Discrete-time switched linear systems state feedback design with application to networked control

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Abstract

This paper addresses the state feedback switched control design problem for discrete-time switched linear systems. More specifically, the control goal is to jointly design a set of state feedback gains and a state dependent switching function, ensuring $H_2$ and $H_\infty$ guaranteed performance. The conditions are based on Lyapunov or Riccati-Metzler inequalities, which allow the derivation of simpler alternative conditions that are expressed as LMIs whenever a scalar variable is fixed. The theoretical results are well adapted to deal with the self-triggered control design problem, where the switching rule is responsible for the scheduling of multiple sampling periods, to be considered in the communication channel in order to improve performance. This method is compared to others from the literature. Examples show the validity of the proposed technique in both contexts, switched and networked control systems.

Index Terms

Switched systems, state feedback control, networked control

I. INTRODUCTION

Switched linear systems represent an important subclass of hybrid systems characterized by presenting several subsystems and a rule orchestrating the switching among them. These systems have attracted the attention of the scientific community in the last decades due to their intrinsic characteristics and their wide field of applications. For instance, the remarkable consistency

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property, defined in [3] and in [9] for discrete and continuous-time systems, respectively, states that, under certain assumptions, the design of a switching rule improves the performance of the overall system when compared to the one of each isolated subsystem. Thus, this property makes the switching control design problem interesting and rewarding, even if all subsystems are stable. Moreover, consistency is an important property in the design of several real world engineering problems, such as the networked control one to be considered in this paper. References [4], [16], [18], [20] and books [15], [22] are surveys on switched systems.

For discrete-time switched linear systems, the literature presents several important results regarding stability [7], [23] and output feedback control design [2], [8], [13]. However, one should notice that only few results are available regarding the classical problem of discrete-time state feedback control design; see, for instance, [13], [17] and [24]. These references consider the joint design of state feedback gains and a switching rule in order to ensure stability.

In this paper, our main goal is to propose a technique based on a piecewise quadratic Lyapunov function to solve the classical state feedback control design problem for discrete-time switched linear systems. Basically, we want to determine a set of state feedback gains and a switching function that ensure stability and a suitable level of $\mathcal{H}_2$ or $\mathcal{H}_\infty$ guaranteed cost for the closed-loop switched system. The conditions are based on Lyapunov or Riccati-Metzler inequalities and, for a class of Metzler matrices, they are expressed in terms of LMIs when a scalar variable is fixed. An example proposed in [17] is used for comparison and to put in evidence the importance of the joint design of the switching function and the state feedback control law. We have applied this theory to solve the self-triggered control design problem, [19], [21]. The study of networked control systems is a topic of great interest nowadays, since most of the current engineering applications require control signals to be transmitted through real communication channels. This fact imposes new challenges to the classical control theory, which has to be adapted to deal with channel imperfections, such as limited computational resources, time delay and limited bandwidth; see [10] for details. Our main goal is to apply the proposed state feedback switched control technique to solve the self-triggered control design problem, in which the sampling periods are chosen dynamically among those belonging to an available set in order to ensure an $\mathcal{H}_2$ performance level, satisfying the limited bandwidth constraints. The literature to date presents several results on sampled-data state-feedback control; see [5], [6], [11], [12], [14], [21] as examples. All these references but [21] consider only stability and adopt a control input...
with a single state feedback gain. In [21], the state feedback gains are calculated using the classical discrete-time LQR problem sequentially with respect to the switching rule. This rule is responsible for the scheduling of the time-varying sampling period to ensure a guaranteed $H_2$ performance level. Our objective is to design the feedback gains and the switching rule jointly in order to optimize performance. We have compared theoretically our technique with [11] and, more specifically, with the two recent papers [12], [21].

