Semi-global Leaderless Consensus with Input Saturation Constraints via Adaptive Protocols

Zhiyang Wu\textsuperscript{1}, Jian Li\textsuperscript{1}, Fang Song\textsuperscript{1,2}

\textsuperscript{1}(Laboratory of Intelligent Control and Robotics, Shanghai University of Engineering Science, Shanghai, 201620, P. R. China)
\textsuperscript{2}(State Key Laboratory of Robotics and System, Harbin Institute of Technology, Harbin, 150001, P. R. China)

\textbf{ABSTRACT:} In this paper, the problems of semi-global leaderless consensus for continuous-time multi-agent systems under input saturation restraints are investigated. We consider the leaderless consensus problems under fixed undirected communication topology. New necessary synthesis conditions are established for achieving semi-global leaderless consensus via the distributed adaptive protocols, which is designed by utilizing low gain feedback tactics and the relative state measurements of the neighboring agents. Finally, simulation results illustrate the theoretic developments.

\textbf{Keywords} - Semi-global consensus; Multi-agent systems; Input saturation; Adaptive protocols

\section{INTRODUCTION}

During the last few decades, consensus has been become a very hot research topic in many scientific fields, such as computer science, physics, biology, and system control. Roughly speaking, consensus means the agreement of agents on the joint status value via utilizing local interaction from the neighboring agents. In recent years, extensive research attention has been paid into the consensus problem of multi-agent systems\cite{1-5}. It should be noted that, although the power of each individual agent is quite limited, the whole interaction network system may accomplish complex tasks in an coordinated fashion. Therefore, the consensus problem has a widenage of potential applications, to mention just a few, including cooperation of unmanned air vehicles (UAVs) in \cite{6}, distributed sensor networks in \cite{7} and \cite{8}, attitude alignment for a flock of satellites in \cite{9}, flocking control in \cite{10} and \cite{11}, congestion control under communication topology \cite{12}. Interested readers are referred to the recent survey articles \cite{14}, \cite{15} and the references cited therein for details. It is known that consensus has been studied for several decades in computer science. Broadly speaking, there are two classes of consensus, that is, the leaderless consensus and the leader-following one. Usually, the multi-agent system dynamic models are described by double-integrator dynamics or single-integrator kinematics \cite{13}-\cite{17}. In recent literature, the general linear dynamic model has been considered to address more complicated system dynamics in engineering practice. For instance, in \cite{18} the authors studied the consensus problems of both leaderless and leader-following cases in the situation of fixed interaction networks. In \cite{19} the consensus problem in the case of leader-following were discussed under a switching communication network. The leaderless nonlinear consensus subject to parametric uncertainties was achieved by developing adaptive continuous finite-time distributed protocols in \cite{20}. The consensus of multi agents with second-order, linear and Lipschitz nonlinear dynamics via utilizing the adaptive algorithms were studied in \cite{21} and \cite{22}, respectively. A distributed observer based algorithm was proposed in \cite{23} for leader-following consensus of networked multiple agents systems.

Indeed, up to now, the investigation on consensus has many comprehensive achievements. However, in practice networked multi-agent systems may have more complicated dynamics. The industrial applications of consensus are limited by all kinds of physical conditions, such as the input saturation constraints. Till now, more and more research activities has paid close attention to the consensus problems for networked multiple agents systems under input saturation constraints because of its unique characteristic and practical applications. In \cite{24}, the authors investigated the global consensus of linear networked multiple agents dynamics with input saturation by distributed observer-based algorithm over fixed and switching interaction networks. For discrete-time systems, the global consensus problem under input saturation constraints was addressed in \cite{25}. In \cite{26} Suet al. proposed a linear low gain feedback algorithm to reach semi-global consensus for networked multi agents dynamics under the assumption of agents are linear asymptotically null controllable with bounded controls (ANCBC) \cite{27}. While in recent paper \cite{28}, the semi-global consensus of nonlinear dynamics in the case of input saturation was studied by utilizing observer-based algorithms and the low gain feedback technique, under the assumptions that the communication networks were jointly connected or connected and each agent was detectable. With the input saturation restricts, the semi-global synchronization problem for multiple agents with linear systems was studied in \cite{29}.

In the present paper, we revisit the problem of leaderless consensus of linear multi agents dynamics with input saturation constraints. New semi-global consensus conditions are established by using distributed adaptive
protocols. Compared with some existing results, the distinctive points of the paper can be expressed as the following sentences. Under the assumption that the agents are ANCBC, without realizing any information of the networks, all agents achieve agreement by utilizing the distributed adaptive protocols and using the low gain feedback technique over both fixed and switching interaction networks. Also, we provide simulation results on two numerical examples to verify the theoretical developments.

