ON BIRATIONAL GEOMETRY OF SINGULAR DEL PEZZO FIBRATIONS

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Abstract. In this article, birational geometry of singular 3-fold del Pezzo fibrations of degree two is considered. After reviewing the very limited literature on this topic, I investigate three aspects of research closely related to this problem. First, I consider a known conjecture about birational rigidity that holds for smooth models and is conjectured in general. I provide a counterexample for the singular case. Second, I highlight the relation between the study of some conjugacy classes in the rank three Cremona group and birational properties of singular del Pezzo fibrations. Third, I study the degree of irrationality of smooth Fano 3-folds and show that the study can be completed if one obtains certain rigidity results for singular del Pezzo fibrations.

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1. Introduction

A smooth surface $S$ with ample anti canonical class, $-K_S$, is called a del Pezzo surface. The degree of $S$ is defined to be the self-intersection number $K_S^2$. A pencil of such surfaces typically forms a so-called Mori fibre space. Mori fibre spaces are a possible outcome of the minimal model program. See Definition 1.1 below for a precise definition and details of conditions for being a Mori fibre space. These varieties play a central role in birational geometry of 3-folds. The study of relations between outcomes of the minimal model program is a rather modern problem in algebraic geometry. In this article, I present some aspects of this problem for pencils of del Pezzo surfaces of degree 2, equipped with some singular points that are allowed in the Mori category.

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Key words and phrases. Birational Automorphism; Mori Fibre Space; Sarkisov Program; Del Pezzo Fibration; Birational Rigidity; Cremona Group; Degree of Irrationality.
Definition 1.1. A Mori fibre space is a variety $X$ together with a morphism $\varphi: X \rightarrow Z$ such that

(i) $X$ is $\mathbb{Q}$-factorial and has at worst terminal singularities,
(ii) $-K_X$, the anti-canonical class of $X$, is $\varphi$-ample,
(iii) $\dim Z < \dim X$,
(iv) $\text{rank Pic}(X) - \text{rank Pic}(Z) = 1$.

A $G$-Mori fibre space for a group $G$, is a variety $X$, where $G$ acts faithfully on $X$, together with a $G$-equivariant morphism $\varphi: X \rightarrow Z$ such that

(i) $X$ is $\mathbb{Q}$-factorial and has at worst terminal singularities,
(ii) $-K_X$, the anti-canonical class of $X$, is $\varphi$-ample,
(iii) $\dim Z < \dim X$,
(iv) $\text{rank Pic}^G(X) - \text{rank Pic}^G(Z) = 1$.

I use abbreviations Mfs, $G$-Mfs and MMP for, respectively, Mori fibre spaces, $G$-Mori fibre spaces and minimal model program, when the group $G$ is known.

A main goal in the birational geometry of 3-folds is to study the geometry of Mfs in dimension three. Here, I consider the case where the fibration is over a curve, in particular the projective line. The other cases are when $\dim Z = 2$, the conic bundle case, or when $Z$ is a point, the Fano case, which will not be treated in this article, although some of our results relate directly to these cases. For the $\dim Z = 1$, it follows from the definition that the fibres are del Pezzo surfaces. If $X$ is smooth and the degree of the del Pezzo surface is $d = K^2_{\eta} \geq 5$, where $\eta$ the generic fibre, then $X_{\eta}$ is rational over the base and hence $X$ is rational. It is known that this is not always the case for $d \leq 4$, see [5] and [13]. Far away from the concept of rationality lies the notion of birational rigidity.

Definition 1.2. Let $X \rightarrow Z$ and $X' \rightarrow Z'$ be Mori fibre spaces. A birational map $f: X \dasharrow X'$ is square if it fits into a commutative diagram

$$
\begin{array}{ccc}
X & \xrightarrow{f} & X' \\
\downarrow & & \downarrow \\
Z & \xrightarrow{g} & Z'
\end{array}
$$

where $g$ is birational and, in addition, the map $f_L: X_L \dasharrow X'_L$ induced on generic fibres is biregular. In this case we say that $X/Z$ and $X'/Z'$ are square birational. We denote this by $X/Z \sim X'/Z'$.

Definition 1.3. The pliability of a Mfs $X \rightarrow Z$ is the set

$$
\mathcal{P}(X/Z) = \{\text{Mfs } Y \rightarrow T \mid X \text{ is birational to } Y \}/ \sim
$$

A Mfs $X \rightarrow Z$ is said to be birationally rigid if $\mathcal{P}(X/Z)$ contains a single element. It is called birationally super-rigid if $\text{Bir}(X) = \text{Aut}(X)$.

It is easy to deduce that a fibration of del Pezzo surfaces of degree $d$ over $\mathbb{P}^1$, denoted from now on by $dP_d/\mathbb{P}^1$, is not birationally rigid, when the total space is smooth and
For $d \geq 4$, the rationality of the 3-fold implies non-rigidity and it was shown in [5] that $dP_3/\mathbb{P}^1$ are birational to conic bundles. Understanding conditions under which a $dP_n/\mathbb{P}^1$, for $n \leq 3$, is birational rigid is a key step in providing the full picture of MMP, and hence the classification, in dimension three. Birational rigidity for the smooth models of degree 1, 2 and 3 is well studied, see for example [40]. However, as explained in Section 2, while the smoothness assumption for degree $s$ is only a generality assumption, considering the smooth case for $d = 1, 2$ is not very natural. Hence the necessity of considering singular cases is apparent. On the other hand, as shown in Sections 3 and 4, birational geometry of singular $dP_2/\mathbb{P}^1$ ties closely with questions in conjugacy classes in the Cremona group and the degree of irrationality of a smooth quartic 3-fold.