The notation is standard. For square matrices, $\text{Tr}(\cdot)$ denotes the trace function. For real matrices or vectors, $(\cdot)'$ indicates transpose. For symmetric matrices, the symbol $(\bullet)$ denotes each of its symmetric blocks. The set of natural number is denoted by $\mathbb{N}$. We also define an important subset of $\mathbb{N}$, given by $\mathbb{K} = \{1, 2, \cdots, N\}$. We consider the set $\mathcal{M}$ composed of all Metzler matrices $\Pi = \{\pi_{ij}\} \in \mathbb{R}^{N \times N}$ with nonnegative elements satisfying the normalization constraint $\sum_{j \in \mathbb{K}} \pi_{ji} = 1, \forall i \in \mathbb{K}$. As a result, each column of $\Pi \in \mathcal{M}$ belongs to the unitary simplex. Linear combinations of positive definite matrices and their inverses are denoted by $P_{pi} = \sum_{j \in \mathbb{K}} \pi_{ji}P_j$ and $P_{qi} = \sum_{j \in \mathbb{K}} \pi_{ji}P_j^{-1}$ for each $i \in \mathbb{K}$, respectively. A matrix is said to be Schur stable if all of its eigenvalues are in the region $\{z \in \mathbb{C} : |z| < 1\}$.

II. Problem Statement and Preliminaries

Consider a discrete-time switched linear system with the following state space realization

$$x(k + 1) = A_\sigma x(k) + B_\sigma u(k) + H_\sigma w(k)$$

$$z(k) = E_\sigma x(k) + F_\sigma u(k) + G_\sigma w(k)$$

evolving from zero initial condition $x(0) = 0$ where, for all $t \geq 0$, the vectors $x(t) \in \mathbb{R}^n$, $w(t) \in \mathbb{R}^r$, $u(t) \in \mathbb{R}^m$ and $z(t) \in \mathbb{R}^p$ are the state, the exogenous perturbation, the control input and the controlled output, respectively. The switching function denoted by $\sigma$ selects at each time instant $k \in \mathbb{N}$ one of the $N$ available subsystems. The main goal is to determine a set of state feedback gains $\{K_1, \cdots, K_N\}$ together with a state dependent switching function $\sigma(\cdot) : \mathbb{R}^n \rightarrow \mathbb{K}$ such that the control law

$$u(k) = K_{\sigma(x(k))}x(k)$$

ensures global asymptotic stability of the origin and a minimum $\mathcal{H}_2$ or $\mathcal{H}_\infty$ guaranteed cost. This problem appears in many practical applications. In special, it is well adapted to deal with the networked control problem to be addressed later. Throughout the paper, the piecewise quadratic Lyapunov function $v(x) = \min_{i \in \mathbb{K}} x^t P_i x$ is used to define the min-type switching strategy

$$\sigma(x) = \arg \min_{i \in \mathbb{K}} x^t P_i x$$

(4)

in which matrices $P_i > 0, \forall i \in \mathbb{K}$ satisfy some specific conditions in order to guarantee stability and performance. Considering system (1)-(2) with $u(k) = 0, \forall k \in \mathbb{N}$, depending on the class of the external perturbation $w$, the two performance indices to be considered are the $\mathcal{H}_2$ performance index, which is associated to an external perturbation of impulsive type $w(k) = e_q \delta(k)$, where $e_q \in \mathbb{R}^r$ is the $q^{th}$ column of the identity matrix, and the $\mathcal{H}_\infty$ performance index, which considers arbitrary perturbations of finite $\ell_2$ norm; both are defined in [3]. These indices are very difficult to calculate and, for this reason, the idea is to obtain an adequate upper bound for both of them, as those presented in [8] for the $\mathcal{H}_2$ case and in [2] for the $\mathcal{H}_\infty$ case.

According to [8], the existence of matrices $P_i > 0$ and a Metzler matrix $\Pi \in \mathcal{M}$ satisfying the so-called Lyapunov-Metzler inequalities

$$\begin{bmatrix} P_i & \ast & \ast \\ P_{pi} A_i & P_{pi} & \ast \\ E_i & 0 & I \end{bmatrix} > 0, \ i \in \mathbb{K}$$