The remainder of this paper is arranged as follows. Preliminaries and notations are introduced in Section 2. In Section 3, based on the distributed adaptive protocols, we address the problem of semi-global leaderless consensus under fixed undirected graph. The simulation results on two numerical examples are given in Section 4 to verify the theoretical developments. Finally, a brief conclusion is drawn in Section 5.

II. PROBLEM BACKGROUND AND PRELIMINARIES

In this paper, we suppose that the information exchange between agents and the leader is described by a fixed undirected graph \( G = (V, E, A) \), with the set of vertices \( V = \{1, 2, \ldots, N\} \) whose elements represent agents of the group, the set of undirected edges \( E \subseteq V \times V \) represent neighboring relations among agents, and a weighted adjacency matrix \( A = [a_{ij}] \in \mathbb{R}^{N \times N} \) is defined as \( a_{ij} > 0 \) if and only if \( (j, i) \in E \); otherwise, \( a_{ij} = 0 \). Here, we also assume that \( a_{ii} = a_0 \) for all \( i = 1, 2, \ldots, n \).

For an undirected graph \( G \), a Laplacian matrix \( L = [l_{ij}] \in \mathbb{R}^{N \times N} \) of graph \( G \) is defined as \( L = D - A \), with the degree matrix \( D = \text{diag} \{d_1, \ldots, d_N\} \) defined as \( i \) th diagonal elements \( \sum_{j=1, j \neq i}^{N} a_{ij} \). Define \( B_0 = \text{diag} \{b_{01}(t), b_{02}(t), \ldots, b_{0N}(t)\} \) as \( b_{0i} > 0 \) if the leader is available by agent \( i \) and \( b_{0i} = 0 \) otherwise. The eigenvalues of \( L \) can be ordered as \( \eta_1 \leq \eta_2 \leq \cdots \leq \eta_N \). Then, \( \eta_1 = 0 \) with its corresponding right eigenvector \( 1 = [1, 1, \ldots, 1]^T \in \mathbb{R}^N \). Moreover, \( \eta_2 > 0 \) if \( G \) is a connected graph.

Assume that \( M \) is a matrix, the notation \( M^T \) denotes its transpose matrix. A symmetric positive matrix \( M \) indicates that all its eigenvalues are positive, a positive semi-definite \( M \) indicates that all its eigenvalues are non-negative. The symbol \( \otimes \) is the Kronecker product. \( I_n \) denotes the identity matrix of dimension \( N \times N \).

Lemma 1 [4]: Denote \( L \) as the Laplacian of a fixed undirected graph \( G \) include \( N \) agents, denote \( \tilde{L} \) as the Laplacian of the graph consisting of \( N \) agents and one leader, \( b_{0i} > 0 \) if there is a spanning tree with the leader as the root vertex.

Lemma 2 [26]: For any \( \psi \in \mathbb{R}^{m \times m} \) and \( \zeta, \ell \in \mathbb{R}^m, i = 1, 2, \ldots, N \)
\[
\frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij}(t)(\zeta_i - \zeta_j)^T (\psi(\varepsilon)(\ell_i - \ell_j)) = \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij}(t)\zeta_i^T (\psi(\varepsilon)(\ell_i - \ell_j))
\]

In this paper, we consider an MAS of \( N \) identical continuous-time agents subject to input saturation constraints, moving in an \( n \)-dimensional Euclidean space. The dynamic of each agent is modeled by
\[
\dot{x}_i = Ax_i + Bu_i, \quad y_i = Cx_i, \quad i = 1, 2, \ldots, N
\]  
(2)
where \( x_i \in \mathbb{R}^p, y_i \in \mathbb{R}^p \) are the state of agent \( i \) and the measurement output of agent \( i \) respectively, \( u_i \in \mathbb{R}^m \) is the control input, \( \sigma_\Delta : \mathbb{R}^m \rightarrow \mathbb{R}^m \) is a saturation function defined as \( \sigma_\Delta(u_i) = \text{sgn}(u_{i,j}) \min \{u_{ij}, \Delta\} \) with a scalar value \( \Delta > 0 \). For \( u_i = (u_{i,1}, u_{i,2}, \ldots, u_{i,m})^T \), \( \sigma_\Delta(u_i) = (\sigma_\Delta(u_{i,1}), \sigma_\Delta(u_{i,2}), \ldots, \sigma_\Delta(u_{i,m}))^T \).

