R_i	Reflectivity
ξ_i	Received power (or radiosity) due to self-emitted unit power (or emittance)

Table 2: Meaning of the different quantities appearing in table 1. The suffix i means for patch i .

3 Comparison of the estimators

We will derive here heuristic formulae for the MSE of the Φ_T and infinite estimators. Based on these formulae, we will obtain the theoretically expected relative efficiencies.

3.1 Heuristic formulae for the MSE

We will derive first an heuristic formula for the Φ_T estimator. From the formula for the MSE and the variance in table 1 we have :

$$E(MSE) = \frac{1}{A_T}\sum_i A_i Var(B_i) = \frac{1}{A_T}\sum_i A_i \left(\frac{b_i R_i \Phi_T}{A_i}(1+2R_i\xi_i) - b_i^2\right) \approx \frac{\Phi_T}{A_T}\sum_i b_i R_i \quad (1)$$

neglecting the b_i^2 and ξ_i terms. Now an average value for b_i can be estimated as $\frac{R_i\Phi_T}{A_T(1-R_{ave})}$, where R_{ave} is the average reflectivity in the scene. Substituting also R_i by R_{ave}, we obtain

$$E(MSE) \approx \frac{\Phi_T^2}{A_T A_{ave}}\frac{R_{ave}^2}{(1-R_{ave})} \quad (2)$$

As for the infinite estimator, we proceed in the same way as before. Instead of b_i we have now β_i, the radiosity in an environment with each reflectivity substituted by its square. Thus an average value for β_i will be $\frac{R_{ave}\Phi_T}{A_T(1-R_{ave}^2)}$, and we obtain:

$$E(MSE) \approx \frac{\Phi_T^2}{A_T A_{ave}}\frac{R_{ave}^2}{(1-R_{ave}^2)} \quad (3)$$

Formulae 1 and 2 refer to one path. As for N paths the MSE gets divided by N (as each variance does) they thus give us also a prediction of the number of paths (or particles) to trace to obtain a given accuracy. For instance, with the Φ_T estimator, to obtain an accuracy of $\mathcal{E}$ we must cast N paths, where N is given by:

$$N \approx \frac{A_T A_{ave}(1-R_{ave})}{\mathcal{E}\Phi_T^2 R_{ave}^2}$$

3.2 Relative efficiencies

Let us now stablish a formula for the relative efficiency. The relative efficiency of two Monte Carlo estimators is the quotient of the products of variance and time cost [6]. We will consider here a somewhat modified relative efficiency. Instead of variance (which is referred to the estimator of the radiosity of a single patch) we will use the expected values of the Mean Square Error found in the previous section. This has two advantatges. First, we can not compare directly variances as we do not know the exact relation between b_i and β_i. But we do know the exact relation between both MSE's. Second, using the MSE we obtain a single relative efficiency for the scene. Using variances, the relative efficiency might depend on the patch considered.
We must obtain first the cost for one path in the Φ_T estimator. The average cost is the expected number of segments the path will have, which can be approximated by $\frac{1}{(1-R_{ave})}$ [11], times the cost of one line. For the infinite estimator we must first select a way of biasing it. We choose to bias it when the accumulated reflectivity (product of reflectivities along a path) is below some threshold. This is the same to say that the relation of the power carried by a particle respective to the initial power has reached a lower bound (the threshold).

The average length with the *biased* estimator can be computed in the following way: The value of the length is minimum 1, as the path ever exits from the source. Now, it will experiment n bounces on the average according to $R_{ave}^n \leq t$. This would do an average length of $1 + \frac{\log t}{\log R_{ave}}$. However, this will result sometimes in a value superior to the experimental one. Take for instance the case were R_{ave}^n is slightly inferior to t. With our formula, the theoretical value and the experimental value will have a difference of about 1.
In the formula for the variance we must introduce the effect of biasing. However, as far as the bias is small, we can still consider formula in table 2 as a good approximation (see section 4). The relative efficiency, re, can then be expressed with the following formula

$$re = \frac{\frac{\Phi_T^2}{A_T A_{ave}} \frac{R_{ave}^2}{(1-R_{ave}^2)} (1 + \frac{\log t}{\log R_{ave}})}{\frac{\Phi_T^2}{A_T A_{ave}} \frac{R_{ave}^2}{(1-R_{ave})} \frac{1}{(1-R_{ave})}} = \frac{(1-R_{ave})(1 + \frac{\log t}{\log R_{ave}})}{1 + R_{ave}} \tag{4}$$