(5)

ensures that the switching function (4) is globally stabilizing and the following upper bound

$$J_2(\sigma) < \min_{i \in \mathbb{K}} \text{tr}(H_i^t P_i H_i + G_i^t G_i)$$

(6)

holds, provided that we impose $\sigma(0) = i^*$, which is the optimal index associated to the right hand side of (6). On the other hand, for the $\mathcal{H}_\infty$ case, reference [2] establishes that the existence of the same previous matrices and a scalar $\rho > 0$ satisfying the so-called Riccati-Metzler inequalities

$$\begin{bmatrix} P_i & \ast & \ast & \ast \\ 0 & \rho I & \ast & \ast \\ P_{pi} A_i & P_{pi} H_i & P_{pi} & \ast \\ E_i & G_i & 0 & I \end{bmatrix} > 0, \ i \in \mathbb{K}$$

(7)
ensures that the switching function (4) is globally stabilizing and the upper bound \( J_\infty(\sigma) < \rho \) is valid. Note that no stability property is imposed to any of the matrices \( \{A_1, \cdots, A_N\} \) individually and, therefore, all of them can be unstable. Indeed, a necessary condition for feasibility of (5) and (7) is that each matrix \( \sqrt{\pi_{ii}} A_i, \ i \in \mathbb{K} \), must be Schur stable, but as \( 0 \leq \pi_{ii} \leq 1 \), since \( \Pi \in \mathcal{M} \), nothing is imposed to each isolated subsystem matrix \( A_i, \ i \in \mathbb{K} \).

### III. State Feedback Switched Control

Connecting the control law (3) to the system (1)-(2) we obtain

\[
\begin{align*}
x(k+1) &= (A_\sigma + B_\sigma K_\sigma)x(k) + H_\sigma w(k) \\
z(k) &= (E_\sigma + F_\sigma K_\sigma)x(k) + G_\sigma w(k)
\end{align*}
\]  

(8)  

(9)

evolving from \( x(0) = 0 \). The idea is to generalize the conditions of (5) and (7) to obtain the gains \( \{K_1, \cdots, K_N\} \) and the switching function \( \sigma(\cdot) \) ensuring an \( \mathcal{H}_2 \) or \( \mathcal{H}_\infty \) guaranteed cost for the closed-loop system. The next theorem considers the \( \mathcal{H}_2 \) case.

**Theorem 1:** If there exist symmetric matrices \( S_i, V_{ij} \), matrices \( J_i, Y_i \) for all \( i, j \in \mathbb{K} \times \mathbb{K} \) and a Metzler matrix \( \Pi \in \mathcal{M} \) satisfying the Lyapunov-Metzler inequalities

\[
\begin{bmatrix}
S_i & \bullet & \bullet \\
A_i S_i + B_i Y_i & V_i & \bullet \\
E_i S_i + F_i Y_i & 0 & I
\end{bmatrix} > 0, \ i \in \mathbb{K} 
\]  

(10)

\[
\begin{bmatrix}
V_{ij} & \bullet \\
J_i & S_j
\end{bmatrix} > 0, \ i, j \in \mathbb{K} \times \mathbb{K} 
\]  

(11)

with \( V_i = J_i + J_i' - \sum_{j=1}^{N} \pi_{ji} V_{ij} \), then the switching function (4) with \( S_i = P_i^{-1}, \ i \in \mathbb{K} \), and the state feedback gains \( K_i = Y_i S_i^{-1}, \ i \in \mathbb{K} \), make the origin a globally asymptotically stable equilibrium point for the closed-loop system (8)-(9), satisfying the inequality

\[
J_2(\sigma) < \min_{i \in \mathbb{K}} \text{tr}(H_i' S_i^{-1} H_i + G_i' G_i)
\]  

(12)

**Proof:** Assume that inequalities (10) and (11) are verified. From the fact that, for each \( i \in \mathbb{K} \),
the inequality $R_i^{-1} \geq Z_i + Z'_i - Z'_i R_i Z_i$ holds for any square matrices $Z_i$ and $R_i > 0$, we have

$$\left( \sum_{j=1}^{N} \pi_{ji} S_j^{-1} \right)^{-1} \geq J_i + J'_i - J'_i \left( \sum_{j=1}^{N} \pi_{ji} S_j^{-1} \right) J_i$$

$$> J_i + J'_i - \sum_{j=1}^{N} \pi_{ji} V_{ij}$$

$$> \mathcal{V}_i$$

(13)

where the second inequality comes from (11), by applying the Schur Complement with respect to the last row and column and remembering that $\Pi \in \mathcal{M}$. Consequently, from the validity of (10), the inequalities

$$\begin{bmatrix}
S_i & \bullet & \bullet \\
A_i S_i + B_i Y_i & S_{qi}^{-1} & \bullet \\
E_i S_i + F_i Y_i & 0 & I
\end{bmatrix} > 0, \ i \in \mathbb{K}$$

