Increasing, continuous operations in fuzzy
\textsc{max} −\textsc{∗} equations and inequalities

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**Notations**

Let \( * : [0, 1]^2 \rightarrow [0, 1] \) and \( A \in [0, 1]^{m \times n} \), \( b \in [0, 1]^m \), \( a, c \in [0, 1] \). Vectors \( x, y \in [0, 1]^n \) are ordered by

\[
(x \leq y) \iff (\forall 1 \leq j \leq m \ x_j \leq y_j).
\]

We use notation

- \( a \lor b = \max(a, b), \ a \land b = \min(a, b), \ a, b \in [0, 1] \),
- \( \bigvee_{1 \leq i \leq n} x_i = \max x_i, \ \bigwedge_{1 \leq i \leq n} x_i = \min x_i, \ x_i \in [0, 1] \),
- \( \max -* \) product of a matrix \( A \) and a vector \( x \) (Zadeh 1971) we call \( A \circ x \in [0, 1]^m \), where

\[
(A \circ x)_i = \bigvee_{j=1}^{n} (a_{ij} * x_j), \quad i \in \{1, \ldots, m\}.
\]

Families of solutions:

- \( S_{\leq}(A, b, *) = \{x \in [0, 1]^n : A \circ x \leq b\} \),
- \( S_{\geq}(A, b, *) = \{x \in [0, 1]^n : A \circ x \geq b\} \),
- \( S(A, b, *) = \{x \in [0, 1]^n : A \circ x = b\} = S_{\geq}(A, b, *) \cap S_{\leq}(A, b, *) \),
- induced implication (Drewniak 1984) \( a \rightarrow c = \max\{t \in [0, 1] : a * t \leq c\} \),
- dual induced implication \( a \leftarrow c = \min\{t \in [0, 1] : a * t \geq c\} \).
**Induced implications**

**Lemma 1**
If an increasing operation \( \ast \) is left continuous and \( 1 \ast 0 = 0 \), then it induces implication in \([0, 1]\).

**Lemma 2**
Let \( a, b \in [0, 1] \), \( \{ t \in [0, 1] : a \ast t \geq b \} \neq \emptyset \). If an increasing operation \( \ast \) is right continuous, then exists \( a \leftarrow b \).

**Example 1**
The binary operations and theirs implications:

\[
T_P(x, y) = x \cdot y, \quad a \xrightarrow{T_P} b = \begin{cases} 
1, & a \leq b \\
\frac{b}{a}, & a > b
\end{cases}
\]

and \( a \xleftarrow{T_P} b = \begin{cases} 
\frac{b}{a}, & a \neq 0 \\
0, & a = 0
\end{cases} \) for \( a \geq b \).

\[
T_L(x, y) = 0 \lor (x + y - 1), \quad a \xrightarrow{T_L} b = 1 \land (1 - a + b)
\]

and \( a \xleftarrow{T_L} b = \begin{cases} 
1 \land (1 - a + b), & b \neq 0 \\
0, & b = 0
\end{cases} \), \( a \geq b \),

\[
T_M(x, y) = x \land y, \quad a \xrightarrow{T_M} b = \begin{cases} 
1, & a \leq b \\
b, & a > b
\end{cases}, \quad a \xleftarrow{T_M} b = b, \ a \geq b
\]

for all \( x, y, a, b \in [0, 1] \).
Convexity properties

Lemma 3 (cf. Drewniak 1989)

Let \(*\) be increasing operation. Families of solutions of \(A \circ x = b\), \(A \circ x \leq b\) and \(A \circ x \geq b\) have the convexity property, i.e.

\[
x \in S_{\leq}(A, b, *) \implies [0, x] \subset S_{\leq}(A, b, *),
\]

\[
x \in S_{\geq}(A, b, *) \implies [x, 1] \subset S_{\geq}(A, b, *),
\]

\[
x \leq y, \ x, y \in S(A, b, *) \implies [x, y] \subset S(A, b, *),
\]

where \([0, x], [x, y], [x, 1]\) are intervals in \(([0, 1]^n, \leq)\).