Structure of the article

Section 2: I focus on a well-known conjecture (2.5) on this topic that connects birational rigidity of del Pezzo fibrations of low degree to the structure of their mobile cone. A counterexample to this conjecture is provided when the 3-fold admits certain singularities. The construction of the counterexample is very precise and technical so that it is natural to expect that the conjecture holds after some modifications.

Section 3: The study of the Cremona group $\text{Cr}_n$, the group of birational self-maps of the projective space $\mathbb{P}^n$, and the plane $\mathbb{P}^2$ in particular, is a classical problem. In Section 3, I show the relation between the embedding of certain subgroups of the Cremona group of rank three $\text{Cr}_3$ and the birational rigidity of certain singular $dP_2/\mathbb{P}^1$.

Section 4: Another classical problem in algebraic geometry is the rationality of a smooth quartic 3-fold, which was proven to be non-rational in [29]. A follow up question is to ask what is the smallest number $m$ for which there exists an $m$-to-1 rational dominant map from a smooth quartic 3-fold to $\mathbb{P}^3$, this number measures how irrational the variety is, and it can be viewed as the generalisation of the notion of gonality for algebraic curves. In Section 4, I study this number for Fano 3-folds, and the quartic 3-fold in particular. It is shown that this number is 2 for a general smooth quartic 3-fold, and all other smooth Fano 3-folds. Then I show that the generality assumption for the quartic case can be lifted if one obtains some rigidity results on a singular $dP_2/\mathbb{P}^1$.

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2. Birational rigidity, mobile and Mori cones

There are already two comprehensive and beautifully written surveys on birational rigidity of Mori fibre spaces by Pukhlikov [41,42]. I strongly recommend the reader interested in this topic to consult these two articles. Here, I concentrate on birational geometry of low degree del Pezzo fibrations, with emphasis on their singularities. Pukhlikov in [40] proved that a general smooth $dP_3/\mathbb{P}^1$ is birationally rigid if the class of 1-cycles
$mK_X - L$ is not effective for any $m \in \mathbb{Z}$, where $L$ is the class of a line in a fibre. This condition is famously known as the $K^2$-condition.

**Definition 2.1.** A del Pezzo fibration is said to satisfy $K^2$-condition if the 1-cycle $K^2$ does not lie in the interior of the Mori cone.

The rationality question for these varieties was investigated in [13]. The birational rigidity of smooth $dP_d/\mathbb{P}^1$ for $d = 1, 2$ was also considered in [40] and the criteria for rigidity are similar to that for $d = 3$. However, it is not natural to only consider the smooth case for $d = 1, 2$ as these varieties very often carry some orbifold singularities inherited from the ambient space. For example a del Pezzo surface of degree 2 is naturally embedded as a quartic hypersurface in the weighted projective space $\mathbb{P}(1, 1, 1, 2)$. It is natural that a family of these surfaces meets the singular point $1/2(1, 1, 1)$. See [1] for construction of models and the study of their birational structure. Some explicit examples of this phenomenon are also presented within the body of this article.

In a sequential work [20–26], Grinenko realised and argued evidently that it is more natural to consider $K$-condition instead of the $K^2$-condition.

**Definition 2.2.** A del Pezzo fibration is said to satisfy $K$-condition if the anticanonical divisor does not lie in the interior of the Mobile cone.

**Lemma 2.3.** $K^2$-condition implies $K$-condition.

*Proof.* This is an easy exercise. \qed

One of the most significant observations of Grinenko was the following theorem.

**Theorem 2.4.** [25, 26] Let $X$ be a smooth 3-fold Mori fibration of del Pezzo surfaces of degree 1 or 2, or a general degree 3, over $\mathbb{P}^1$. Then $X$ is birationally rigid if $-K_X \notin \text{Int}(\text{Mob}(X))$.

He then speculated that this must hold in general, as formulated in the conjecture below, with no restriction on the singularities.

**Conjecture 2.5.** ([26] Conjecture 1.5 and [23] Conjecture 1.6) Let $X$ be a 3-fold Mori fibration of del Pezzo surfaces of degree 1, 2 or 3 over $\mathbb{P}^1$. Then $X$ is birationally rigid if and only if $-K_X \notin \text{Int}(\text{Mob}(X))$.

Grinenko also constructed many nontrivial examples, which supported his arguments. The study of quasi-smooth models of $dP_2/\mathbb{P}^1$ in [1], i.e. models that typically carry a quotient singularity, also gives evidence that the relation between birational rigidity and the position of $-K$ in the mobile cone is not affected by the presence of the non-Gorenstien point. On the other hand, in [10], Example 4.4.4, it was shown that this conjecture does not hold in general for the degree 3 case (of course in the singular case) and suggested that one must consider the semi-stability condition on the 3-fold $X$ in order to state an updated conjecture:

**Conjecture 2.6** ([10], Conjecture 2.7). Let $X$ be a $dP_3/\mathbb{P}^1$ which is semistable in the sense of Kollár [31]. Then $X$ is birationally rigid if $-K_X \notin \text{Int}(\text{Mob}(X))$. 
The general speculation is that Grinenko’s conjecture might hold if one only considers Gorenstien singularities. Below in Example 2.7 I give a counterexample to Conjecture 2.5 for a Gorenstien singular degree 2 del Pezzo fibration. Although this type of (counter)examples are very difficult to produce, it is expected that it is possible to produce one also in degree 1. On the other hand, a notion of (semi)stability for del Pezzo fibrations of degree 1 and 2 seems necessary (as already noted in [17] Problem 5.9.1) and yet there has been no serious attempt in this direction.