4 Results

In figure 1 we compare the expected errors against experimental ones for the Φ_T estimator and the scene in figure 1, a cube with 27 cubes inside and a whole face as emitter. The average reflectivity is 0.2 (figure 1b) 0.5 (figure 1c) and 0.8 (figure 1d). An average of 5 executions was taken for each case.
In figure 2 we compare the expected errors against experimental ones for the biased infinite estimator, with two bias, 0.01 and 0.001, and the same scene as before. The average reflectivity is 0.2 (figure 2a) 0.5 (figure 2b) and 0.8 (figure 2c). An average of 5 executions was taken for each case. The greater bias in the 0.01 case accounts for the deviation of the 0.01 curves with respect to the expected values.
In figure 3 we compare the errors of both methods against the scene of figure 4. The biased method corresponds to $t = 0.001$. We also present another kind of error, the % of area of the scene surface such that its radiosity has a relative error of at least 10%. This error measure is proposed in [1]. We call that error %A. The images obtained with the biased method, are presented in figure 4(a), and those obtained with the Φ_T method in figures 4(b). Each light source is meshed in 225 patches. The number of patches is 66.572, with each wall meshed into 5000 square patches, and ceiling and floor in 10000 each. The average reflectivity of the scene is $R_{ave} = 0.5$, and the expected length of a path is in accordance with this reflectivity. In the biased case, the discrepancy of about 1 can be explained by the stated in the previous section. The errors in figure 3 are found respective to a reference solution obtained with the Φ_T estimator. The relative efficiency, about 3, is in accordance with the expected value (see figure 2d). In table 3 we present the statistics of rays cast in the images of figure 4(a) (biased), and 4(b) (Φ_T), as well as expected and experimental errors, which are in great concordance. The experimental efficiency is also in accordance with the expected one (which depends only on t and R_{ave}, see formula 4 and figure 2(d)). In figure 5 we present results corresponding to a scene with 2200 patches (each wall meshed in 200 square patches, and ceiling and floor in 400 each), and we present the results corresponding to four different number of lines for both estimators. The statistic of rays is given in table 4. The average value for the reflectivities was of $R_{ave} = 0.58$. One thing to be remarked is a small discrepancy between the expected length of a path, which according to the formula $\frac{1}{(1-R_{ave})}$ should be 2.3, and the experimentally obtained one, 2.1. This can be justified by the fact that most