To address the leaderless consensus, the average state of \( N \) agents is defined as follows
\[
\dot{x}_i = \frac{1}{N} \sum_{i=1}^{N} x_i, \quad i = 1, 2, \ldots, N
\]  
(3)

www.ijres.org 37 | Page
Definition 1. To address the semi-global leaderless consensus problem, it means that, for any prior given compact set \( \chi \in \mathbb{R}^n \), we can design a distributed control law \( u_i \) by only using the partial information of agent \( i \) from neighbors, such that all the agents can reach the same state asymptotically, i.e.,

\[
\lim_{t \to \infty} \left\| x_i(t) - x_j(t) \right\| = 0, \quad i, j = 1, 2, \ldots, N
\]

with any initial condition \( x_i(0) \in \chi, i = 0, 1, 2, \ldots, N \)

In order to employ the low gain feedback technique, we need to suppose that the pair \((A, B)\) satisfies the following Assumption 1. For more details, the reader can refer to the book [27].

Assumption 1: The pair \((A, B)\) is ANCBC and each eigenvalue of the matrix \( A \) is on the imaginary axis simple.

Lemma 3[29]: Under Assumption 2. A unique solution \( P(\varepsilon) \) is derived by the following parametric ARE.

\[
A^T P(\varepsilon) + P(\varepsilon) A - \gamma P(\varepsilon) B B^T P(\varepsilon) = -\varepsilon P(\varepsilon)
\]  

(4)

where \( \gamma > 0 \) is a positive scalar. Furthermore, we have \( \lim_{\varepsilon \to 0} P(\varepsilon) = 0 \).

Lemma 4[29]: Let Assumption 1 hold, for any \( \gamma > 0 \) and \( \varepsilon \in (0, 1] \), the unique solution in Lemma 3 is \( P(\varepsilon) = W^{-1}(\varepsilon) \) which satisfies a Lyapunov function

\[
W(A + \frac{\varepsilon}{2} I_n)^T + (A + \frac{\varepsilon}{2} I_n)W = \gamma BB^T
\]  

(5)

Assumption 2: The graph \( G \) is connected at all time.

III. MAIN RESULTS

This section investigates the semi-global leaderless consensus with input saturation under a fixed interaction network topology. The adaptive control algorithm will be employed to address the problem. Now, motivated by the distributed consensus protocol with an adaptive control law in [22], we are ready to design a distributed consensus algorithm as follows:

\[
u_i = K \sum_{j=1}^{N} a_{ij} m_j (x_i - x_j) \quad \dot{m}_j = \beta_j a_{ij} (x_i - x_j)^T \hat{K} (x_i - x_j)
\]  

(6)

where \( \beta_{ij} = \beta_{ji} \) are the plus quantities, \( m_j \) is the adaptive coupling weights, the feedback matrix \( K \) and \( \hat{K} \) are designed as \( K = -B^T P(\varepsilon) \) and \( \hat{K} = -P(\varepsilon) B B^T P(\varepsilon) \) respectively, \( P(\varepsilon) \) is the unique solution that can be obtained from the algebra Riccati equation (4).

By introducing the above contents, now we propose the results as follows.

Theorem 1. Suppose that Assumptions 1 and 2 are satisfied. Then the multi-agent system (2) of \( N \) agents achieve semi-global leaderless consensus under the distributed control law (6).

Proof: Let us define \( \delta_i = [\delta_1^T, \delta_2^T, \ldots, \delta_N^T]^T \) with \( \delta_i = x_i - \bar{x} \). Then, we have the following equations

\[
\dot{\delta}_i = \dot{x}_i - \dot{x} = A \delta_i + B \sigma(\omega_i) - \frac{1}{N} \sum_{i=1}^{N} B \sigma(\omega_i)
\]

\[
\dot{m}_j = \beta_j a_{ij} (\delta_i - \delta_j)^T \hat{K} (\delta_i - \delta_j)
\]  

(7)

Now, for the multi-agent system (2), let us consider a common Lyapunov function candidate

\[
V = \sum_{i=1}^{N} \delta_i^T P(\varepsilon) \delta_i + \sum_{i=1}^{N} \sum_{j=1, j \neq i}^{N} \frac{(m_{ij} - \omega_i)^2}{2\beta_{ij}}
\]  

(8)

The matrix \( P(\varepsilon) \) is the unique solution from the parametric algebra Riccati equation (6), \( \omega \) is a positive parameter to be designed. By way of convenient notation, define \( c \) be a positive number such that...
\[ c \geq \sup_{\varepsilon \in (0,1], \chi(0) \in \chi, i=1,2,...,N} \left\{ \delta^T P(\varepsilon) \delta + \sum_{i=1}^{N} \beta_i^2 \left( \frac{m_{ij} - \omega}{\beta_{ij}} \right)^2 \right\} \]  