The most natural construction of smooth $dP_2/\mathbb{P}^1$ is the following (see [26]).

Let $E = \mathcal{O} \oplus \mathcal{O}(a) \oplus \mathcal{O}(b)$ be a rank 3 vector bundle over $\mathbb{P}^1$ for some positive integers $a, b$, and let $V = \text{Proj}_{\mathbb{P}^1} E$. Denote the class of the tautological bundle on $V$ by $M$ and the class of a fibre by $L$ so that $\text{Pic}(V) = \mathbb{Z}[M] + \mathbb{Z}[L]$

Assume $\sigma: X \to V$ is a double cover branched over a smooth divisor $R \sim 4M - 2eL$, for some integer $e$. The natural projection $p: V \to \mathbb{P}^1$ induces a morphism $\pi: X \to \mathbb{P}^1$, such that the fibres are del Pezzo surfaces of degree 2 embedded as quartic surfaces in $\mathbb{P}(1, 1, 1, 2)$. This 3-fold $X$ can also be viewed as a hypersurface of a rank two toric variety. Let $T$ be a toric fourfold with Cox ring $\mathbb{C}[u, v, x, y, z, t]$, that is $\mathbb{Z}_2$-graded by

$$
\begin{pmatrix}
u & x & y & z & t \\
1 & 1 & 0 & -a & -b & -e \\
0 & 0 & 1 & 1 & 1 & 2
\end{pmatrix}
$$

The 3-fold $X$ is defined by the vanishing of a general polynomial of degree $(-e, 4)$. This construction can now be generalised to non-Gorenstein models. As before, let $X$ be defined by the vanishing of a polynomial of degree $(-e, 4)$ but change the grading on $T$ to

$$
\begin{pmatrix}
u & x & y & z & t \\
1 & 1 & -a & -b & -c & -d \\
0 & 0 & 1 & 1 & 1 & 2
\end{pmatrix}
$$

where $c$ and $d$ are positive integers. It is easy to check that, assuming $f$ imposes no singularities, $X$ is smooth if and only if $e = 2d$. See [1] for more details and construction.

The cone of numerically effective divisors on $T$ is generated by the toric principal divisors, associate to columns of the matrix above, and we have $\text{Eff}(T) \subset \mathbb{Z}^2$. This cone decomposes, as a chamber, into a finite union of subcones

$$\text{Eff}(T) = \bigcup \text{Nef}(T_i)$$

where $T_i$ are toric varieties isomorphic to $T$ in codimension 1. In other words, they are obtained by the variation of geometric invariant theory on the Cox ring of $T$, see [18] and [11]. This is in principle the toric 2-ray game on $T$, see [17] for an explanation of 2-ray games and Sarkisov program. In certain cases, when the del Pezzo fibration is a Mori dream space, i.e. it has a finitely generated Cox ring, its 2-ray game is realised by restricting the 2-ray game of the toric ambient space. In other words, in order to trace the Sarkisov link one runs the 2-ray game on $T$, restricts it to $X$ and checks whether the game remains in the Sarkisov category, in which case a winning game is obtained and
a birational map to another Mfs is constructed. See [1–4, 11] for explicit constructions of these models for del Pezzo fibrations or blow ups of Fano 3-folds. I demonstrate this method in the following example, which also shows that Conjecture 2.5 does not hold.

**Example 2.7.** Assume the grading on $\text{Cox}(T) = \mathbb{C}[u, v, x, t, y, z]$, that is the Cox ring of the toric variety $T$, is given by the matrix

$$A = \begin{pmatrix} u & v & x & t & y & z \\ 1 & 1 & 0 & -2 & -2 & -4 \\ 0 & 0 & 1 & 2 & 1 & 1 \end{pmatrix}$$

and let the irrelevant ideal defining $T$ be $I = (u, v) \cap (x, y, z, t)$. In other words $T$ is the geometric quotient with character $\psi = (-1, -1)$. Let $X \subset \mathbb{F}$ be a hypersurface defined by a polynomial $f$ of bi-degree $(-4, 4)$. A general $f$ is obtained by a combination of monomials from the following table

<table>
<thead>
<tr>
<th>deg of $u, v$ coefficient</th>
<th>0</th>
<th>2</th>
<th>2</th>
<th>6</th>
<th>8</th>
<th>10</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>fibre monomials</td>
<td>$x^3z$</td>
<td>$xy^3$</td>
<td>$yzt$</td>
<td>$xy^2z$</td>
<td>$tx^2$</td>
<td>$y^2z^2$</td>
<td>$y^3z^2$</td>
</tr>
<tr>
<td></td>
<td>$t^2$</td>
<td>$ty^2$</td>
<td>$x^2yz$</td>
<td>$y^4z$</td>
<td>$x^2z^2$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Now let $X$ be defined by a special $f$ with the property that $u^i$ divides the coefficient polynomials according to the table

