Measuring Inequality with Interval Data

by

Steve Beckman, W. James Smith and Buhong Zheng
Measuring Inequality with Interval Data

Steve Beckman, W. James Smith and Buhong Zheng

Department of Economics
University of Colorado Denver
Denver, CO, USA

Revised
November 2008

Abstract: This paper employs the axiomatic approach underpinning the literature on income inequality measurement to analyze measures of dispersion in interval data. We find that some widely employed measures fail to properly measure dispersion when data are not of the ratio type. We go on to prove that, under reasonable conditions, variance is the only decomposable measure that can be used to consistently measure inequality of interval data. Moreover, the only proper Lorenz dominance condition for interval data is absolute Lorenz dominance that Moyes (1987) introduced.

Key Words: unit-consistency, unit-invariance, decomposable measures, variance, inequality orderings, absolute Lorenz dominance.
Measuring Inequality with Interval Data

I. Introduction

Evaluating and understanding the variability of natural phenomena and social activities has played a critical role in human development. By studying variabilities, we deduce the laws governing the various activities/phenomena and derive their implications for nature and for society. An important issue arising naturally in the process is the quantification of variability which is an essential step in answering the question whether the variability is increased or decreased. In the natural sciences, the variance and the coefficient of variation are commonly used to measure variability while in social sciences the issue of measurement becomes much more complex.

The most comprehensively developed measurement of variability is in the area of income distributions where “variability” is referred to as “income inequality.” The measurement of income inequality concerns how unequally incomes are distributed among the recipients of a social group. Although the issue of measuring income inequality is age-old and various axioms and measures had been introduced, the literature took a giant step forward only after Atkinson’s classical contribution which laid the welfare-economics foundation for income inequality measurement. Built upon Rothschild and Stiglitz’s (1970) seminal work measuring risk-aversion, Atkinson (1970) established the important connection between Lorenz dominance and the Pigou-Dalton principle of transfers which both had been in the literature for more than a half century. With almost 40 years of further research, we now have a much better understanding of the various issues in measuring income inequality. Inequality measurement criteria such as the Lorenz curve and Atkinson inequality indices are now standard tools for research and economic policy analyses.

These results apply to ratio data, ones with a natural zero or origin. The purpose of this paper is to examine the extent to which an axiomatic approach can be applied to another important class of data, namely, interval data, for which no natural origin exists. In social sciences as well as in natural sciences, interval data are often encountered. Temperature, for example, is measured under different systems (e.g., Fahrenheit and Celsius). Intelligence quotient (IQ) scores (a person’s measured rank on the normal distribution with an average value of 100 and a standard deviation of 15), and standardized test scores (SAT and ACT) serve as examples of interval data. Even for ratio data, it is common to adjust the origin by subtracting the mean from individual datum. This of course changes the origin in a manner similar to interval data.

Does the approach so useful in income inequality measurement lend itself to the analysis, for example, of temperature variation and IQ inequality? Or is such an approach even necessary? To illustrate the necessity of the proposed analysis, take the very commonly used coefficient of variation, CV, and consider a very simple example involving two sets of temperatures with two different measuring units. Specifically, in Fahrenheit $x_F = (40, 50, 60, 70, 200)$ and $y_F = (10, 20, 30, 40, 50)$. For the coefficient
of variation $CV$, we have $CV(x_F) > CV(y_F)$. But when we convert the temperature from Fahrenheit into Celsius and obtain $x_C$ and $y_C$, respectively, we find $CV(x_C) < CV(y_C)$. This certainly should give pause when variability is to be measured in a consistent fashion with respect to different measuring units.

In this paper, we propose a unit-consistency requirement for inequality measurement with interval data. Using the aforementioned axiomatic approach from the literature of income inequality measurement, we prove that variance is the sole measure that satisfies both unit-consistency and decomposability. We also show that absolute Lorenz dominance (Moyes, 1987) is the only Lorenz dominance condition that satisfies unit-consistency. The importance of these results lies in the fact that consistent characterization of variability is critical if phenomenon and policies are to be evaluated with confidence.