Corollary 1

If \(*\) is an increasing operation, then

- \(1 \in S_{\geq}(A, b, *) \iff S_{\geq}(A, b, *) \neq \emptyset\),
- \(0 \in S_{\leq}(A, b, *) \iff S_{\leq}(A, b, *) \neq \emptyset\).

Definition 1

By greatest solutions of system \(A \circ x \leq b\) (and \(A \circ x = b\)) with \(\max-*\) product we call minimal elements in \(S_{\leq}(A, b, *)\) (in \(S(A, b, *)\)).

Theorem 1

If an operation \(*\) is increasing, left-continuous on the second argument and \(1 \ast 0 = 0\), then \(S_{\leq}(A, b, *)\) the complete lattice. Moreover, if \(S_{\geq}(A, b, *) \neq \emptyset\) and \(S(A, b, *) \neq \emptyset\), then \(S_{\geq}(A, b, *) \neq \emptyset\) and \(S(A, b, *) \neq \emptyset\) are closed under arbitrary suprema.
The greatest solution

Let \( u = \max S_{\leq}(A, b, \ast) = \max\{x \in [0, 1]^n : A \circ x \leq b\}. \)

**Theorem 2**

*If an operation \( \ast \) is increasing, left-continuous on the second argument and \( 1 \ast 0 = 0 \), then there \( u \) is the greatest element of \( S_{\leq}(A, b, \ast) \), where*

\[
u_j = \bigwedge_{i=1}^{m} (a_{ij} \ast b_i), \quad j \in \{1, \ldots, n\}.
\]

*It means \( u = A \rightarrow b \).*

**Corollary 2**

*If an operation \( \ast \) is increasing, left-continuous on the second argument and \( 1 \ast 0 = 0 \), then \( S_{\leq}(A, b, \ast) = [0, A \rightarrow b] \).*

**Theorem 3**

*If an operation \( \ast \) is increasing, left-continuous on the second argument, \( 1 \ast 0 = 0 \) and \( S(A, b, \ast) \neq \emptyset \), then \( \max S(A, b, \ast) = A \rightarrow b \).*
Reduced matrix

Definition 2
Let $x \in S(A, b, \ast)$. By reduced matrix of equation system $A \circ x = b$ we call the matrix $A'_{b}(x)$, where

$$
a'_{ij}(x) = \begin{cases} a_{ij} & \text{, if } a_{ij} \ast x_{j} = b_{i} \\
0 & \text{, in other case} \end{cases}, i \in \{1, \ldots, m\}, j \in \{1, \ldots, n\}.
$$

Let $x \in S_{\geq}(A, b, \ast)$. By reduced matrix of system of inequalities $A \circ x \geq b$ we call $A'_{\geq b}(x)$, where

$$
a'_{ij}(x) = \begin{cases} a_{ij} & \text{, if } a_{ij} \ast x_{j} \geq b_{i} \\
0 & \text{, in other case} \end{cases}, i \in \{1, \ldots, m\}, j \in \{1, \ldots, n\}.
$$

Theorem 4 (Drewniak, Matusiewicz 2010)

*If an operation $\ast$ is increasing, left-continuous on the second argument, and it has neutral element $e = 1$, then $S(A, b, \ast) = S(A'_{b}(u), b, \ast)$.***
Minimal solutions (1)

**Definition 3**
By minimal solutions of system \( A \circ x \geq b \) (and \( A \circ x = b \)) with max product we call minimal elements in \( S_\geq (A, b, \ast) \) (in \( S(A, b, \ast) \)). The set of all minimal solution is denoted by \( S^0_\geq (A, b, \ast) \) \( (S^0(A, b, \ast)) \).