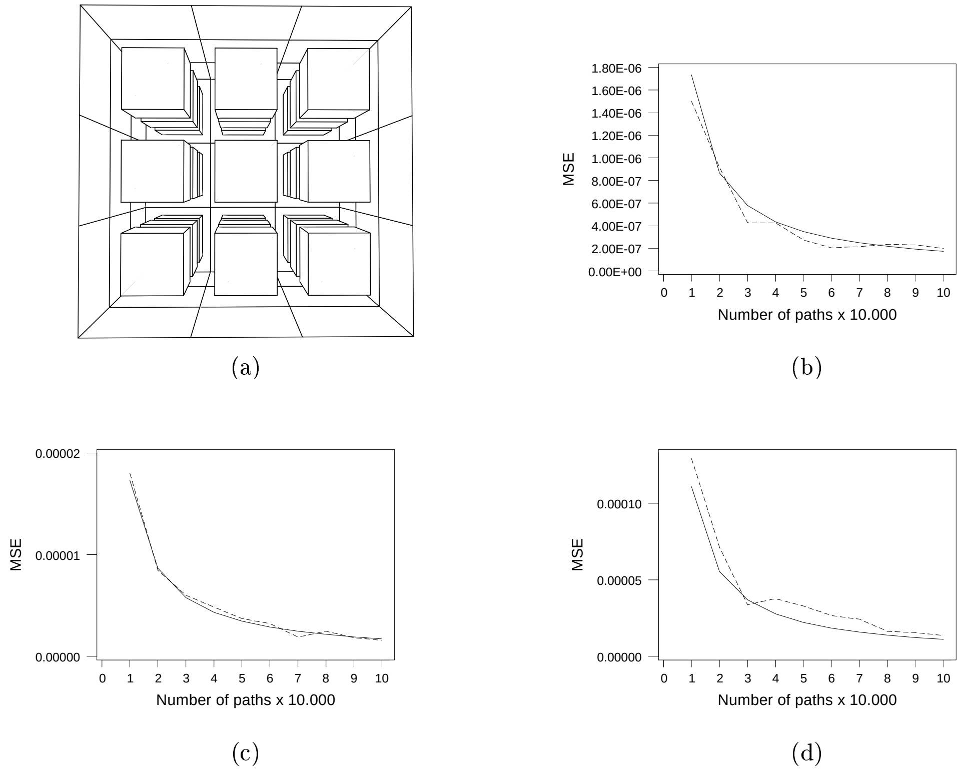


Figure 1: Expected (continuous curves) versus experimental (dotted curves) MSE error for the Φ_T estimator and reflectivities 0.2 (b), 0.5 (c) and 0.8 (d). The scene is in (a), and a whole face of the enclosing cube was taken as emitter. The experimental values are the average of five tests for each reflectivity.