II. Inequality Measurement: a Very Brief Review

Let $x \in \mathbb{R}$ be an interval-data variable and $\mathbf{x} = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$ be a distribution of $x$ at $n$ points. To facilitate proofs of the main results, we require $n \geq 3$. We also denote the dimension of any distribution $\mathbf{x}$ as $n(\mathbf{x})$ and the mean income of $\mathbf{x}$ as $\mu(\mathbf{x})$. The notation $\hat{\mathbf{x}}$ denotes the “equalized version” of $\mathbf{x}$, i.e., $n(\hat{\mathbf{x}}) = n(\mathbf{x})$ and $\hat{x}_i = \mu(\mathbf{x})$ for all $i$. Because the aim of this paper is to characterize inequality measures that are applicable to interval data, we first briefly present the basic elements of income inequality measurement.$^1$

Denote the set of all distributions under consideration $\mathcal{D} = \cup_{n \geq 3} \mathbb{R}^n$, an inequality measure is defined as a function $I : \mathcal{D} \rightarrow \mathbb{R}_+$ which, for each distribution $\mathbf{x}$, indicates a level of inequality in the distribution and $I(\mathbf{x})$ possesses the following properties:

- **Symmetry**: $I(\mathbf{x})$ is a symmetric function of $\mathbf{x}$.
- **Normalization**: $I(\hat{\mathbf{x}}) = 0$.
- **Differentiability**: $I(\mathbf{x})$ has continuous first-order and second-order partial derivatives $I_i(\mathbf{x})$ and $I_{ij}(\mathbf{x})$.
- **Replication Invariance**: $I(\mathbf{y}) = I(\mathbf{x})$ if $\mathbf{y}$ is obtained from $\mathbf{x}$ by replication.
- **Strict Schur-Concavity**: $I(\mathbf{Bx}) < I(\mathbf{x})$ for all bistochastic matrices $\mathbf{B}$ that are not permutation matrices.

**Symmetry** is a standard assumption in measurement studies. **Normalization**, together with **strict Schur-concavity**, requires that inequality attain its minimum value of zero when all individuals have the same value. **Differentiability** requires $I(\mathbf{x})$ to be differentiable in each argument; it also implies that $I(\mathbf{x})$ is a continuous function of $\mathbf{x}$. **Replication invariance** states that inequality remains unchanged if the population expands itself through copying the original distribution; **Strict Schur-concavity** is the Pigou-Dalton principle of transfers.$^1$

---

$^1$To conserve space, the review here is kept extremely brief. A comprehensive review can be found in, for example, Lambert (2001).

3
An important class of inequality measures developed in the literature is that satisfying the decomposability condition elaborated in Bourguignon (1979), Cowell (1980) and Shorrocks (1980). A decomposable inequality measure enables inequality in a population to be partitioned according to some identifiable characteristics such as geographic areas.

- **Decomposability**: If an income distribution \( x \) is partitioned into \( m \) non-empty subgroups, \( x = (x^1, x^2, ..., x^m) \), decomposability requires the following relationship between the total inequality value \( I(x) \) and the subgroup inequality values \( I(x^j) \):

\[
I(x) = I(x^1, x^2, ..., x^m) = \sum_{j=1}^{m} w^j(\mu, n)I(x^j) + I(\mu_11_1, \mu_21_2, ..., \mu_m1_m)
\]  

where \( w^j \) is the weight attached to subgroup \( j \) and \( \mu = (\mu_1, ..., \mu_m) \) and \( n = (n_1, ..., n_m) \) with \( \mu_j = \mu(x^j) \) and \( n_j = n(x^j) \). The notation \( 1_j \) represents the unit vector \((1, 1, ..., 1)\) with \( n_j \) components.

Condition (2.1) places significant restrictions on the functional form that an inequality measure can take. In an important contribution, Shorrocks (1980) shows that if \( I(x) \) satisfies symmetry, normalization, differentiability, replication-invariance and decomposability then there exist functions \( \lambda(\cdot) \) and \( \phi(\cdot) \) such that

\[
I(x) = \frac{1}{n(x)\lambda(\mu(x))} \sum_{i=1}^{n(x)} (\phi(x_i) - \phi(\mu(x)))
\]  

where \( \lambda(\cdot) \) is positive and differentiable; \( \phi'(x) \) is continuous; and \( \phi(x) \) is strictly convex.