(14)

are also verified. Multiplying both sides of (14) by $\text{diag}\{P_i, P_{pi}, I\}$ and making the replacements $A_i \rightarrow A_i + B_i K_i$ and $E_i \rightarrow E_i + F_i K_i$ we obtain (5) proving, thus, the theorem.

This result was obtained without introducing any conservatism in the Lyapunov-Metzler inequalities (5). Indeed, making the feasible choices $J_i = S_{qi}^{-1}$ and $V_{ij} = S_{qi}^{-1} S_j^{-1} S_{qi}^{-1} + \varepsilon I$, with $\varepsilon > 0$ arbitrarily small, we have $\mathcal{V}_i = S_{qi}^{-1} - \varepsilon I$, which indicates that no conservatism was introduced in (13) and, therefore, the conditions are necessary and sufficient for the existence of a solution to the Lyapunov-Metzler inequalities. The minimum $\mathcal{H}_2$ guaranteed cost can be obtained by solving the following optimization problem

$$\min_{i \in \mathbb{K}} \inf_{\{S_i, V_{ij}, J_i, Y_i, \Pi\} \in \Psi_2} \text{Tr}(H_i' S_i^{-1} H_i + G_i' G_i)$$

(15)

where $\Psi_2$ is the set of all feasible solutions to inequalities (10)-(11). Note that solving this problem is not a simple task since inequalities (10) are nonconvex due to the product of variables $\pi_{ji} V_{ij}$ for all $i, j \in \mathbb{K} \times \mathbb{K}$. However, an alternative condition was proposed in [7] based on Metzler matrices with the same diagonal elements $\pi_{ii} = \gamma > 0, \ \forall i \in \mathbb{K}$. Although being more conservative, this condition, obtained from Theorem 1 replacing $\mathcal{V}_i$ by $\gamma V_i + (1 - \gamma) V_j$ for all $i \neq j \in \mathbb{K} \times \mathbb{K}$ and making $V_{ij} = V_j$ for all $i, j \in \mathbb{K} \times \mathbb{K}$, can be solved using a line search with respect to the scalar $0 \leq \gamma < 1$ and a set of LMIs.
The next theorem presents the results for the $\mathcal{H}_\infty$ case.

**Theorem 2:** For a given scalar $\rho > 0$, if there exist symmetric matrices $S_i$, $V_{ij}$, matrices $J_i$, $Y_i$ for all $i, j \in \mathbb{K} \times \mathbb{K}$ and a Metzler matrix $\Pi \in \mathcal{M}$ satisfying the Riccati-Metzler inequalities

\[
\begin{bmatrix}
S_i & \bullet & \bullet & \bullet \\
0 & \rho I & \bullet & \bullet \\
A_i S_i + B_i Y_i & H_i & V_i & \bullet \\
E_i S_i + F_i Y_i & G_i & 0 & I
\end{bmatrix} > 0, \ i \in \mathbb{K} \quad (16)
\]

and (11) with $V_i = J_i + J_i' - \sum_{j=1}^{N} \pi_{ij} V_{ij}$, then the switching function (4) with $S_i = P_i^{-1}$, $i \in \mathbb{K}$, and the state feedback gains $K_i = Y_i S_i^{-1}$, $i \in \mathbb{K}$, make the origin a globally asymptotically stable equilibrium point for the closed-loop system (8)-(9) and the inequality $J_\infty(\sigma) < \rho$ holds.