(9)

Such a \( c \) exists since \( \chi \) is bounded and \( \lim_{\varepsilon \to 0} P(\varepsilon) = 0 \). Let

\[ \zeta_\varepsilon := \{ \delta \in \mathbb{R}^N : V(\delta) \leq c \} \]  

(10)

and \( \varepsilon^* \in (0,1] \) be such that, for each \( \varepsilon \in (0, \varepsilon^*], \) \( e \in \zeta_\varepsilon \) implies that

\[ \left\| B^T P(\varepsilon) \sum_{j=1}^{N} m_{ij} a_{ij} (x_i - x_j) \right\|_{\infty} \leq \Delta, \quad i = 1,2,...,N \]  

(11)

Therefore, by Lemma 3, consider any \( \varepsilon \in (0, \varepsilon^*] \), for any \( \varepsilon \in \zeta_\varepsilon \), the derivative of \( V \) can be evaluated as

\[ \dot{V} = \sum_{i=1}^{N} \delta_i^T P(\varepsilon) \dot{\delta}_i + \sum_{i=1}^{N} \delta_i^T P(\varepsilon) \dot{\delta}_i + \sum_{i=1}^{N} \sum_{j=1}^{N} \left( \frac{m_{ij} - \omega}{\beta_{ij}} \right) m_{ij} \]

\[ = \sum_{i=1}^{N} \delta_i^T (P(\varepsilon) A + A^T P(\varepsilon)) \delta_i - 2 \sum_{i=1}^{N} \delta_i^T P(\varepsilon) BB^T P(\varepsilon) \sum_{j=1}^{N} m_{ij} a_{ij} (\delta_i - \delta_j) \]

\[ + \sum_{i=1}^{N} \sum_{j=1}^{N} \left( \frac{m_{ij} - \omega}{\beta_{ij}} \right) \beta_j a_{ij} (\delta_i - \delta_j)^T P(\varepsilon) BB^T P(\varepsilon) (\delta_i - \delta_j) \]  

(12)

By Lemma 2, for any \( \zeta_\varepsilon \in \mathbb{R}^m, \ell_i \in \mathbb{R}^m, i = 1,2,...,N \), we have

\[ \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij} (\xi_i - \xi_j)^T (\ell_i - \ell_j) = \xi^T (L(t) \otimes I_m) \ell \]

where \( \xi = [\xi_1, \xi_2, \ldots, \xi_N]^T \) and \( \ell = [\ell_1, \ell_2, \ldots, \ell_N]^T \). Therefore, we can continue (12) as follows:

\[ \dot{V} = \sum_{i=1}^{N} \delta_i^T P(\varepsilon) \dot{\delta}_i + \sum_{i=1}^{N} \delta_i^T P(\varepsilon) \dot{\delta}_i - 2 \sum_{i=1}^{N} \sum_{j=1}^{N} \omega a_{ij} \delta_i^T P(\varepsilon) BB^T P(\varepsilon) (\delta_i - \delta_j) \]

\[ = \delta^T (I_N \otimes (P(\varepsilon) A + A^T P(\varepsilon) - 2\omega L \otimes P(\varepsilon) BB^T P(\varepsilon))) \delta \]

Since the graph is connected, zero is a simple eigenvalue of \( L \) and all the other eigenvalues are positive. Let \( U \in \mathbb{R}^{N \times N} \) be such a unitary matrix that \( U^T LU = \Lambda = \text{diag} (0, \eta_2, \ldots, \eta_N) \). Because the right and left eigenvectors of \( L \) corresponding to the zero eigenvalue are \( 1 \) and \( 1^T \), respectively, we can choose

\[ U = \begin{bmatrix} \frac{1}{\sqrt{N}} & \cdots & \frac{1}{\sqrt{N}} \end{bmatrix} \quad \text{and} \quad U^T = \begin{bmatrix} 1^T \\ \vdots \\ Y_N \end{bmatrix}, \quad \text{with} \quad Y_i \in \mathbb{R}^{N(N-1)} \quad \text{and} \quad Y_2 \in \mathbb{R}^{(N-1) \times N} \].

Let

\[ \xi = \begin{bmatrix} \xi_1^T \\ \cdots \\ \xi_N^T \end{bmatrix} = (U^T \otimes I_n) \delta \].