<table>
<thead>
<tr>
<th>power of $u$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>8</th>
<th>9</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>monomial</td>
<td>$x^2yz$</td>
<td>$xzt$</td>
<td>$yzt$</td>
<td>$x^2z^2$</td>
<td>$xy^2z$</td>
<td>$tx^2$</td>
<td>$y^2z^2$</td>
<td>$y^3z^2$</td>
<td>$z^4$</td>
</tr>
<tr>
<td></td>
<td>$xy^2$</td>
<td>$y^3z$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

and general coefficients otherwise. A simple computation using the Bertini theorem shows that $X$ is smooth away from the point $p = (0 : 1 : 0 : 0 : 1)$. The germ at this point is $0 \in (\mathbb{C}^4, t^2 + x^3 + y^4 + u^{12})$; in particular $p$ is a $cE_6$ singularity. The 2-ray game of $T$ restricts to a game on $X$, guaranteed by the appearance of the monomials $t^2$ and $x^3z$ for instance, and end with a bad link to a weak Fano 3-fold. In particular the cone of mobile divisors on $X$ is provided by $T$ and is described as

$$\text{Mob}(X) = \text{Convex} \langle (1, 0), (-2, 1) \rangle$$

By adjunction formula $K_X = (K_T + X)|_{X}$, which implies that $-K_X \sim \mathcal{O}(-2, 1) \in \partial\text{Mob}(X)$.

Now consider the fibrewise transform

$$(u, v, x, t, y, z) \mapsto (u, v, u^4x, u^6t, u^3y, z)$$

$X$ maps to another $dP_2/\mathbb{P}^1$, $X'$, defined as a hypersurface of bidegree $(0, 4)$ in $\mathbb{F}'$, where $\mathbb{F}'$ is constructed similar to $\mathbb{F}$, with grading on the Cox ring

$$A' = \begin{pmatrix} u & v & x & t & z & y \\ 1 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 2 & 1 & 1 \end{pmatrix}$$
In particular, $X'$ is smooth. The non-fibrewise transform of $X'$, that is the restriction of the 2-ray game of $F'$ show that $X'$ is birational to a Fano 3-fold quartic hypersurface in the weighted projective space $\mathbb{P}(1,1,1,2)$, see [1] Theorem 3.3. In particular, $X$ is not birationally rigid.

3. G-equivariant rigidity and conjugacy in Cremona group

In this section, I concentrate on $G$-Mori fibre spaces over $\mathbb{P}^1$, for the group $G = \text{PSL}_2(\mathbb{F}_7)$. The motivation for this particular case is explained below, in Subsection 3.2. Let us begin with the construction, which is interesting on its own.

3.1. $\text{PSL}_2(\mathbb{F}_7)$-Mori fibre spaces over $\mathbb{P}^1$. Let $\mathbb{P} = \mathbb{P}^1 \times \mathbb{P}(1,1,1,2)$, and denote by $\pi$, the natural projection $\pi: \mathbb{P} \rightarrow \mathbb{P}^1$. Suppose

$$f(x,y,z) = x^3y + y^3z + z^3x$$

and let $X'_n \subset \mathbb{P}$, for a non-negative integer $n$, be a 3-fold given by the equation

$$\alpha_n(u,v)t^2 + \beta_n(u,v)f(x,y,z) = 0,$$

where

(i) $u$ and $v$ are the homogeneous coordinates on $\mathbb{P}^1$,
(ii) $x$, $y$ and $z$ are weighted homogeneous coordinates of weight 1 on $\mathbb{P}(1,1,1,2)$, and $t$ is a weighted homogeneous coordinate of weight 2 on $\mathbb{P}(1,1,1,2)$,
(iii) $\alpha_n$ and $\beta_n$ are general homogeneous polynomials of degree $n$, and
(iv) $|Z_{\alpha}| = |Z_{\beta}| = n$ and $Z_{\alpha} \cap Z_{\beta} = \emptyset$, where $Z_{\alpha}: (\alpha_n = 0) \subset \mathbb{P}^1$ and $Z_{\beta}: (\beta_n = 0) \subset \mathbb{P}^1$.

There is an action of the group $\text{PSL}_2(\mathbb{F}_7)$ on $X'_n$, induced from the natural action on the fibres.

Remark 1. The variety $X'_n$ is just a direct product $\mathbb{P}^1 \times S$, where $S$ is the unique del Pezzo surface of degree 2 with an action of the group $\text{PSL}_2(\mathbb{F}_7)$. The variety $X'_n$ is unique, since there is only one pair of linear forms with distinct zeroes on $\mathbb{P}^1$ up to a change of coordinates.

Let $T \subset X'_n$ be the divisor defined by the equation $f(x,y,z) = 0$. Denote by $t_1^n, \ldots, t_n^n \in \mathbb{P}^1$ the points of the set $Z_{\alpha}$, and by $S_1, \ldots, S_n$ the fibres of the induced fibration $\pi: X'_n \rightarrow \mathbb{P}^1$ over these points, and let $C_i = S_i \cap T$. Let $p_1, \ldots, p_n$ be the points given by

$$\{\beta = x = y = z = 0\} \subset X'_n,$$

and denote by $S'_1, \ldots, S'_n$ the fibres of $\pi$ passing through these points.

The following lemma follows from the construction of $X'_n$.

Lemma 3.1. For $X'_n$, constructed as above, we have

(i) $\text{Sing}(X'_n) = \{p_1, \ldots, p_n\} \cup C_1 \cup \cdots \cup C_n$,
(ii) each of the points $p_j$ is a singular point of type $\frac{1}{2}(1,1,1)$ on $X'_n$, and $X'_n$ is locally isomorphic to $\mathbb{A}^1 \times \mathbb{C}$ along the curves $C_i$,
(iii) the fibres $S_1, \ldots, S_n$ are non-reduced fibres of $\pi$, and $S_i \cong \mathbb{P}^2$,
(iv) each of the fibres $S'_1, \ldots, S'_n$ has a unique singularity at the point $p_j$, and is isomorphic to the cone over the plane quartic curve $f(x,y,z) = 0$. 