The most general criterion of income inequality measurement is the Lorenz curve which forms the basis for Lorenz dominance comparisons. Generally speaking, distribution \( x \) exhibits less inequality than distribution \( y \) if the Lorenz curve of \( x \) lies above that of \( y \). There exist various versions of Lorenz dominance, but a unified definition can be given as follows:

**Definition 2.1.** For any distribution \( x \in \mathcal{D} \) and \( y \in \mathcal{D} \), \( x \) Lorenz dominates \( y \) if and only if

\[
L(x; l) := \frac{1}{n} \sum_{i=1}^{l} m(\tilde{x}_i, \mu(x)) \geq \frac{1}{n} \sum_{i=1}^{l} m(\tilde{y}_i, \mu(y)) =: L(y; l)
\]  

for all \( l = 1, 2, ..., n - 1 \) with the strict inequality holding for some \( l \). Also, \( \tilde{x}_i \) and \( \tilde{y}_i \) are \( i \)th values in the increasingly sorted distributions \( x \) and \( y \), i.e., \( \tilde{x}_i \leq \tilde{x}_{i+1} \) and \( \tilde{y}_i \leq \tilde{y}_{i+1} \); \( m(x_i, \mu(x)) \) is continuous and strictly increasing in \( x_i \) with

\[
\frac{1}{n} \sum_{i=1}^{n} m(\tilde{x}_i, \mu(x)) = \frac{1}{n} \sum_{i=1}^{n} m(\tilde{y}_i, \mu(y)) = c
\]
for some nonnegative constant $c$.

For $m(\tilde{x}_i, \mu(x)) = \tilde{x}_i / \mu(x)$ and $c = 1$ definition (2.3) corresponds to the familiar (relative) Lorenz dominance; for $m(\tilde{x}_i, \mu(x)) = \mu(x) - \tilde{x}_i$ and $c = 0$ it leads to the absolute Lorenz dominance as introduced in Moyes (1987); and for $m(\tilde{x}_i, \mu(x)) = (\tilde{x}_i - \mu(x)) / (\mu(x))^a$ and $c = 0$ it gives rise to the Krtscha-type intermediate Lorenz dominance (e.g., Zheng, 2007b).

### III. Interval Data, Unit-Invariance and Unit-Consistency

In general, for two interval variables $x$ and $y$, which measure the same subject,

$$y = ax + b$$  \hspace{1cm} (3.1)

where $a$ is the scale factor and $b$ is the location factor. For example, if $x$ is the degree of temperature measured in Fahrenheit and $y$ is the temperature in Celsius, then $x$ and $y$ are related via

$$y = \frac{5}{9}x - 17\frac{7}{9}.$$  

Thus, an interval value is unique up to an affine transformation (in contrast, a ratio value is unique up to a linear transformation while an ordinal value is unique up to a monotonic transformation).

In elementary statistics textbooks, it is usually assumed that the data type involved must be at least interval. The following result shows that the interval data type is required for the mean to be a consistent measure of central location.\(^2\)

**Proposition 3.1.** Suppose $x$ and $y$ are two ways of measuring the same subject with $y = f(x)$. For any two stages ($A$ and $B$) of the subject, we have four data sets $x_A$, $x_B$, $y_A$, and $y_B$. The comparison of the means between the two stages is consistent, i.e.,

$$\mu(x_A) \geq \mu(x_B) \iff \mu(y_A) \geq \mu(y_B)$$  \hspace{1cm} (3.2)

if and only if $f(x) = ax + b$ for some constants $a \in \mathbb{R}_{++}$ and $b \in \mathbb{R}$.

In measuring temperature, it is clear that both Fahrenheit ($x$) and Celsius ($y$) must be consistent in reporting the change in the average temperature from year to year or from place to place ($A$ to $B$). The above result states that in order to have consistent measurement, temperature (or any other subject) must be measured at least as an interval variable. In other words, if the variable is not at least interval then one cannot use the mean as a meaningful measure of central location. Other measures such as the median may be more appropriate.

Now we turn to the measurement of variability/inequality of interval data. Once a measuring unit is chosen – say Fahrenheit – the results from income inequality measurement can be applied to interval data (note that some inequality measures

\(^2\)All proofs are in the Appendix.
require positive values). Suppose one distribution of temperatures exhibits more variation (inequality) than another distribution of temperatures, clearly the same conclusion should hold when temperature is expressed in Celsius.