Then we have

\[ \dot{V} = \delta^T (I_N \otimes (P(\varepsilon) A + A^T P(\varepsilon) - 2\omega L \otimes P(\varepsilon) BB^T P(\varepsilon))) \delta \]

\[ \leq \sum_{i=1}^{N} \xi_i^T \left( P(\varepsilon) A + A^T P(\varepsilon) - 2\omega \eta_i P(\varepsilon) BB^T P(\varepsilon) \right) \xi_i \]  

(13)

By Lemma 3 choosing \( \gamma \leq 2\omega \eta_i, i = 2, \ldots, N \), (13) can be continued as:

\[ \dot{V} \leq \sum_{i=2}^{N} \xi_i^T \left( P(\varepsilon) A + A^T P(\varepsilon) - 2\gamma P(\varepsilon) BB^T P(\varepsilon) \right) \xi_i = -\sum_{i=1}^{N} \delta_i^T P(\varepsilon) \delta_i < 0 \]

Thus, semi-global consensus is achieved, i.e., such that for a set of original condition \( x_i(0) \in \chi \)
lim_{t \to \infty} x_i(t) - x_j(t) = 0, \quad i, j = 1, 2, ..., N

This completes the proof.

IV. SIMULAITON STUDY

Herein the simulation experiments are supplied to clarify the theoretic results on the leaderless consensus developed in Section 3. Consider a multi agents system consisting of eight agents. Assume that the dynamic of every agent holds the form of (2) with the following parameters:

\[
A = \begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}
\]

Obviously the pair \((A, B)\) satisfies Assumption 1. Suppose the topology is considered as in Fig.1. The initial states of eight agents are randomly assigned. Thus, We chose \(\beta_{ij} = 1\) and \(\omega = 5\), \(\eta_2 = 0.2907\) can be obtained by using an exhaustive method. Then \(\gamma \leq 2\omega\eta_2 = 2.907\). For the situation of \(\varepsilon = 0.2\), then we get a set of solutions as

\[
P(0.2) = \begin{bmatrix} 0.3429 & -0.3086 \\ -0.3086 & 0.5556 \end{bmatrix}
\]

Thus, the feedback gain \(K\) and \(\hat{K}\) as

\[
K = \begin{bmatrix} 0.3086 & -0.5556 \end{bmatrix}, \quad \hat{K} = \begin{bmatrix} 0.0952 & -0.1715 \\ -0.1715 & 0.3087 \end{bmatrix}
\]

Fig. 1: The interaction communication network.

Fig. 2: The states of eight agents. Fig. 3: The adaptive weights \(m_{ij}\).

Under the control law (6), the trajectories of the state difference between agents are shown in Fig. 2. It is obvious that the semi-global leaderless consensus can be achieved.

For the situation of \(\varepsilon = 0.2\), we can get the unique solution
Semi-global Leaderless Consensus with Input Saturation Constraints via Adaptive Protocols

\[
P(0.4) = \begin{bmatrix} 0.4883 & -0.3906 \\ -0.3906 & 0.6250 \end{bmatrix}
\]

Thus, the feedback gain \( K \) and \( \hat{K} \) as

\[
K = \begin{bmatrix} 0.3906 \\ -0.6250 \end{bmatrix}, \quad \hat{K} = \begin{bmatrix} 0.1526 & -0.2441 \\ -0.2441 & 0.3906 \end{bmatrix}
\]

Under the control law (6), it is obvious that the semi-global leaderless consensus can be achieved. The simulation results are in Fig. 4. The adaptive weights \( m_{ij} \) under \( \varepsilon = 0.2 \) and \( \varepsilon = 0.4 \) are shown in Fig. 3 and Fig. 5 respectively.

V. CONCLUSION

In this paper, we consider the semi-global leaderless consensus problem of ANCBC multiple-agents system subject to input saturation restraints both over fixed and switching topology. The adaptive control law is proposed by utilizing a low gain feedback strategy. The connectivity of the graph in fixed networks and the joint connectivity of the graph in switching networks are the key requirements to guarantee the semi-global leaderless consensus. Finally, the theoretical results were verified by simulation study.

VI. Acknowledgements

This work was partly supported by the State Key Laboratory of Robotics and System (HIT) under Grant SKLRS-2014-MS-10, the Jiangsu Provincial Key Laboratory of Advanced Robotics Fund Projects under Grant JAR201401, and the Foundation of Shanghai University of Engineering Science under Grant nhky-2015-06, 14KY0130.

REFERENCES

Semi-global Leaderless Consensus with Input Saturation Constraints via Adaptive Protocols