Let $v: \tilde{X}_n \to X'_n$ be a blow up of the 3-fold $X'_n$ at the curves $C_1, \ldots, C_n$, and $\mu: \tilde{X}_n \to X_n$ be a contraction of the strict transforms $\tilde{S}_i$ of the non-reduced fibres $S_i$ on $X_n$. Both $v$ and $\mu$ are $\text{PSL}_2(F_7)$-equivariant birational morphisms:

\[ \begin{array}{c}
\tilde{X}_n \\
\downarrow v \\
X'_n \\
\downarrow \mu \\
X_n
\end{array} \]

Let $Q_i = \mu(\tilde{S}_i)$ for $1 \leq i \leq n$. Since $\mu \circ v^{-1}$ is an isomorphism in the neighbourhood of the points $P_j \in X'_n$, I will use the same letters $P_j$ to denote the corresponding points of $X_n$.

**Lemma 3.2.** For $X_n$ as above, we have

(i) $\text{Sing}(X_n) = \{P_1, \ldots, P_n, Q_1, \ldots, Q_n\}$,

(ii) each of the points $P_j$ and $Q_i$ is a singular point of type $\frac{1}{2}(1,1,1)$ on $X_n$, and in particular

(iii) the variety $X_n$ is a $\text{PSL}_2(F_7)$-Mori fibre space.

The group $\text{PSL}_2(F_7)$ acts on $\mathbb{P}^2$ and hence on $\mathcal{Y} \cong \mathbb{P}^1 \times \mathbb{P}^2$.

G. Belousov has recently announced the following.

**Theorem-Conjecture 3.3** (Belousov). The varieties $\mathcal{Y}$ and $X'_n$, for $n \geq 0$, are the only $\text{PSL}_2(F_7)$-Mori fibre spaces over $\mathbb{P}^1$ in dimension 3.
are relevant to the embeddings of $\text{PSL}_2(\mathbb{F}_7)$ into $\text{Cr}_3$ remains unsolved. Below I will discuss some results about the birational structure of the fibrations $\mathcal{Y}$ and $\mathcal{X}_n$.

**Lemma 3.4.** There is a $\text{PSL}_2(\mathbb{F}_7)$-equivariant birational equivalence between the varieties $\mathcal{Y}$ and $\mathcal{X}_1$.

**Proof.** It is immediate to see that $\mathcal{X}'_1$ (and thus also $\mathcal{X}_1$) is $\text{PSL}_2(\mathbb{F}_7)$-equivariantly birational to $\mathbb{P}(1, 1, 1, 2)$. Blowing up the singular point of $\mathbb{P}(1, 1, 1, 2)$, we obtain a $\text{PSL}_2(\mathbb{F}_7)$-equivariant birational morphism

$$\mathbb{P}_{p_2}(\mathcal{O}_{p_2} + \mathcal{O}_{p_2}(2)) \to \mathbb{P}(1, 1, 1, 2).$$

Note that for any $n \in \mathbb{Z}$ there is an action of $\text{PSL}_2(\mathbb{F}_7)$ on the projectivization

$$\mathcal{P}_n \cong \mathbb{P}_{p_2}(\mathcal{O}_{p_2} + \mathcal{O}_{p_2}(n))$$

arising from the action of $\text{PSL}_2(\mathbb{F}_7)$ on the base $\mathbb{P}^2$, since all line bundles on $\mathbb{P}^2$ are naturally $\text{PSL}_2(\mathbb{F}_7)$-equivariant. The ($\text{PSL}_2(\mathbb{F}_7)$-equivariant) projection $\pi_n: \mathcal{P}_n \to \mathbb{P}^2$ has a section $\Sigma_0$ with the following property: $\Sigma_0$ is $\text{PSL}_2(\mathbb{F}_7)$-invariant, and if $C \subset \Sigma_0 \cong \mathbb{P}^2$ is a smooth $\text{PSL}_2(\mathbb{F}_7)$-invariant curve of degree $d$, one has a diagram of $\text{PSL}_2(\mathbb{F}_7)$-equivariant morphisms

$$\begin{array}{ccc}
\mathcal{P}_n & \xrightarrow{\beta_0} & \mathcal{P}_0
\\
\downarrow & & \downarrow
\\
\mathcal{P}_n \oplus d & \xrightarrow{\beta'_0} & \mathcal{P}_0 \oplus d
\end{array}$$

where $\beta_0$ is a blow up of the curve $C$, and $\beta'_0$ is a contraction of the strict transform of the divisor $\pi_n^{-1}(\pi_n(C))$. Moreover, the projection $\pi_n$ has a bunch of $\text{PSL}_2(\mathbb{F}_7)$-invariant sections that do not intersect $\Sigma_0$. Let $\Sigma_\infty$ be one of these sections. If $C \subset \Sigma_\infty \cong \mathbb{P}^2$ is a smooth $\text{PSL}_2(\mathbb{F}_7)$-invariant curve of degree $d$, one has a diagram of $\text{PSL}_2(\mathbb{F}_7)$-equivariant morphisms

$$\begin{array}{ccc}
\mathcal{P}_n & \xrightarrow{\beta_\infty} & \mathcal{P}_0
\\
\downarrow & & \downarrow
\\
\mathcal{P}_n \oplus d & \xrightarrow{\beta'_\infty} & \mathcal{P}_0 \oplus d
\end{array}$$

where $\beta_\infty$ is a blow up of the curve $C$, and $\beta'_\infty$ is a contraction of the strict transform of the divisor $\pi_n^{-1}(\pi_n(C))$.