In income inequality measurement, the traditional remedy to ensure consistency is to impose a “unit-invariance” condition on inequality measures. An income inequality measure is unit-invariant if the inequality value remains the same when incomes are expressed in different units such as dollars and pounds. Such a unit-invariance condition leads to the class of relative inequality measures. Given this, it is tempting to consider “unit-invariance” for inequality measurement with other types of data. In measuring variation in temperatures, an invariance condition guarantees that the measure of inequality be the same whether temperature is measured in Fahrenheit or Celsius. It is obvious that, for a unit-invariant inequality measure, the example of inconsistent measurement discussed above cannot occur.

For interval data, however, unit-invariance is not a solution for inconsistency. For ratio values, the unit, for example, dollars versus pounds, is determined by a scale factor \((a)\). For interval data, however, the unit is determined by a scale factor \((a)\) and a location factor \((b)\). Unit-invariance for interval data requires that the inequality level be the same for all permissible values of \(a\) and \(b\). The simultaneous consideration of scale and location leads to the following impossibility result for unit-invariance:

**Proposition 3.2.** No inequality measure for interval data can satisfy unit-invariance for all \(a \in \mathbb{R}_{++}\) and \(b \in \mathbb{R}\).

The proper requirement to address the issue, we argue, is one that is similar to the unit-consistency axiom recently introduced for income inequality measurement (Zheng, 2007a; 2007b). Unit-consistency in income inequality requires that the use of different measurement units should not alter the rankings of income distributions with respect to inequality. It is also important to note, as Kolm (1976) points out, that unit-invariance introduces value judgements into inequality measurement because the property implies what Kolm terms a rightist view of inequality measurement to the exclusion of all other views. In contrast, unit-consistency imposes no such value judgement because it focuses solely on the consistency of using different units of measurement. Within the context of interval data, the unit-consistency axiom can be stated as follows (the notation \(\mathbf{1}\) represents the unit vector \((1,1,\ldots,1)\) with \(n\) components):

- **Unit-Consistency:** For any two distributions \(x, y \in \mathcal{D}\) and any inequality measure \(I(\cdot)\), if \(I(x) < I(y)\), then \(I(ax + b\mathbf{1}) < I(ay + b\mathbf{1})\) for any \(a \in \mathbb{R}_{++}\) and \(b \in \mathbb{R}\).

Clearly, unit-invariance implies unit-consistency, but not the converse. In what follows, we investigate the implications of unit-consistency for inequality measurement with interval data. The investigation is carried out separately for decomposable inequality measures and for partial inequality orderings with Lorenz curves.

### 3.1. Unit-consistent decomposable inequality measures

For any inequality measure, the following lemma demonstrates that unit-consistency
for interval data consists of two separate sub-layers of consistency, scale-consistency and location-consistency:

**Lemma 3.1.** For any two distributions \( x, y \in \mathcal{D} \) and any inequality measure \( I(\cdot) \), the following two conditions are equivalent:

\[
I(x) < I(y) \iff I(ax + b1) < I(ay + b1) \quad \text{for any } a \in \mathbb{R}_+ \text{ and } b \in \mathbb{R}
\]

(3.3)

and

\[
\begin{align*}
I(x) < I(y) & \iff I(ax) < I(ay) \quad \text{for any } a \in \mathbb{R}_+ \\
\text{and } I(x + b1) & < I(y + b1) \quad \text{for any } b \in \mathbb{R}.
\end{align*}
\]

(3.4)

The first part of (3.4) constitutes unit-consistency in income inequality measurement which we refer to as “scale-consistency.” The second part of (3.4) is referred to as “location-consistency.” Two variables \( x \) and \( y \) are said to be different only in location if \( y = x + b \) for some constant \( b \). It follows from Lemma 3.1 that the set of unit-consistent inequality measures is the intersection between the set of scale-consistent inequality measures and the set of location-consistent inequality measures.

Following the discussions of the unit-consistency axiom in Zheng (2007a), an immediate implication of the axiom is that \( I(\cdot) \) is unit-consistent if and only if, for all \( x \in \mathcal{D}, a \in \mathbb{R}_+ \) and \( b \in \mathbb{R} \), there exist continuous functions \( f(\cdot, \cdot) \) and \( g(\cdot, \cdot) \) which are increasing in the second arguments such that

\[
I(ax) = f(a, I(x)) \quad \text{and} \quad I(x + b1) = g(b, I(x)).
\]

(3.5)

With decomposability and other conditions, the functions \( f(\cdot, \cdot) \) and \( g(\cdot, \cdot) \) can be further specified as follows.

