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Computing the R^4 term at Two Super-string Loops

Roberto Iengo*
INFN, Sezione di Trieste and Scuola Internazionale Superiore di Studi Avanzati (SISSA) Via Beirut 4, I-34013 Trieste, Italy

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Abstract

We use a previously derived integral representation for the four graviton amplitude at two loops in Super-string theory, whose leading term for vanishing momenta gives the two-loop contribution to the R^4 term in the Effective Action. We find by an explicit computation that this contribution is zero, in agreement with a general argument implying the vanishing of the R^4 term beyond one loop.

As it is well known, Super-string perturbation theory at two loops implies computing a sigma-model functional integration on a genus two Riemann surface (meaning the functional integration over the Super-string fields $X^\mu(z)$ and $\psi^\mu(z)$, where z is a complex coordinate on the surface).

We use the hyper-elliptic formalism, in which the genus two surface is represented as a two sheets covering of the complex plane, described by the equation:

$$y^2(z) = \prod_{i=1}^{6}(z - a_i).$$

The complex numbers $a_i,\ i = 1 \cdots 6$, are six branch points, by going around them one passes from one sheet to the other.

In a previous paper [1] (based on older work [2, 3, 4]) the coefficient of the R^4 term in the Effective Action at two loops was derived in the form of a certain amplitude A. Here R^4 means a particular invariant contraction of four curvature tensors, sometimes also indicated as $t_8 t_8 R^4$, see ref.[5, 6, 7].

*E-mail: iengo@he.sissa.it

Up to an overall constant, A turns out to be expressed by the following integral representation (see eq.(2) of ref.[1]):

$$A = V \cdot I \tag{1}$$

where V is the factor

$$V = |z_1 - z_2|^4 (z_1 - z_3)|z_1 - z_3|^2 (\bar{z}_2 - \bar{z}_3)|z_2 - z_3|^2 \tag{2}$$

and I is the integral:

$$\int \prod_1^6 d^2 a_i d^2 z \frac{(z - z_1)\bar{z} - \bar{z}_2)}{T^5 \prod_{i<j} |a_i - a_j|^2 |y(z)y(z_1)y(z_2)y(z_3)|^2} \cdot LSC \cdot RSC. \tag{3}$$

In this integral, LSC represents the left super-current contribution and it is:

$$LSC = \frac{1}{2} \sum_{i=1}^{6} \frac{1}{z_1 - a_i} - \frac{1}{z_1 - z} - \frac{1}{z_1 - z_3} - \frac{1}{z_1 - z_2}.$$

The right super-current contribution RSC is:

$$RSC = \frac{1}{2} \sum_{i=1}^{6} \frac{1}{\bar{z}_2 - \bar{a}_i} - \frac{1}{\bar{z}_2 - \bar{z}} - \frac{1}{\bar{z}_2 - \bar{z}_1} - \frac{1}{\bar{z}_2 - \bar{z}_3}.$$

Also, T is the determinant of the genus two period matrix, which with our variables takes the form

$$T(a_i, \bar{a}_i) = \int \frac{d^2 w_1 d^2 w_2 |w_1 - w_2|^2}{|y(w_1)y(w_2)|^2}.$$

The points z_i are arbitrary, except that we avoid taking $z_1 = z_2$.

Let us explain the main points of this formula. We recall (see for instance ref.[8, 9]) that one starts by computing the sigma-model expectation value (meaning the functional integration over the Super-String fields X^μ, ψ^μ and ghosts) of four graviton vertex operators and a left and a right super-current operator. The graviton polarizations and momenta combine in an expression corresponding to the previously recalled relativistic invariant R^4, which appear as a factor of an amplitude to be evaluated for our purpose in the limit of vanishing momenta. One has then to perform the integration over the Riemann surface moduli (that is, the branching points of $y(z)$) and also the puncture moduli (that is, the position of the vertex operators on the surface). The position on the surface of the super-current operators can be arbitrarily fixed. Different choices for the position of the super-current operators are related by a total derivative in the integration moduli, which is in general irrelevant.

However, if the left and right super-current operators are taken at the same position, the integration over the moduli appears to diverge. Actually, in this case we have seen in ref.[1] that this divergence is compensated by a boundary term, which is in this case the non-vanishing contribution of the total derivative. Since the integration over the moduli is found to be convergent for generic left and right super-current points, this fact is consistent with the arbitrariness of the choices. Thus, we avoid taking the same position for the left and right super-current operators, as otherwise we should include an additional contribution from the boundary term.