By now I have shown that $\mathcal{X}_1$ is $\text{PSL}_2(\mathbb{F}_7)$-equivariantly birationally equivalent to $\mathcal{P}_2$. Let $C_4 \subset \Sigma_0 \subset \mathcal{P}_2$ be the Klein quartic curve (that is, in the above notations, the plane curve given by the equation $f_4(x, y, z) = 0$). Since $C_4$ is $\text{PSL}_2(\mathbb{F}_7)$-invariant, it gives rise to a $\text{PSL}_2(\mathbb{F}_7)$-equivariant birational map $\mathcal{P}_2 \dashrightarrow \mathcal{P}_6$. Let $\Sigma_\infty \subset \mathcal{P}_6$ be one of the $\text{PSL}_2(\mathbb{F}_7)$-equivariant sections of $\pi_6$, and let $C_6 \subset \Sigma_\infty$ be the Hessian curve of the Klein quartic. Then $C_6$ is a $\text{PSL}_2(\mathbb{F}_7)$-invariant curve of degree 6, so that it gives rise to a $\text{PSL}_2(\mathbb{F}_7)$-equivariant birational map $\mathcal{P}_6 \dashrightarrow \mathcal{P}_0$.

Combining the birational maps obtained above, we finally get a $\text{PSL}_2(\mathbb{F}_7)$-equivariant birational map

$$\mathcal{X}_1 \dashrightarrow \mathcal{X}'_1 \dashrightarrow \mathbb{P}(1, 1, 1, 2) \dashrightarrow \mathcal{P}_2 \dashrightarrow \mathcal{P}_6 \dashrightarrow \mathcal{P}_0 \cong \mathbb{P}^1 \times \mathbb{P}^2 \cong \mathcal{Y}.$$
The results obtained in [9] by Bogomolov and Prokhorov imply the following.

**Proposition 3.5.** There is no $\mathrm{PSL}_2(F_7)$-equivariant birational equivalence between the varieties $Y$ and $X_0$.

**Sketch of proof.** Let $G$ be a finite group acting on a (smooth) variety $M$. Consider the collection of cohomology groups

$$\varphi(M, G) = \left\{ H^1(H, \operatorname{Pic}(M)) \mid H \subset G \right\}.$$ 

One can check that if $M'$ is another (smooth) variety with an action of $G$, and there exists a $G$-equivariant birational equivalence $M \times \mathbb{P}^n \dashrightarrow M' \times \mathbb{P}^{n'}$ for some $n, n' \geq 0$, where the action of $G$ on $\mathbb{P}^n$ and $\mathbb{P}^{n'}$ is trivial, then

$$\varphi(M, G) = \varphi(M', G).$$

On the other hand, one can check that $\varphi(S, \mathrm{PSL}_2(F_7))$ is non-trivial, where $S$ is the unique del Pezzo surface of degree 2 with an action of $\mathrm{PSL}_2(F_7)$, while $\varphi(\mathbb{P}^2, \mathrm{PSL}_2(F_7))$ is trivial. (Note that at the same time one has

$$H^1(\mathrm{PSL}_2(F_7), \operatorname{Pic}(S)) = H^1(\mathrm{PSL}_2(F_7), \operatorname{Pic}(\mathbb{P}^2)),$$

and what really matters here are the cyclic subgroups of the group $\mathrm{PSL}_2(F_7)$.)

I expect the following to hold. In fact, I believe that the methods of [40] can be adopted to prove birational rigidity of these models, and hence the conjecture. At the moment, the biggest difficulty seems to be the exclusion of the $1/2$-lines in the singular fibres.

**Conjecture 3.6.** The varieties $X_n$ are non-rational for $n \geq 2$.

**Remark 2.** In the next section, I explore the degree of irrationality for Fano 3-folds, and in particular the smooth quartic. I would like to bring your attention to the close relation between Conjecture 3.6 and Conjecture 4.9, as explained in Subsection 4.1.

### 4. Degree of irrationality of a smooth quartic

The notion of gonality for algebraic curves is a well-studied topic. The higher dimensional analogue is the **degree of irrationality** of an algebraic variety. In this section we survey the literature on this notion and provide some results for the degree of irrationality of smooth Fano 3-folds. We show that the picture for Fano 3-folds can be completed upon obtaining certain rigidity results on singular $dP_2/\mathbb{P}^1$.

Let $V$ be an algebraic variety defined over $\mathbb{C}$ and $K(V)$ be its field of rational functions.

**Definition 4.1.** The degree of irrationality $d(V)$ of the variety $V$ is the minimal degree of a field extension $\mathbb{C}(x_1, \ldots, x_n) \subset K(V)$, where $x_1, \ldots, x_n$ are algebraically independent rational functions on $V$. 

The degree of irrationality $d(V)$ of the variety $V$, as a birational invariant of $V$, is nothing but the minimal degree of a generically finite dominant rational map $V \dashrightarrow \mathbb{P}^n$, where $n = \dim(V)$. In particular, the variety $V$ is rational if and only if $d(V) = 1$. The degree of irrationality defined above has been introduced in [33], where the following result is proved.

**Theorem 4.2.** Let $\pi : V \dashrightarrow C$ be a dominant map and $\dim(C) = 1$. Then $d(V) \geq d(C)$.

Moreover, the following result is proved in [50].