**Lemma 3.2.** For any \( x \in \mathcal{D} \), \( I(x) \) satisfies symmetry, normalization, decomposability, strict Schur-concavity and unit-consistency if and only if

\[
I(ax) = a^\tau I(x) \quad \text{and} \quad I(x + b1) = \gamma^b I(x)
\]

for all \( a \in \mathbb{R}_+, b \in \mathbb{R} \) and some constants \( \tau \in \mathbb{R}, \gamma \in \mathbb{R}_+ \).

With lemma 3.2, we are able to characterize the unit-consistent decomposable inequality indices for interval data:

**Proposition 3.3.** For any \( x \in \mathcal{D} \), suppose \( I(x) \) satisfies symmetry, normalization, differentiability, replication-invariance, decomposability, and strict Schur-concavity. (1) \( I(x) \) satisfies location-consistency if and only if it is a positive multiple of the form

\[
I^*_\alpha(x) = \frac{1}{n(x)e^{\beta\mu(x)}} \sum_{i=1}^{n(x)} \left\{ e^{\alpha x_i} - e^{\alpha \mu(x)} \right\} \quad \text{with } \alpha \neq 0,
\]

(3.7a)

Note that the location-consistency condition is weaker than the translation-invariance condition \( I(x) = I(x + b1) \) which characterizes the class of absolute inequality measures. In the measurement literature, this type of data is referred to as **difference** data and an example is the so-called Thurstone Case V scale, which measures response strength, in psychology (Suppes and Zinnes, 1963).
and

\[ I_v^l(x) = \frac{1}{n(x) \sigma(x)} \sum_{i=1}^{n(x)} (x_i - \mu(x))^2, \]  

(3.7b)

for some \( \alpha, \beta \in \mathbb{R} \); (2) \( I(x) \) satisfies unit-consistency (i.e., both scale-consistency and location-consistency) if and only if it is a positive multiple of the variance

\[ I(x) = \frac{1}{n(x)} \sum_{i=1}^{n(x)} (x_i - \mu(x))^2. \]  

(3.8)

Note that if \( \alpha = \beta \) then (3.7a) becomes the well-known Kolm-Pollak absolute inequality index which has been characterized by, for instance, Ebert (1988). The absolute version of (3.7a) and (3.7b) has also been characterized by Chakravarty and Tyagarupananda (1998); our result generalizes their findings when location (or translation)-invariance (\( I(x+b1) = I(x) \)) is replaced with location-consistency.

Finally, we briefly characterize unit-consistent Lorenz dominance.

3.2. Unit-consistent Lorenz dominance

Definition 2.1 provides the general definition of Lorenz dominance that we will follow in the following characterization. Following Zheng (2007b), it is clear that a Lorenz dominance is unit-consistent if and only if each Lorenz ordinate \( L(x;l) \) defined in (2.3) is unit-consistent. Results similar to Lemmas 3.1 and 3.2 can also be easily established for Lorenz dominance. That is, Lorenz dominance is unit-consistent if and only if it is both scale-consistent and location-consistent; a Lorenz ordinate is unit-consistent if and only if there exist functions \( f(\cdot, \cdot) \) and \( g(\cdot, \cdot) \) which are increasing in the second arguments such that

\[ L(ax; l) = f(a, L(x; l)) \quad \text{and} \quad L(x + b1; l) = g(b, L(x; l)) \]  

(3.9)

Unit-consistent Lorenz dominance is characterized as follows:

**Proposition 3.4.** For all \( x \in \mathcal{D} \), (1) \( L(x; l) \) is location-consistent if and only if

\[ m(\tilde{x}_i, \mu(x)) = \varepsilon \alpha(x)(\tilde{x}_i - \mu(x)) + c \]  

(3.10)

for some constants \( \varepsilon \in \mathbb{R}^{++} \) and \( c \in \mathbb{R} \); and (2) \( L(x; l) \) is unit-consistent (i.e., both scale-consistent and location-consistent) if and only if it is a positive multiple of

\[ \frac{1}{n(x)} \sum_{i=1}^{l} (x_i - \mu(x) + c). \]  

(3.11)

Therefore the only partial inequality ordering criterion for interval data is the absolute Lorenz dominance condition that Moyes (1987) introduced.
IV. Conclusion