**Proof:** The proof is similar to that of Theorem 1. \[\square\]

One should note that the existence of state feedback gains $\{K_1, \ldots, K_N\}$ such that the closed-loop systems $A_i + B_i K_i$, for $i \in \mathbb{K}$, are Schur stable is not required by neither of the theorems. Hence, without this requirement, the gains can be calculated more freely in order to enhance performance. Moreover, similarly to the $\mathcal{H}_2$ case, the best $\mathcal{H}_\infty$ guaranteed cost can be obtained by solving the following optimization problem

\[
\rho^* = \inf_{\{S_i, V_{ij}, J_i, Y_i, \Pi\} \in \Psi_\infty} \rho \quad (17)
\]

where $\Psi_\infty$ is the set of all feasible solutions to inequalities (11) and (16). Once again, it is worth remembering that although these inequalities are difficult to solve due to the product of variables $\pi_{ij} V_{ij}$ for all $i, j \in \mathbb{K} \times \mathbb{K}$, simpler alternative conditions can be obtained by adopting Metzler matrices with $\pi_{ii} = \gamma > 0$. The next example emphasizes the importance of the joint design of the switching rule and the state feedback gains.

**Example 1:** Consider the data from Example 1 of [17]. Theorem 2 has been slightly modified in order to cope with exponential stability and $\mathcal{H}_\infty$ performance simultaneously. The first diagonal element of inequality (16) has been changed to $\xi S_i$ with $0 < \xi \leq 1$. Figure 1 shows the curves $\sqrt{\rho_{\text{sub}}}$ against $\xi \in [0.25, 1]$ obtained from the suboptimal solution to the convex problem (17).
with given Metzler matrices of the form

$$\Pi = \begin{bmatrix} p & (1-q) \\ (1-p) & q \end{bmatrix}$$

(18)

Four cases corresponding to the vertices \((p, q)\) of the box \([0, 1] \times [0, 1]\) have been considered and the following remarks can be drawn. First, feasibility is preserved for \(\xi = 0.25\) which is much smaller than the minimum value \(\xi = 0.7275\) reported in [17]. Second, for \(\xi = 1\) the suboptimal solution provided by (18) with \((p, q) = (1, 0)\) ensures \(\sqrt{\rho_{sub}} = 1.0404\), which is significantly better than the value \(\sqrt{\rho} = 1.6508\) reported in [17]. Hence, even though the optimal solution to problem (17) is not exactly calculated, our design conditions outperform the ones proposed in [17]. Figure 1 also puts in clear evidence that the choice of \(\Pi \in \mathcal{M}\) is a key issue to improve performance. As we have commented before the determination of the optimal Metzler matrix is not a simple task due to the nonconvex nature of the optimization problem to be handled. Hence, suboptimal solutions as those proposed in this example are useful.

**IV. Networked Control**

Consider a continuous-time system with the state space realization

$$\dot{x}(t) = Ax(t) + Bu_k(t) + Hw(t)$$

(19)

$$z(t) = Ex(t) + Fu_k(t)$$

(20)
evolving from zero initial condition, where the control input $u_k(t)$ is transmitted through a limited bandwidth channel whose sampling periods $T_i$, $i \in \mathbb{K}$, are defined by the designer and respect the minimum period $T_\ast > 0$ allowed for transmission, that means, $T_i \geq T_\ast$ for all $i \in \mathbb{K}$. These sampling periods constitute the set $\mathcal{T} = \{T_i, i \in \mathbb{K}\}$. Assuming that the quantization effects can be ignored, the control input is modeled as a piecewise constant signal $u_k(t) = u(t_k)$, $\forall t \in [t_k, t_{k+1})$, for all $k \in \mathbb{N}$, where it is assumed that each time interval between two successive sampling instants is such that $t_{k+1} - t_k \in \mathcal{T}$, $\forall k \in \mathbb{N}$. As in [21], we can define the self-triggered control design problem, where each sampling period has to be selected taking into account two important (and possibly conflicting) characteristics that are inherent in the networked control problem: $\mathcal{H}_2$ performance, which generally induces small values for the sampling period, and limited bandwidth, which constrains this behavior. The main goal is to design a control law $u_k(t)$ and a switching rule $\sigma(x(t_k))$ that selects, at each time instant $k \in \mathbb{N}$, one of the $N$ sampling periods $T_i$, $i \in \mathbb{K}$, belonging to $\mathcal{T}$.