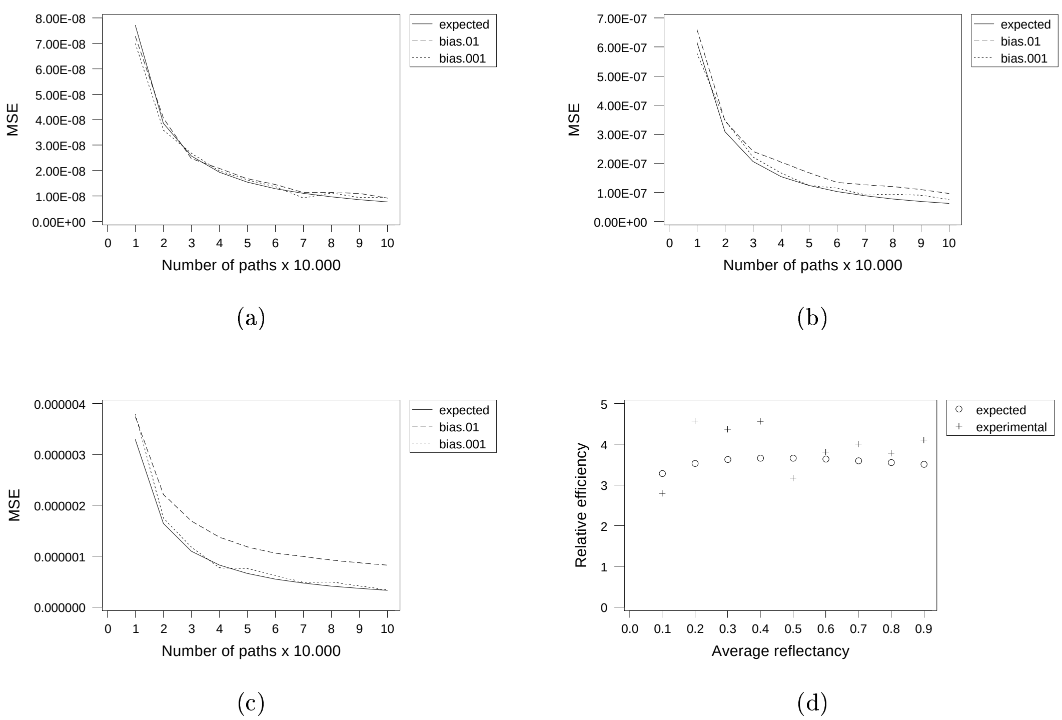


Figure 2: Expected versus experimental MSE error when biasing the infinite estimator to $t = 0.01$ and to $t = 0.001$, for reflectivities 0.2 (a), 0.5 (b) and 0.8 (c). The scene is the one from figure 1(a). The experimental values are the average of five tests for each reflectivity. In (d) we have the relative efficiency.

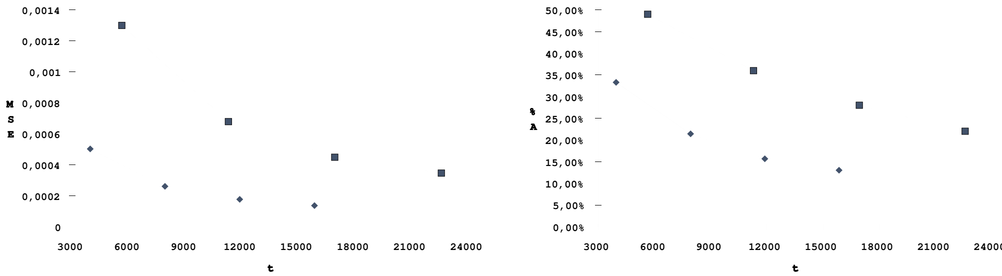
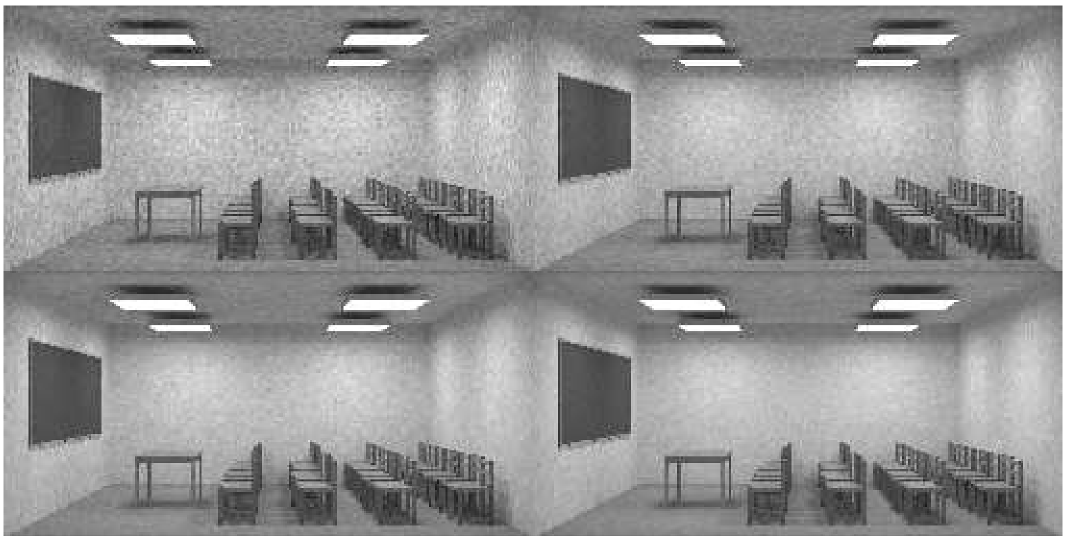
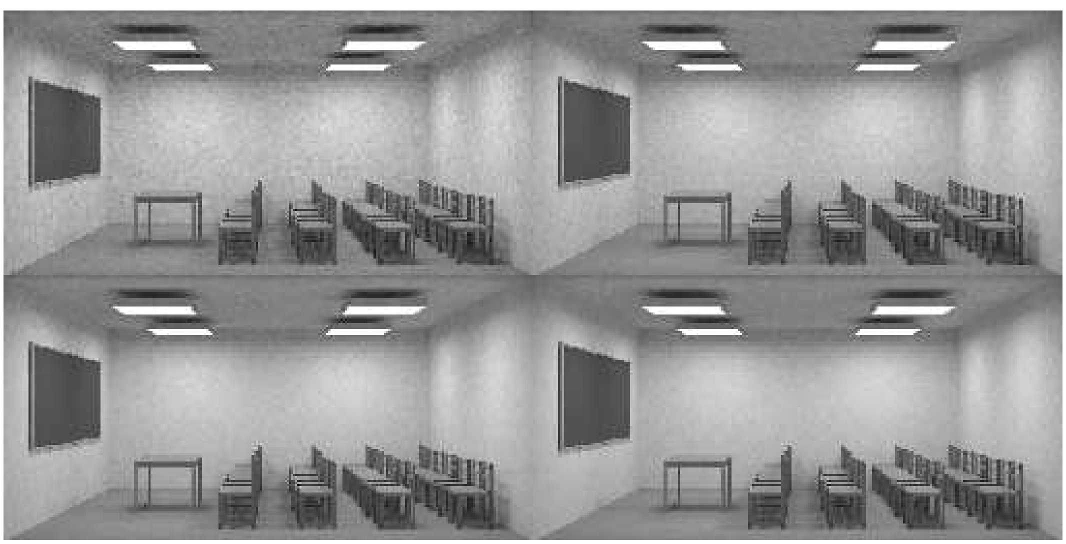