The integral is invariant under simultaneous $SL(2C)$ transformations of the integration variables and of the (arbitrary) super-current positions. Therefore one can, also arbitrarily, fix three among the integration variables.

One can fix three of the six branching points and thus integrate over the remaining three (in agreement with three complex moduli describing the deformations of a genus two surface) and also over the four puncture moduli (that is the vertex positions).

Another possible choice is to fix three among the puncture positions and integrate over the remaining one and the six branching points. This is what we have done to get eq.(3), in which $z_{1,2,3}$ represent the fixed puncture positions, whereas z (the remaining puncture modulus) and $a_{1,..,6}$ (the branching points) are integration variables. In eq.(3) we have further made the allowed arbitrary choice of fixing the left and right super-current at z_1 and z_2 respectively.

It appears to be difficult to perform the integration, even numerically, because it is a multiple integral over many variables with oscillating phases, thus possibly giving many cancellations. In the previous paper [1] the convergence properties of the integral were thoroughly analyzed, with the conclusion that the integral is convergent and thus gives a finite or zero result.

Notice that A is invariant under a transformation:

$$z_i \to \frac{\alpha z_i + \beta}{\gamma z_i + \delta} \qquad \alpha\delta - \beta\gamma = 1$$

for $i = 1, 2, 3$ (this can be shown by making the same transformation on the integration variables). Thus, we have the freedom of choosing $z_{1,2,3}$.

A standard choice is to take $z_1 \to \infty, z_2 \to 0$ and take finite $x \equiv z_3$.

In this limit we have:

$$LSC \to \frac{1}{z_1^2}(\frac{1}{2}\sum_{i=1}^{6} a_i - z - x)$$

and

$$RSC \to -\frac{1}{2}\sum_{i=1}^{6}\frac{1}{\bar{a}_i} + \frac{1}{\bar{z}} + \frac{1}{\bar{x}}.$$

Also: $V \to -|z_1|^6 z_1 \bar{x}|x|^2$ and $|y(z_1)|^2 \to |z_1|^6$.

Thus the amplitude reduces to:

$$\Rightarrow A = -\bar{x}|x|^2 \int d\mu \int \frac{d^2z\ \bar{z}}{|y(z)y(0)y(x)|^2}(RL - \frac{1}{\bar{x}}L - xR + \frac{x}{\bar{x}}).$$

Here, we called:

$$d\mu \equiv \frac{\prod_1^6 d^2a_i}{T^5 \prod |a_i - a_j|^2}, \qquad L \equiv \frac{1}{2}\sum_{i=1}^{6} a_i - z, \qquad R \equiv \frac{1}{2}\sum_{i=1}^{6}\frac{1}{\bar{a}_i} - \frac{1}{\bar{z}}.$$

It can be checked that A is independent of x, by rescaling the integration variables, and that the integral is convergent, by the same analysis summarized in the Sect.3 of ref.[1] (see the tables there).

For instance, let us analyze the potentially dangerous corner where every $a_i \to 0$: by putting $a_1 = u$, and $a_i = u\alpha_i$ for $i \geq 2$, we get

$$d\mu \sim |u|^{10} d^2u\ , \qquad \int \frac{d^2z\ \bar{z}}{|y(z)y(0)y(x)|^2} \sim \frac{\bar{u}}{|u|^{10}}\ ;$$

since L is regular and $R \sim 1/\bar{u}$, we finally get in the corner $u \to 0$ the convergent expression $\sim \int d^2u$.

Of course, this analysis does not take into account possible cancellations which could make the total result equal to zero. We will indeed prove that it is zero.

We begin by observing that:

$$\frac{1}{|y(0)|^2}(\frac{1}{2}\sum_i \frac{1}{\bar{a}_i}) = -\sum_i \frac{\partial}{\partial \bar{a}_i}\frac{1}{|y(0)|^2}.$$

Therefore, the following identity holds for an integral expression which we call Q:

$$Q \equiv \int d\mu \int \frac{d^2z\ \bar{z}}{|y(z)y(0)y(x)|^2}(\frac{1}{2}\sum_i \frac{1}{\bar{a}_i})(L-x) = \int d\mu \int \frac{d^2z\ \bar{z}}{|y(0)|^2}\sum_i \frac{\partial}{\partial \bar{a}_i}\frac{L-x}{|y(z)y(x)|^2}.$$

We integrated by parts, observing that $\sum_i \frac{\partial}{\partial \bar{a}_i}\frac{1}{T^5 \prod |a_i - a_j|^2} = 0$.