Considering that the external input $w(t)$, $t \geq 0$, is of the impulsive type $w(t) = e_q \delta(t)$, with $e_q$ being the $q$-th column of the identity matrix of compatible dimension, the continuous-time system (19)-(20) with $x(0) = 0$, can be alternatively written as the same equations but with $w(t) = 0$ for all $t \geq 0$ and evolving from $x(0) = H e_q$. Introducing the augmented matrices

$$
\mathcal{A} = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}, \quad \mathcal{C} = \begin{bmatrix} E & F \end{bmatrix}
$$

the next lemma presents, for each element of the set $\mathcal{T}$, what we call an exact discretization of the continuous-time system.

**Lemma 1:** Consider the continuous-time system (19)-(20) with $w(t) = 0$ for all $t \geq 0$ and evolving from the initial condition $x(0) = x_0 \in \mathbb{R}^n$. For each $T_i \in \mathcal{T}$, it is possible to define matrices $(A_{di}, B_{di}, E_{di}, F_{di})$ with compatible dimensions such that

$$
e^{\mathcal{A}T_i} = \begin{bmatrix} A_{di} & B_{di} \\ 0 & I \end{bmatrix}
$$

and

$$
\int_0^{T_i} e^{\mathcal{A}'t'} \mathcal{C} e^{\mathcal{A}t} dt = \begin{bmatrix} E_{di}' \\ F_{di}' \end{bmatrix}
$$

(23)
Moreover, denoting $u(k) = u(t_k)$, the equivalent discrete-time system defined by

\[ x(k + 1) = A_{di}x(k) + B_{di}u(k), \quad x(0) = x_0 \tag{24} \]

\[ z(k) = E_{di}x(k) + F_{di}u(k) \tag{25} \]

is such that the following equality holds

\[ \int_0^\infty z(t)'z(t)\,dt = \sum_{k=0}^\infty z(k)'z(k) \tag{26} \]

**Proof:** The proof is given in [1]. \qed

This lemma presents a discretization method for which, given a sampling period $T_s$, the norm of the discrete-time output trajectory $z(k)$, $\forall k \in \mathbb{N}$, equals the norm of the continuous-time output trajectory $z(t)$, $\forall t \geq 0$, and, for this reason, we call it an exact discretization technique. In addition, remembering that the sampling period must be conveniently selected by a switching rule $\sigma(\cdot)$, we can define the following discrete-time switched linear system

\[ x(k + 1) = A_{\sigma}x(k) + B_{\sigma}u(k), \quad x(0) = x_0 \tag{27} \]

\[ z(k) = E_{\sigma}x(k) + F_{\sigma}u(k) \tag{28} \]

After obtaining the equivalent discrete-time system, our proposal is to design a set of state feedback gains $\{K_1, \cdots, K_N\}$ together with a switching rule $\sigma(x)$ such that the control law (4) ensures stability and a guaranteed $\mathcal{H}_2$ performance level for the system (27)-(28). Note that, this problem is exactly the same as stated in Section II and its solution is obtained from the result of Theorem 1. Similar procedure can be done for the $\mathcal{H}_\infty$ case whenever the external input is supposed to be transmitted through the limited bandwidth channel; see [21] for details.

A. State dependent sequential design

Instead of a joint design, a set of state feedback control gains $K_i$, $i \in \mathbb{K}$, and a state dependent switching rule $\sigma(x(k))$ are designed sequentially. That is, first the gains $K_i$, $i \in \mathbb{K}$, are chosen and then the switching rule is designed for that set of specific state feedback gains, which are naturally chosen to be the optimal ones. Indeed, from Lemma 1 the optimal state feedback gain
$K_i$ associated with the sampling period $T_i$, $i \in \mathbb{K}$, is given from the solution to the problem

$$\min_{K_i} \| (E_{di} + F_{di}K_i)(zI - (A_{di} + B_{di}K_i))^{-1}H \|^2_2$$

(29)

Hence, for each subsystem, the optimal state feedback gain is calculated by solving the well-known LQR design problem for an LTI discrete-time system. The switching rule to be implemented is given in (4), where matrices $P_i > 0$, $i \in \mathbb{K}$, are obtained from the minimization of the guaranteed cost (6) subject to the Lyapunov-Metzler inequalities (5), making the following replacements: $A_{di} + B_{di}K_i \rightarrow A_i$ and $E_{di} + F_{di}K_i \rightarrow E_i$, $H_i \rightarrow H$ and $G_i = 0$, for all $i \in \mathbb{K}$.