Figure 3: MSE and %A errors for the images in figures 4(a) (biased, dotted line) and 4(b) (Φ_T estimator, continuous line). Times are given in seconds.



(a)



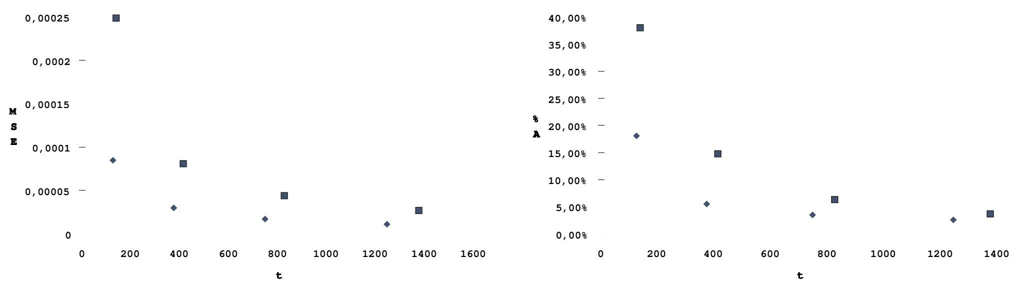
(b)

Figure 4: Images obtained with the infinite biased with $t = 0.001$ (a) and Φ_T estimator (b), with increasing number of lines (see table 3). The fourth image is obtained with four times the lines of the first one. No Gouraud shading is done.

Table 3: Number of rays cast in figures 4(a) *biased* with $t = 0.001$ infinite estimator and (b) Φ_T estimator. (1) An average of 9.7 bounces per ray. (2) An average of 2.0 bounces per ray.

biased (figure4(a))	paths traced(1)	total rays	expected error	experimental error
top left	900.000	8.771.091	0,012	0,0013
top right	1.800.000	17.542.182	0,00061	0,00068
bottom left	2.700.000	26.313.273	0,00041	0,00045
bottom right	3.600.000	35.084.364	0,00030	0,00034
Φ_T (figure4(b))	paths traced(2)	total rays	expected error	experimental error
top left	3.600.000	7.216.927	0,00045	0,00050
top right	7.200.000	14.437.655	0,00023	0,00026
bottom left	10.800.000	21.664.960	0,00015	0,00018
bottom right	14.400.000	28.880.485	0,00011	0,00013

of the power is first directed to the floor, with very low reflectivity (0.3). The same thing happens with the biased estimator, which also shows an important deviation (about 40%) of the expected error against the experimental one. This has been the most important deviation we have found in our tests. The wall patches of this scene are much bigger than in the previous scene, and we have only one small area light source. The relative efficiency of both methods is again in accordance with the expected value (see figure 2d).