Also,

$$\sum_i \frac{\partial}{\partial \bar{a}_i}\frac{L-x}{|y(z)y(x)|^2} = -(L-x)(\frac{\partial}{\partial \bar{z}} + \frac{\partial}{\partial \bar{x}})\frac{1}{|y(z)y(x)|^2}.$$

Thus, by integrating by parts in d^2z we get:

$$Q = \int d\mu \int d^2z\ \frac{(L-x)}{|y(0)y(z)|^2}(1 - \bar{z}\frac{\partial}{\partial \bar{x}})\frac{1}{|y(x)|^2}.$$

The result of the above steps is that:

$$\begin{aligned}
\int d\mu \int d^2z \, \frac{\bar{z}}{|y(0)y(z)y(x)|^2} R(L-x) &= \\
= \int d\mu \int d^2z \, \frac{(L-x)}{|y(0)y(z)|^2}(1 - \bar{z}\frac{\partial}{\partial \bar{x}} - \frac{\bar{z}}{\bar{z}})\frac{1}{|y(x)|^2} &= \\
= -\frac{\partial}{\partial \bar{x}} \int d\mu \int d^2z \, \frac{\bar{z}}{|y(0)y(z)y(x)|^2}(L-x).
\end{aligned}$$

By using the previous results, we conclude that we can write our amplitude in the form:

$$\begin{aligned}
A = \quad & |x|^2(\bar{x}\frac{\partial}{\partial \bar{x}} + 1) \int d\mu \int \frac{d^2z \; \bar{z}}{|y(z)y(0)y(x)|^2} L \\
- \quad & |x|^4(\frac{\partial}{\partial \bar{x}} + \frac{1}{\bar{x}}) \int d\mu \int \frac{d^2z \; \bar{z}}{|y(z)y(0)y(x)|^2}. \qquad (4)
\end{aligned}$$

Now we perform a rescaling of the integration variables:

$$a_i \rightarrow x a_i \qquad z \rightarrow xz$$

and observe that under this rescaling we have:

$$\begin{aligned}
d\mu \quad &\rightarrow \quad |x|^{12} d\mu, \quad d^2z\bar{z} \rightarrow |x|^2 \bar{x} d^2z\bar{z} \\
\frac{1}{|y(z)y(0)y(x)|^2} \quad &\rightarrow \quad \frac{1}{|x|^{18}} \frac{1}{|y(z)y(0)y(1)|^2}, \quad L \rightarrow xL.
\end{aligned}$$

Therefore:

$$\begin{aligned}
(\bar{x}\tfrac{\partial}{\partial \bar{x}} + 1) \quad & \int d\mu \int \frac{d^2z \; \bar{z}}{|y(z)y(0)y(x)|^2} L = \\
(\bar{x}\tfrac{\partial}{\partial \bar{x}} + 1) \quad & \frac{1}{|x|^2} \int d\mu \int \frac{d^2z \; \bar{z}}{|y(z)y(0)y(1)|^2} L = 0, \qquad (5)
\end{aligned}$$

and

$$\begin{aligned}
(\tfrac{\partial}{\partial \bar{x}} + \tfrac{1}{\bar{x}}) \quad & \int d\mu \int \frac{d^2z \; \bar{z}}{|y(z)y(0)y(x)|^2} = \\
(\tfrac{\partial}{\partial \bar{x}} + \tfrac{1}{\bar{x}}) \quad & \frac{1}{\bar{x}x^2} \int d\mu \int \frac{d^2z \; \bar{z}}{|y(z)y(0)y(1)|^2} = 0. \qquad (6)
\end{aligned}$$

In conclusion, from eqs.(4,5,6), we get that the amplitude A is zero, and therefore there is no contribution to the invariant R^4 at two string loops.

This is in agreement with the indirect argument of Green and Gutperle, Green,Gutperle and Vanhove, and Green and Sethi [5, 6, 7] that the R^4 term does not receive contributions beyond one loop.

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