**Example 2:** Consider the system given in (19)-(20) with

$$A = \begin{bmatrix} 0 & 1 \\ -6 & 1 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, H = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, F = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

and sampling periods $T_1 = 1.35$ [s] and $T_2 = 2.40$ [s], see [21]. Solving problem (29), we obtain a minimum LQR cost equal to $J_2 = 482.5776$ for $i = 1$ and $J_2 = 476.9576$ for $i = 2$. Plugging these gains into the sequential procedure with a Metzler matrix of the form (18) with $p = 1$ and $q = 0$ we are able to calculate the state dependent sampling strategy $\sigma(x)$ given in (4) that reduces significantly the cost to $J_2 = 283.1828$. However, applying Theorem 1, with the Metzler matrix defined by $p = q = 0$, the joint design provides the state feedback gains $K_1$, $K_2$ and state dependent sampling strategy $\sigma(x)$ that promotes again an expressive cost reduction to $J_2 = 256.5348$.

This simple example puts in evidence a much more general property. In fact, comparing to the sequential design, the joint one always produces a state dependent sampling rule and a set of state feedback gains that impose to the closed-loop system a better or at least the same $\mathcal{H}_2$ performance. Indeed, this is true because the sequential control design procedure is alternatively obtained from the optimal solution to problem (15) with the additional linear constraints $K_iS_i = Y_i$ where $K_i$ is the optimal solution to (29), for each $i \in \mathbb{K}$.

**B. State dependent sampling design**

Now, we compare our previous results with those of [12] where the authors consider only the stability problem without defining a performance index. For more details see [12] and the references therein. The system to be handled is (19) with zero input $w(t) = 0$, $t \geq 0$, and control
law defined by a single state feedback gain \( u_k(t) = Kx_k(t) \) such that the closed-loop system matrix \((A + BK)\) is Hurwitz. For the stability analysis, the system of interest is \( x(k+1) = \Phi_x x(k) \), where \( \Phi_x = \prod_{j=1}^{l_\ell} (A_{dj} + B_{dj}K) \) and where each pair \((A_{dj}, B_{dj})\), \( j \in \mathbb{K}\), can be readily determined from Lemma 1 with the adoption of the sampling period \( T_j \in T, j \in \mathbb{K}\). Moreover, taking into account that \( l_{\min} \leq l_\ell \leq l_{\max} \), for each \( \ell \in \mathbb{N}\), define \( \Theta \) as the subset constituted by all matrices \( \Phi_x \) that are Schur stable.

The result reported in [12] is that with matrices \( P_\ell > 0 \) satisfying

\[
\Phi_x'P_\ell\Phi_x - P_\ell < 0, \ \ell \in \Theta
\]  

the state dependent switching rule \( \sigma(x) = \arg\min_{\ell \in \Theta} x'\Phi_x'P_\ell\Phi_x x \) is stabilizing. Now, following the proof of Theorem 1, see also [7], it can be shown that if (30) is replaced by the Lyapunov-Metzler inequalities

\[
\Phi_x'P_{p\ell}\Phi_x - P_\ell < 0, \ \ell \in \Theta
\]  

where \( P_{p\ell} = \sum_{j \in \Theta} \pi_{j\ell}P_j \), with \( P_j > 0 \), for all \( j \in \Theta \), and \( \Pi \in \mathcal{M} \), then the state dependent strategy \( \sigma(x) = \arg\min_{\ell \in \Theta} x'P_\ell x \) is stabilizing. Indeed, adopting the min-type Lyapunov function \( v(x) = \min_{i \in \Theta} x'P_i x \) and assuming that \( \sigma(x(k)) = \arg\min_{i \in \Theta} x(k)'P_i x(k) = \ell \), we have