(a) (b)

Figure 5: MSE and A% errors for the images in figure 6(a) (biased estimator, dotted line) and 6(b) (Φ_T estimator, continuous line). Times are given in seconds.

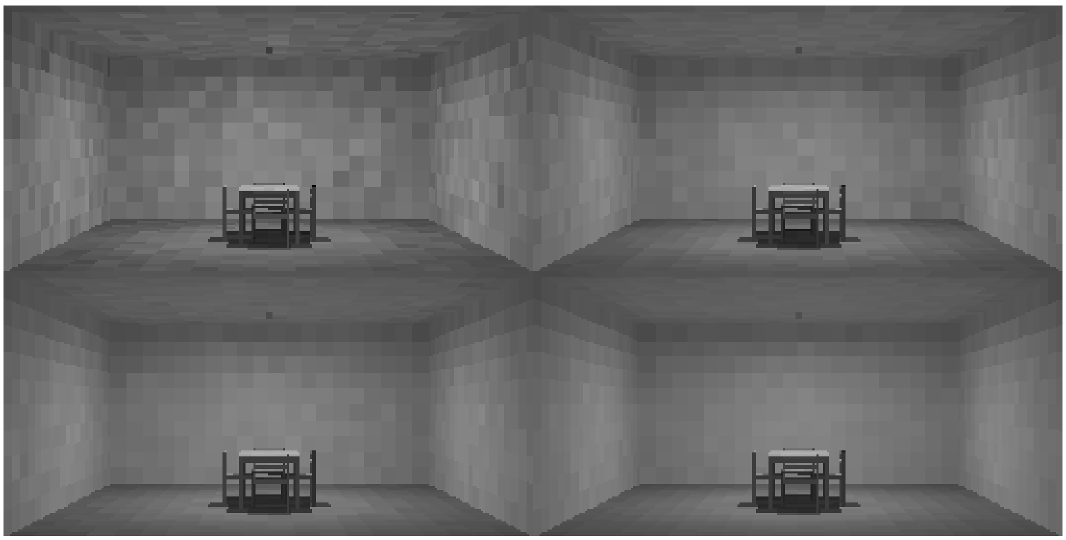
Table 4: Number of rays cast in figures 6(a) biased estimator and (b) Φ_T estimator. (1) An average of 11.9 rays per path. (2) An average of 2.1 rays per path.

biased (figure6(a))	paths traced(1)	total rays	expected error	experimental error
top left	40.000	476.902	0,000366	0,00249
top right	120.000	1.430.706	0,00012	0,000081
bottom left	240.000	2.861.412	0,000061	0,000044
bottom right	400.000	4.769.020	0,000036	0,000027
Local-Φ_T (figure6(b))	paths traced(2)	total rays	expected error	experimental error
top left	200.000	425.406	0,00011	0,000085
top right	600.000	1.274.494	0,000038	0,00003
bottom left	1.200.000	2.554.032	0,000019	0,000017
bottom right	2.000.000	4.251.203	0,000012	0,000011