An interval value is determined by a scale factor and a location factor, unlike a ratio value which relies only on a scale factor. We find that joint consideration of these two factors makes inequality measurement of interval data quite different from that of ratio data. These differences arise because of the requirement of consistency in inequality comparisons when different measuring units such as Fahrenheit and Celsius are adopted. In income inequality measurement, consistency is traditionally guaranteed through a unit-invariance condition. In contrast, for any interval data, we show that it is impossible to construct a unit-invariant inequality measure. With unit-consistency, which is a weaker and more reasonable requirement than unit-invariance, a much broader set of income inequality measures than the relative class has been characterized (Zheng, 2007a; 2007b). For interval data, however, we find in this paper that the only unit-consistent decomposable inequality measure is variance and the only unit-consistent Lorenz dominance is absolute Lorenz dominance that Moyes (1987) introduced. This is quite remarkable given the large number of candidates for measurement of dispersion.
Appendix

Proof of Proposition 3.1.

Condition (3.2) implies that \( \mu(y) \) is an increasing function of \( \mu(x) \), i.e.,

\[
\mu(y) = g(\mu(x))
\]
or

\[
\frac{1}{n} \sum_{i=1}^{n} f(x_i) = g\left(\frac{1}{n} \sum_{i=1}^{n} x_i\right) \tag{A1}
\]

which is a generalized Jensen equation. The solution to (A1) is (Aczél, 1966, p.145)

\[
f(x) = g(x) = ax + b \text{ for some constants } a \text{ and } b.
\]

To ensure \( g(\cdot) \) to be an increasing function, \( a \) must also be positive. □

Proof of Proposition 3.2.

For any inequality measure \( I(\cdot) \), unit-invariance requires

\[
I(ax + b1) = I(x) \tag{A2}
\]

for all \( x \in \mathcal{D} \) and all values of \( a \in \mathbb{R}_{++} \) and \( b \in \mathbb{R} \). Clearly, letting \( b = 0 \) and \( a = 1 \) respectively, we have for all \( x \in \mathcal{D} \),

\[
I(ax) = I(x) \text{ and } I(x + b1) = I(x). \tag{A3}
\]

Conversely, if (A3) holds, then (A2) will hold as well because

\[
I(ax + b1) = I[a(x + (b/a)1)] = I(x + (b/a)1) = I(x).
\]

Condition (A3) means that inequality measure \( I(\cdot) \) is both relative and absolute. But Zheng (1994) showed that there is no nontrivial inequality measure (i.e., non-constant) that can be both relative and absolute. Note that the proof remains valid even if \( x, ax + b1 \in \mathcal{D}^+ =: \cup_{n \geq 3} \mathbb{R}^n_{++} \) instead of \( \mathcal{D} \) as is usually the case for income distributions. □

Proof of Lemma 3.1.

For any inequality measure \( I(\cdot) \), unit-consistency requires that if \( I(x) < I(y) \), then \( I(ax + b1) < I(ay + b1) \) for all values of \( a \in \mathbb{R}_{++} \) and \( b \in \mathbb{R} \). Clearly, letting \( b = 0 \) and \( a = 1 \) respectively, we have

\[
I(ax) < I(ay) \text{ and } I(x + b1) < I(y + b1). \tag{A4}
\]

That is, \( I(\cdot) \) must be both scale-consistent and location-consistent. Conversely, if (A4) holds, then \( I(ax + b1) < I(ay + b1) \) will hold as well: \( I(x + b1) < I(y + b1) \) for all \( b \) entails

\[
I(x + (b/a)1) < I(y + (b/a)1); \tag{A5}
\]

10
then applying scale-consistency of $I(\cdot)$ to (A5) leads to

$$I(ax + b1) = I(a(x + (b/a)1)) < I(a(y + (b/a)1)) = I(ay + b1).$$

\[\text{\hspace{1cm} ■}\]

**Proof of Lemma 3.2.**

The first part of the lemma is Proposition 3 of Zheng (2007a); the second part can be proved similarly. For completeness, we sketch a proof here.