\[
v(x(k+1)) = \min_{i \in \Theta} x(k)'\Phi_x'P_i\Phi_x x(k) \\
\leq \sum_{j \in \Theta} \pi_{j\ell} x(k)'\Phi_x'P_j\Phi_x x(k) \\
< v(x(k)) \ \forall x(k) \neq 0
\]

where we have used the fact that (31) holds and each column of \( \Pi \in \mathcal{M} \) belongs to the unitary simplex, that is \( \pi_{j\ell} \geq 0, \forall j \in \Theta \) and \( \sum_{j \in \Theta} \pi_{j\ell} = 1 \). From this we can draw the following conclusions:

- Setting \( \Pi = I \in \mathcal{M} \) which leads to \( P_{p\ell} = P_\ell \) for all \( \ell \in \Theta \), we conclude that (30) is a particular case of (31) which means that our approach produces less conservative stability conditions. Furthermore, the set \( \Theta \) does not need to be constituted by Schur stable matrices, exclusively. This additional degree of freedom represented by the choice of the Metzler matrix \( \Pi \in \mathcal{M} \) is particularly important when a performance index has to be taken into account due to the consistency property of this class of min-type switching function; see [3] for details.
• The product that appears in the calculation of the transition matrix \( \Phi_\ell \) makes virtually impossible to jointly determine the matrices \( P_\ell \) and \( K \). The same seems to be true if we wish to optimize some performance index. From the previous discussion (see Example 2), this is a design limitation that is not present in the conditions of Theorems 1 and 2.

**Example 3:** This example has been considered in [12]. The state space matrices of the continuous-time system are

\[
A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0.6 \end{bmatrix}, \quad K = \begin{bmatrix} -1 \\ -6 \end{bmatrix}
\]

As indicated in [12], constant sampling periods preserve stability in the interval \( T \in (0, 0.59] \).

Considering \( \mathcal{T} = \{T_a, T_b\} \) we have verified that the Lyapunov-Metzler inequalities \( \tilde{A}_{di}P_{pi}\tilde{A}_{di} - P_i < 0 \) with \( \Pi \in \mathcal{M} \), \( P_i > 0 \) and \( \tilde{A}_{di} = A_{di} + B_{di}K \) for both \( i = 1, 2 \) are feasible in all points of the box \((T_a, T_b) \in (0, 1] \times (0, 1] \) except those such that \( \min\{T_a, T_b\} > 0.59 \).

Compared to the stability region provided in [12] this is an important improvement. It is interesting to mention that, for the constant matrix \( \Pi \) with structure (18) and \((p, q) = (0, 0)\), the stability region given in [12] coincides with the feasibility region of the Lyapunov-Metzler inequalities. In addition, for \((T_a, T_b) = (0.28, 0.58)\), which is outside of the stability region provided in [12], with \((p, q) = (0.8, 0.3)\) the previous Lyapunov-Metzler inequalities are feasible being thus possible to stabilize the system with a sampling period determined by the state dependent switching strategy provided by Theorem 1 with the additional linear constraints \( KS_i = Y_i, i \in \mathbb{K} \).

Figure 2 shows the closed-loop system trajectories and the time varying sampling period evolving from \( x(0) = [-1 \ 1]' \). The choice of matrix \( \Pi \in \mathcal{M} \) is the key issue to reduce conservatism.

**V. Conclusion**

In this paper, we have proposed a state feedback switched control design for discrete-time switched linear systems. The technique consists in designing jointly a set of state feedback gains and a switching function ensuring stability and a suitable \( \mathcal{H}_2 \) or \( \mathcal{H}_\infty \) performance level.

The conditions are based on Lyapunov or Riccati-Metzler inequalities, which are non-convex but allow the derivation of easier to solve alternative conditions. This technique was applied to the self-triggered control design problem, in which the switching function selects, at each time instant, one of the available sampling periods in order to improve \( \mathcal{H}_2 \) performance. As it was
confirmed through numerical examples, this method is less conservative and outperforms other strategies available in the literature.

REFERENCES


