Asymptotic Formula for $(1 + 1/x)^x$, Revisited

Vadim Ponomarenko

In a recent note [1] appearing in this Monthly, Chen and Choi calculated $a_j$ in

\[
(1 + \frac{1}{x})^x = \sum_{j=0}^{\infty} \frac{a_j}{x^j}
\]  

We offer a briefer calculation of these same coefficients. Applying the change of variables $y = \frac{1}{x}$, and the two functions $f(y) = e^y$, $g(y) = \frac{\ln(1+y)}{y}$, we rewrite (1) as

\[
f(g(y)) = \sum_{j=0}^{\infty} (ea_j)y^j
\]

We take derivatives of this formal power series $j$ times, and substitute $y = 0$, to get

\[
\frac{d^j}{dy^j} f(g(y))|_{y=0} = e(j!)a_j
\]  

The left side of (2) may be calculated with Faa di Bruno’s famous formula, which gives

\[
\frac{d^j}{dy^j} f(g(y)) = \sum \frac{j!}{k_1! \cdots k_j!} f^{(k_1 + \cdots + k_j)}(g(y)) \left( \frac{g^{(1)}(y)}{1!} \right)^{k_1} \cdots \left( \frac{g^{(j)}(y)}{j!} \right)^{k_j}
\]

where the sum is taken over all solutions to $k_1 + 2k_2 + \cdots + jk_j = j$. Because $g(y) = 1 - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \cdots$, we have $g^{(l)}(y)|{y=0} = (-1)^l \frac{l!}{l+1}$. We also have $\lim_{y \to 0} f^{(k_1 + \cdots + k_j)}(g(y)) = \lim_{y \to 0} f(g(y)) = e$. Combining, we get

\[
e(j!) \sum \frac{1}{k_1! \cdots k_j!} \left( \frac{1}{2} \right)^{k_1} \cdots \left( \frac{1}{j+1} \right)^{k_j} (-1)^{k_1+2k_2+\cdots+jk_j} = e(j!)a_j
\]

We now cancel $e(j!)$ from both sides, and use the $k_1 + 2k_2 + \cdots + jk_j = j$ restriction, to get the main result from [1]:

\[
(-1)^j \sum \frac{1}{k_1! \cdots k_j!} \left( \frac{1}{2} \right)^{k_1} \cdots \left( \frac{1}{j+1} \right)^{k_j} = a_j
\]

REFERENCES