Choose any two distributions $x$ and $y$ with the same population size and same mean, \textit{i.e.}, $\mu(x) = \mu(y) = \mu$ and $n(x) = n(y) = n$. Then consider $z = (x, y)$. Apply the decomposability condition (2.1) to $z$,

$$I(z) = w^1(\mu, n)I(x) + w^2(\mu, n)I(y).$$

(A6)

Now transforming $x$ and $y$ to $x + b$ and $y + b$, respectively, we have

$$I(z + b1) = w^1(\mu + b, n)I(x + b1) + w^2(\mu + b, n)I(y + b1).$$

(A7)

Applying the unit-consistency condition (3.5) and using (A6), we further have

$$\tilde{g}(w^1I(x) + w^2I(y)) = \tilde{w}^1\tilde{g}(I(x)) + \tilde{w}^2\tilde{g}(I(y)),$$

where $\tilde{g}(\cdot) = g(b, \cdot)$, $\tilde{w}^1 = w^1(\mu, n)$ and $\tilde{w}^2 = w^2(\mu + b, n)$. Following Aczél (1966, p. 66), the above equation can be transformed into a standard Cauchy equation and its nontrivial solution (\textit{i.e.}, non-constant) leads to

$$I(x + b1) = \varphi(b)I(x)$$

(A8)

for some continuous function $\varphi(\cdot)$.

Next, consider $b, b' \in \mathcal{R}$, we have

$$I[x + (b + b')1] = \varphi(b + b')I(x)$$

and

$$I[x + (b + b')1] = I[(x + b1) + b'1] = \varphi(b')I(x + b1) = \varphi(b)\varphi(b')I(x).$$

It follows that

$$\varphi(b + b') = \varphi(b)\varphi(b')$$

(A9)

for all $b, b' \in \mathcal{R}$. The nontrivial solution to (A9) is

$$\varphi(b) = \gamma^b$$

for some $\gamma \in \mathcal{R}_{++}$.  

\[\text{\hspace{1cm} ■}\]

**Proof of Proposition 3.3.**

The proof of location-consistent inequality measures (3.7a) and (3.7b) adopts the approach of Chakravarty and Tyagarupananda (1998) in their characterization of
absolute decomposable inequality measures. For simplicity, we will write \( \mu(x) \) as \( \mu \) and \( n(x) \) as \( n \).

Using (2.2) and Lemma 3.2, we have

\[
I(x+b1) = \frac{1}{n\lambda(\mu+b)} \sum_{i=1}^{n} [\phi(x_i + b) - \phi(\mu + b)] = \gamma^b I(x)
\]

for some \( \gamma \in \mathbb{R}_{++} \). It follows that

\[
\frac{1}{\gamma^b n\lambda(\mu+b)} \sum_{i=1}^{n} [\phi(x_i + b) - \phi(\mu + b)] = I(x) \tag{A10}
\]

for all \( b \in \mathbb{R} \).

Denoting \( \vartheta(\mu, b, n) = \gamma^b n\lambda(\mu+b) \) and differentiating (A10) with respect to \( b \), we have

\[
\vartheta(\mu, b, n) \left[ \sum_{i=1}^{n} \{ \phi'(x_i + b) - \phi'(\mu + b) \} \right] - \left[ \sum_{i=1}^{n} \{ \phi(x_i + b) - \phi(\mu + b) \} \right] \vartheta_2(\mu, b, n) = 0 \tag{A11}
\]

where \( \vartheta_2(\mu, b, n) \) stands for the derivative of \( \vartheta(\mu, b, n) \) with respect to \( b \).

Subtract the derivative of (A11) with respect to \( x_j \) from its derivative with respect to \( x_i \),

\[
\vartheta(\mu, b, n) \left[ \phi''(x_i + b) - \phi''(x_j + b) \right] - [\phi'(x_i + b) - \phi'(x_j + b)] \vartheta_2(\mu, b, n) = 0.
\]

Rearrange,

\[
\phi''(x_i + b) - \frac{\vartheta_2(\mu, b, n)}{\vartheta(\mu, b, n)} \phi'(x_i + b) = \phi''(x_j + b) - \frac{\vartheta_2(\mu, b, n)}{\vartheta(\mu, b, n)} g'(x_j + b) \tag{A12}
\]

holds for any \( n \) and \( \mu \) while keeping \( x_i \) and \( x_j \) constant. This can be possible only for \( n \geq 3 \). It follows that

\[
\frac{\vartheta_2(\mu, b, n)}{\vartheta(\mu, b, n)} = c
\]

for some constant \( c \). Bringing back \( \vartheta(\mu, b, n) = \gamma^b n\lambda(\mu+b) \), we have

\[
\frac{\gamma^b \ln \gamma \lambda(\mu+b) + \gamma^b \lambda_1(\mu+b)}{\gamma^b \lambda(\mu+b)} = c
\]

or

\[
\frac{\lambda_1(\mu+b)}{\lambda(\mu+b)} = c - \ln \gamma \equiv \tilde{c}. \tag{A13}
\]

Let \( \mu + b = s \), the solution to (A13) is

\[
\lambda(s) = \kappa e^{\tilde{c}s} \tag{A14}
\]
where \( \kappa \) is some constant.