**Theorem 4.3.** Let $\pi : V \dashrightarrow W$ be a rational dominant map, where $V$ and $W$ are uniruled algebraic surfaces. Then $d(V) \geq d(W)$.

It is well known that Theorem 4.3 fails if we drop either the uniruledness assumption or the assumption that $\dim(V) = 2$ (see [16], [29], [50]).

The case of curves, the degree of irrationality is nothing but the gonality and its properties are extensively studied. Moreover, the degrees of irrationality of uniruled surfaces have similar nature due to the following result in [48].

**Theorem 4.4.** Let $V = C \times \mathbb{P}^1$, where $C$ is a curve. Then $d(V) = d(C)$.

Unfortunately, only a few sporadic results about the degrees of irrationality of nonuniruled surfaces are known. The following result is due to [45], [49], and [51].

**Theorem 4.5.** Let $S$ be a minimal smooth surface of Kodaira dimension zero.

- If $S$ is an Enriques surface, then $d(S) = 2$.
- If $S$ is a K3 surface with $\text{rk Pic}(S) = 20$, then $d(S) = 2$.
- If $S$ is a quartic in $\mathbb{P}^3$ with $\text{rk Pic}(S) = 1$, then $d(S) = 3$.
- If $S$ is abelian, then $d(S) \geq 3$.
- If $S$ is bielliptic and $2K_S \sim 0$, then $d(S) = 2$.
- If $S$ is bielliptic and $2K_S \not\sim 0$, then $3 \leq d(S) \leq 4$.

There are abelian surfaces whose degree of irrationality is 3 (see [45]) but it is not known whether there is an abelian surface of degree of irrationality greater than 3.

On the other hand, I expect that $d(S) \in \{2, 3\}$, whenever $S$ is a smooth K3 surface. In fact, it is easy to check this for the famous 95 family of K3 surfaces, listed in [27, 13.3]. Moreover, for a minimal smooth K3 surface $S$, the equality $d(S) = 2$ is equivalent to the existence of a birational involution $\tau$ of the surface $S$ such that $\tau$ acts nontrivially on the nonzero holomorphic form of $S$ and the fixed locus of $\tau$ contains a curve (see [6], [7], [35], [36], [37], [52]). Thus, we have $d(S) \geq 3$ when $\text{rk Pic}(S) = 1$ and the surface $S$ is not a double cover of $\mathbb{P}^2$ ramified along a sextic (see Corollary 10.1.3 in [36]). Hence, it follows from [34] that $d(S) \geq 3$ when the surface $S$ is either a very general complete intersection of a quadric and a cubic or a very general complete intersection of 3 quadrics.

In the next dimension up, it is natural to start the investigation on the degree of irrationality on Mori fibre spaces. Assume that $\pi : V \rightarrow Z$ is an Mfs with $\dim(V) = 3$, and suppose $Z$ is rational.

**Proposition 4.6.** If $\dim(Z) > 0$, the inequality $d(V) \leq 2$ holds.
Proposition 4.7. Let $V$ be a smooth Fano 3-fold. Then the inequality $d(V) \leq 3$ holds and the equality $d(V) = 3$ implies that $V$ is a smooth quartic 3-fold in $\mathbb{P}^4$.

Proof. We may assume that $V$ is birational neither to a conic bundle nor to an elliptic fibration with a section. Then $V$ is either a double cover of $\mathbb{P}^3$ ramified along a sextic or a smooth quartic 3-fold (see [30]). In both cases $V$ is not rational (see [28], [29]), in the former case $d(V) = 2$, but in the latter case $d(V) \leq 3$ because a projection $X \rightarrow \mathbb{P}^3$ from a point of $X$ is a dominant rational map of degree 3.

Obviously, the degree of irrationality of a smooth quartic 3-fold is either 2 or 3 (see [29]).

Proposition 4.8. Let $V$ be a general smooth quartic 3-fold in $\mathbb{P}^4$. Then $d(V) = 3$.

Proof. Suppose that $d(V) = 2$. Let $K(V)$ be the field of rational functions of $V$. Then there is a finite field extension $K(V) \supset \mathbb{C}(x_1, x_2, x_3)$ of degree 2 that induces a birational involution $\tau$ of $V$. The birational involution $\tau$ is biregular because $\text{Bir}(V) = \text{Aut}(V)$ by [29]. However, the group $\text{Aut}(V)$ is trivial (see [32], [38]).

It seems reasonable for us to expect the following.

Conjecture 4.9. Let $V$ be a smooth quartic 3-fold in $\mathbb{P}^4$. Then $d(V) = 3$.

4.1. Smooth quartic and birational rigidity of a singular $dP_2/\mathbb{P}^1$. Suppose we have a smooth quartic 3-fold $V$ in $\mathbb{P}^4$ with $d(V) = 2$. Then there is a birational involution $\tau$ of $V$ such that the quotient 3-fold $V/\tau$ is rational. The fixed locus of involution $\tau$ is either a smooth quartic surface or the disjoint union of a smooth plane quartic curve and four distinct points. In the former case the quotient $V/\tau$ is a double cover of $\mathbb{P}^3$ ramified along a smooth quartic, which is not rational (see [46]). Therefore, the involution $\tau$ fixes the disjoint union of a smooth plane quartic curve and four points. We may assume that the quartic $V \subset \text{Proj}(\mathbb{C}[x, y, z, t, w])$ is given by the equation

$$h_4(x, y, z) + t^2a_2(x, y, z) + twb_2(x, y, z) + w^2c_2(x, y, z) + g_4(t, w) = 0,$$

while $\tau$ acts as $\tau(x : y : z : t : w) = (x : y : z : -t : -w)$, where $h_i, a_i, b_i, c_i, g_i$ are homogeneous polynomials of degree $i$. Then $\tau$ fixes the smooth irreducible curve $C$ given by the equations $t = w = 0$ and four points $O_1, O_2, O_3, O_4$ given by $x = y = z = 0$.