**Corollary 3 (cf. Drewniak 1989)**

If \( \ast \) is increasing operation, then
\[
\bigcup_{x \in S^0_\geq (A, b, \ast)} [x, 1] \subset S_\geq (A, b, \ast).
\]

**Theorem 5**

If \( \ast \) is increasing, right-continuous on the second argument, then
- each \( x \in S_\geq (A, b, \ast) \) is bounded from below by some \( v \in S^0_\geq (A, b, \ast) \),
- each \( x \in S(A, b, \ast) \) is bounded from below by some \( v \in S^0(A, b, \ast) \),
- we have \( S^0(A, b, \ast) \subset S^0_\geq (A, b, \ast) \).

**Theorem 6**

If \( \ast \) is increasing, continuous on the second argument and \( 1 \ast 0 = 0 \), then
\[
S(A, b, \ast) = \bigcup_{v \in S^0(A, b, \ast)} [v, A \circ b).
\]
Algorithm of computing minimal solutions (1)

Let $S_{\geq}(A, b, \ast) \neq \emptyset$, an operation $\ast$ be increasing, right-continuous on the second argument one and

$$0 < b_m \leq \ldots \leq b_2 \leq b_1.$$  

ALGORITHM I

Step 1. Determine the reduced matrix $A'_{\geq_b}(x)$. Let $i := 1$, $K := \emptyset$, $V := \{1, \ldots, m\}$.

Step 2. Choose $k_i$ that $a'_{ik_i} > 0$ and calculate $v_{k_i} = a'_{ik_i} \leftarrow b_i$ and $K := K \cup \{k_i\}$.

Step 3. Determine the set

$$V := V \cap \{i < s \leq m \text{ oraz } a'_{sk_i} \ast v_{k_i} < b_s\}.$$  

Step 4. If $V \neq \emptyset$, to $i := \min V$ and return to Step 2.

In other case go to Step 5.

Step 5. If $k \notin K$, then $v_k := 0$.

Let us denote the set of all vectors $v$ from this algorithm obtained for $x \in S_{\geq}(A, b, \ast)$ by $\text{Alg}(x)$ (see Step 2).

Corollary 4

Let $x \in S_{\geq}(A, b, \ast)$. If an operation $\ast$ is increasing, right-continuous on the second argument, then

$$\text{cardAlg}(x) \leq m^n.$$
Theorem 7
If an operation $\ast$ is increasing, right-continuous on the second argument, then $S^0_\geq(A, b, \ast) \subset \text{Alg}(1)$.

Theorem 8
Let $x \in S(A, b, \ast)$. If an operation $\ast$ is increasing, right-continuous on the second argument, then $\text{Alg}(x) \subset S(A, b, \ast)$.

Theorem 9
Let $b \in (0, 1]^n$. If an operation $\ast$ is increasing, continuous on the second argument and $1 \ast 0 = 0$, then $S^0_\leq(A, b, \ast) \subset \text{Alg}(A \rightarrow b)$.

$$S(A, b, \ast) = \bigcup_{v \in S^0_\leq(A, b, \ast)} [v, A \rightarrow b].$$
Example 2

Let \( x \ast y = \sqrt{x \cdot y} \) and

\[
A = \begin{bmatrix}
0.1 & 0.16 & 0.25 \\
0.2 & 0.09 & 0.05
\end{bmatrix}, \quad b = \begin{bmatrix}
0.4 \\
0.3
\end{bmatrix}, \quad A'_b(1) = \begin{bmatrix}
0 & 0.16 & 0.25 \\
0.2 & 0.09 & 0
\end{bmatrix}.
\]

We get \( a\leftarrow b = \frac{b^2}{a} \), \( b^2 \leq a \).