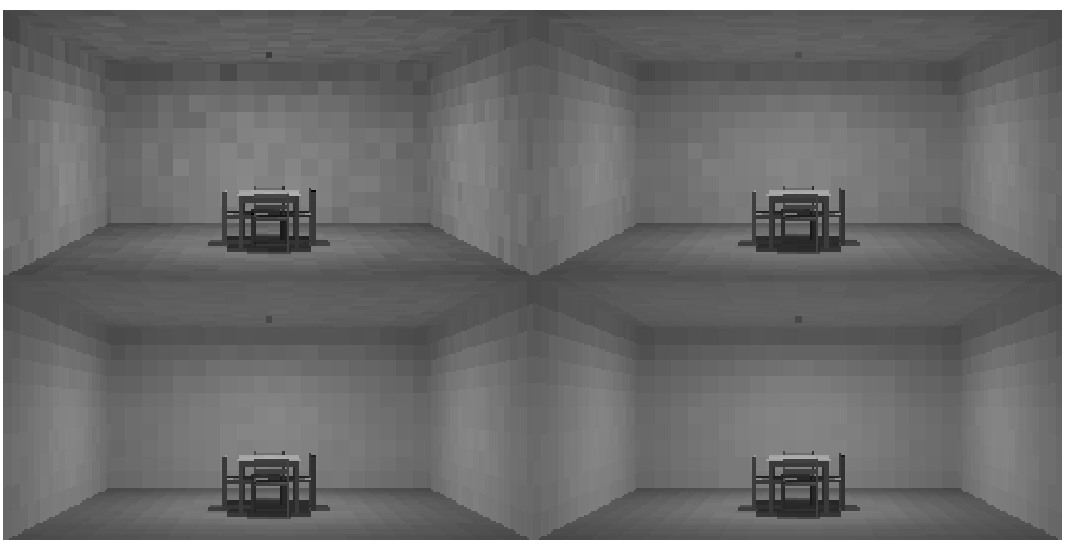
5 Other alternatives for biasing

An alternative approach to bias upon the accumulated reflectivity along the path (or the energy the particle carries) would be to bias upon the path length. That is, to cut off the path at a given predetermined length. However, this biasing is clearly an inferior strategy to the previous. A path along a high reflectivity region will leave a lot of energy to distribute, and conversely, a path along a

low reflectivity region would result in overwork in this region.
Also, we only considered in this paper the case for a small bias (such as $t = 0.01$ and $t = 0.001$). The error done with these biased estimators is good approximated with the error formula for the infinite one, and to practical effects, it can be considered unbiased. We have not considered lower bias for the following reasons. First, a lower bias will cause a lot of energy to be undistributed (although it can be somewhat masked by considering it as an ambient term). Second, a lower bias will not give a much better strategy than the Φ_T method. For instance, supposing a bias of $t = 0.1$, the average length would be 4.3 for the biased estimator, against 2 for the Φ_T one. This proportion between expected lengths keeps more o less constant for different values of R_{ave} (of course between 0 and 1).
Another alternative would be to use Russian Roulette in combination with biasing. However, Russian Roulette is nothing more than the Φ_T estimator. That is, we would use the whole Φ_T estimator to expand only the remaining energy! As the average length is the same independently of the energy to be distributed, thus strategy is clearly inferior to any of the ones considered above.



(a)



(b)

Figure 6: Images obtained with the biased estimator (a) and Φ_T estimator (b), with increasing number of lines (see table 4). The fourth image is obtained with ten times the lines of the first one. No Gouraud shading is done. The main differences with figure 4 are the much larger wall patches and the much smaller area of the source.

6 Conclusions

We have given in this paper heuristic formulae for the MSE for the infinite and Φ_T shooting random walk estimators. In this way we can know a priori the number of paths necessary to obtain a given accuracy. Also, based on these formulae, we obtained the relative efficiency of both estimators (when biasing the infinite one), and showed that the Φ_T estimator was the best of both. We contrasted our formulae against experimental results, and found an almost complete concordance, except in one case, where a 40% in excess of error was predicted. Thus our formulae have given, in the worst case, an accuracy of at least the same order of magnitude, and in the best case they give a very good prediction. We discussed also and discarded other alternatives of biasing the infinite estimator. As the methods in [10], [3] can be considered a breadth-first approach to the Φ_T estimator, which would be a depth-first approach, it is very likely that the results in this paper also apply to them.

7 Acknowledgements

This project has been funded in part with grant number TIC 95-0614-C03-03 of the Spanish Government.

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