Substituting (A14) back into (A12), we then have

\[
\phi''(z) - c\phi'(z) = d
\]

for some constant \( d \). The solution is

\[
\phi(z) = \begin{cases} 
A + Be^{cz} + Cz, & c \neq 0 \\
A + Bz^2 + Cz, & c = 0
\end{cases}
\]

(A15)

where \( A, B \) and \( C \) are constants.

Substituting (A14) and (A15) into (2.2) completes the proof of (3.7a) and (3.7b).

Applying the scale-consistency condition \( I(ax) = a^r I(x) \) to (3.7a) and (3.7b) and taking differentiation with respect to \( a \), we obtain that only (3.7b) with \( \beta = 0 \) can be scale-consistent. This leads to (3.8).

**Proof of Proposition 3.4.**

The proof utilizes the same arguments as those in Zheng (2007b). For completeness, we sketch all elements of the proof as follows.

Applying location-consistency \( L(x+b; l) = g(b, L(x); l) \) to \( nL(x; l) = \sum_{i=1}^l m(\bar{x}_i, \mu) \), we have

\[
g(b, \sum_{i=1}^l m(\bar{x}_i, \mu)) = \sum_{i=1}^l g(b, m(\bar{x}_i, \mu)).
\]

(A16)

The solution to this Pexider equation leads to

\[
m(\bar{x} + b, \mu + b) = \varphi(b)m(\bar{x}, \mu) + \psi(b)
\]

for some continuous functions \( \varphi(\cdot) \) and \( \psi(\cdot) \). For any \( b, b' \in \mathcal{R} \), we further have

\[
\varphi(b + b') = \varphi(b)\varphi(b') \quad \text{and} \quad \psi(b + b') = \varphi(b)\psi(b') + \psi(b')
\]

(A17)

and the solutions to (A17) are

\[
\varphi(b) = \gamma^b \quad \text{and} \quad \psi(b') = \kappa\gamma^b - \kappa
\]

or

\[
m(\bar{x}_i + b, \mu + b) = \gamma^b m(\bar{x}_i, \mu) + \kappa\gamma^b - \kappa
\]

(A18)

for all \( b \in \mathcal{R} \) and for some constants \( \gamma \in \mathcal{R}_{++} \) and \( \kappa \in \mathcal{R} \).

The requirement \( \frac{1}{n} \sum_{i=1}^n m(\bar{x}_i, \mu) = c \) in Definition 2.1 implies ( Zheng, 2007b, Lemma 4.2),

\[
m(\bar{x}_i, \mu) = \psi(\mu)(\bar{x}_i - \mu) + c.
\]

(A19)

Thus,

\[
m(\bar{x}_i + b, \mu + b) = \psi(\mu + b)(\bar{x}_i - \mu) + c.
\]

(A20)
and

\[ m(\tilde{x}_i + b, \mu + b) = \gamma^b m(\tilde{x}_i, \mu) + \kappa \gamma^b - \kappa = \gamma^b \{ \psi(\mu)(\tilde{x}_i - \mu) + c \} + \kappa \gamma^b - \kappa = \gamma^b \psi(\mu)(\tilde{x}_i - \mu) + \gamma^b c + \kappa \gamma^b - \kappa. \]  \quad \text{(A21)}

Equating (A20) with (A21), we obtain

\[ \psi(\mu + b) = \gamma^b \psi(\mu) \text{ and } \gamma^b c + \kappa \gamma^b - \kappa = c. \]

It follows that (by choosing \( \mu = 0 \) in the first equation or solving it as a Cauchy equation)

\[ \kappa = -c \text{ and } \psi(\mu) = \varepsilon \gamma^\mu \]

where \( \varepsilon = \psi(0) \). Substitute \( \psi(\mu) = \varepsilon \gamma^\mu \) into (A19) entails (3.10). Note that \( \varepsilon \) must be positive to ensure that \( m(x, \mu) \) is strictly increasing in \( x \). Finally, when applying the scale-consistency condition to (3.10), we obtain \( \gamma = 1 \) which leads to (3.11).
References