Using \( Alg(1) \) we get:

1. For \( k_1 = 2 \) we get \( K = \{2\} \) and \( V = \emptyset \). We obtain \( v_2^1 = 0.16 \leftarrow 0.4 = 1 \).
2. For \( k_1 = 3 \) we get \( v_3^2 = 0.2 \leftarrow 0.4 = 0.8 \) and \( K = \{3\}, \ V = \{2\}, \ i = 2 \).
Choosing \( k_2 = 1 \), we compute \( v_1^2 = 0.2 \leftarrow 0.3 = 0.45 \), \( K = \{1, 3\}, \ V = \emptyset \).
3. For \( k_1 = 3 \) we get \( v_3^3 = 0.2 \leftarrow 0.4 = 0.8 \) and \( K = \{3\}, \ V = \{2\}, \ i = 2 \).
Choosing \( k_2 = 2 \), we compute \( v_3^3 = 0.09 \leftarrow 0.3 = 1 \), \( K = \{2, 3\}, \ V = \emptyset \).

Thus we have the following projections:

\[
v^1 = \begin{bmatrix}
0 \\
1 \\
0
\end{bmatrix}, \quad v^2 = \begin{bmatrix}
0.45 \\
0 \\
0.8
\end{bmatrix}, \quad v^3 = \begin{bmatrix}
0 \\
1 \\
0.8
\end{bmatrix}.
\]

Since \( v^1 \| v^2 \) and \( v^1 \leq v^3 \), then \( S^0_\geq (A, b, \ast) = \{v^1, v^2\} \).
Algorithm of computing minimal solutions (2)

Let an operation $\ast$ be increasing, continuous on the second argument and have neutral element $e = 1$.

ALGORYTM I’

Step 0. We calculate $u = A \circ \rightarrow b$.

Step 1. We determine $Alg(u)$ from Algorithm I.

Step 2. We determine $S^0(A, b, \ast)$ as a set of minimal elements in $Alg(u)$.

Definition 4
An operation $\ast$ is conditionally cancellative if

$$a \ast x = a \ast y \neq 0 \Rightarrow x = y \quad \text{for } a, x, y \in (0, 1].$$

Theorem 10
Let $\ast$ be increasing, continuous on the second argument and conditionally cancellative operation and $1 \ast 0 = 0$, then If $v \in S^0(A, b, \ast)$, then $v_j \in \{0, u_j\}$ for $j \in \mathbb{N}$, where $u = A \circ \rightarrow b$.

Corollary 5
If $\ast$ is increasing, continuous on the second argument and conditionally cancellative operation and $1 \ast 0 = 0$, then

$$\text{card } S^0(A, b, \ast) \leq \left(\frac{n}{\frac{n}{2}}\right).$$
Example 3

Let $\ast = T_P$ and

$$A = \begin{bmatrix} 1 & 0.8 & 0.5 & 0.5 \\ 0.8 & 0.8 & 0.1 & 0.4 \\ 0.4 & 0.6 & 0.3 & 0.3 \\ 0.4 & 0.4 & 0.2 & 0.1 \end{bmatrix}, \quad b = \begin{bmatrix} 0.5 \\ 0.4 \\ 0.3 \\ 0.2 \end{bmatrix}.$$ 

We determine $Alg(u)$:

$$u = \begin{bmatrix} 0.5 \\ 0.5 \\ 1 \\ 1 \end{bmatrix}, \quad v^1 = \begin{bmatrix} 0.5 \\ 0.5 \\ 0 \\ 0 \end{bmatrix}, \quad v^2 = \begin{bmatrix} 0.5 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad v^3 = \begin{bmatrix} 0.5 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad v^4 = \begin{bmatrix} 0.5 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

$$v^5 = \begin{bmatrix} 0 \\ 0.5 \\ 1 \\ 0 \end{bmatrix}, \quad v^6 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \quad v^7 = \begin{bmatrix} 0.5 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad v^8 = \begin{bmatrix} 0 \\ 0.5 \\ 0 \\ 1 \end{bmatrix}, \quad v^9 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$ 

From Algorithm $I'$ we obtain the solutions of $A \circ x = b$. In this set we have all minimal solution of the system. We get $S^0(A, b, \ast) = \{v^1, v^2, v^3, v^5, v^6, v^8\}$, because $v^2 = v^4$, $v^6 = v^9$, $v^3 = v^7$. 

• J. Drewniak, Fuzzy relation calculus, Silesian University, Katowice 1989.