Let $\psi : V \rightarrow Y$ be the double cover given by involution $\tau$. Then $Y$ is a Fano 3-fold with canonical $\mathbb{Q}$-factorial singularities, $-K_Y^3 = 2$, and $\text{Pic}(Y) = \mathbb{Z}$. The 3-fold $Y$ is locally isomorphic to $\mathbb{A}_1 \times \mathbb{C}$ along the smooth curve $\psi(C)$. On the other hand, the 3-fold $Y$ has 4 isolated singular points $\psi(O_1), \psi(O_2), \psi(O_3)$, and $\psi(O_4)$, which are cyclic quotient singular points of type $\frac{1}{2}(1, 1, 1)$.
The forms \( x, y, \) and \( z \) generate the anticanonical linear system \( |-K_Y| \) which gives a rational map to \( \mathbb{P}^2 \) whose general fibre is a smooth elliptic curve. Let \( \mathcal{H} \) be the pencil on the 3-fold \( Y \) generated by the forms \( t \) and \( w \). Then the base locus of \( \mathcal{H} \) consists of the curve \( \psi(C) \) but a general surface in \( \mathcal{H} \) is a smooth del Pezzo surface of degree 2.

Let \( \rho : \hat{Y} \to Y \) be the blow up at the points \( \psi(O_1) \), \( \psi(O_2) \), \( \psi(O_3) \), \( \psi(O_4) \). Put \( E_i = \rho^{-1}(\psi(O_i)) \). Then the linear system \( |-K_Y| \) has no base points and \( -K_Y^3 = 0 \). Therefore, there is an elliptic fibration \( \eta : \hat{Y} \to \mathbb{P}^2 \) such that \( E_i \) is a section of \( \eta \). In particular, the reflection of the generic fibre of \( \eta \) with respect to the section \( E_i \) induces an involution \( \tau_i \in \text{Bir}(Y) \), which is not birational whenever the polynomials \( a_2, b_2, c_2 \) are sufficiently general. However, it is easy to see that the involution \( \tau_i \) is birational when \( a_2 = b_2 = c_2 = 0 \).

Let \( \nu : Y \to Y \) be the blow up of \( \psi(C) \). Then \( \mathcal{H} \) induces a fibration \( \xi : Y \to \mathbb{P}^1 \) whose general fibre is a del Pezzo surface of degree 2. In particular, the group \( \text{Bir}(Y) \) contains a huge subgroup generated by Bertini involutions of the generic fibre of \( \xi \) (see [40]) but the involution \( \tau_i \) does not commute with \( \xi \) if the polynomials \( a_2, b_2, c_2 \) are general.

The nonrationality of \( Y \) implies Conjecture 4.9. It seems to me that \( Y \) is birationally rigid, or at least it has finite pliability. The only known way to prove the nonrationality of a 3-fold fibred into rational surfaces that is not birational to a conic bundle is the way of [40], but \( \xi \) does not satisfy most of the conditions of [40].

Let \( \alpha = t^2, \beta = u^2, \gamma = tw \). Then there is a homogeneous polynomial \( f_2 \) of degree 2 such that \( f_2(\alpha,\beta,\gamma) = g_4(t,w) \). Hence, the 3-fold \( Y \) is a weighted complete intersection \( H_4(x,y,z) + \alpha a_2(x,y,z) + \beta b_2(x,y,z) + \beta c_2(x,y,z) + f_2(\alpha,\beta) = \alpha\beta - \gamma^2 = 0 \) in \( \mathbb{P}(1^3,2^3) \cong \text{Proj}(\mathbb{C}[x,y,z,\alpha,\beta,\gamma]) \) and the natural projection \( \mathbb{P}(1^3,2^3) \to \mathbb{P}^2 \) induces the rational map \( \eta \circ \rho^{-1} \). The hypersurface \( \alpha\beta - \gamma^2 = 0 \) has a natural projection to \( \mathbb{P}^1 \) which induces the rational map \( \xi \circ \nu^{-1} \). The 3-fold \( Y \) is birational to a hypersurface

\[
\beta^2 h_4(x,y,z) + \beta^2 a_2(x,y,z) + \beta^2 b_2(x,y,z) + \beta^3 c_2(x,y,z) + \beta^2 f_2(\gamma^2/\beta,\beta) = 0
\]

in \( \mathbb{P}(1^3,2^2) \cong \text{Proj}(\mathbb{C}[x,y,z,\beta,\gamma]) \) by putting \( \alpha = \gamma^2/\beta \). In the case of Fermat quartic hypersurface \( x^4 + y^4 + z^4 - t^4 - w^4 = 0 \), the question is reduced to the following: is the field

\[
\mathbb{C}(x,y,\beta) \left( \sqrt[4]{\beta^2(1+x^4+y^4)-\beta^4} \right)
\]

a purely transcendental extension of the field \( \mathbb{C} \)?

**Proposition 4.10.** Conjecture 3.6 implies Conjecture 4.9.

**Proof.** The construction above is a special case in Conjecture 3.6 with \( n = 2 \). \( \square \)

**References**


