GENERALIZED DOMINATION, INDEPENDENCE AND IRREDUNDANCE IN GRAPHS

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Abstract

The purpose of this paper is to present some basic properties of $P$-dominating, $P$-independent, and $P$-irredundant sets in graphs which generalize well-known properties of dominating, independent and irredundant sets, respectively.

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In this paper we will consider finite undirected graphs with no multiple edges, and with no loops. For a graph $G$ we will refer to $V(G)$ (or $V$) and $E(G)$ (or $E$) as the vertex and edge set, respectively.

A nonempty subset $D$ of the vertex set $V$ of a graph $G$ is a dominating set if every vertex in $V - D$ is adjacent to a member of $D$. If $u \in D$ and $v \in V - D$, and $uv \in E$, we say that $u$ dominates $v$ and $v$ is dominated by $u$.

The minimum (maximum) of the cardinalities of the minimal dominating sets in $G$ is called the upper domination number of $G$ and it is denoted by $\gamma(G)$ ($\Gamma(G)$).
We write $H \leq G$ if $H$ is an induced subgraph of $G$. We use the notation $G[A]$ for the subgraph of $G$ induced by $A \subseteq V(G)$.

A set $S \subseteq V(G)$ is said to be independent if $G[S]$ is totally disconnected (i.e., $G[S]$ is an edgeless graph). Obviously, each maximal independent set is a minimal dominating set. If $S$ is a maximal independent set of $G$, then $G[S \cup \{v\}]$ contains as a subgraph $K_2$, i.e., the subgraph which is forbidden for the property "to be totally disconnected".

For $v \in V$, we denote by $N(v)$ a set of vertices adjacent to $v$ (neighbours of $v$) and by $N(A)$ a set of neighbours of vertices of $A$. By $N[v]$ and $N[A]$ we denote $N(v) \cup \{v\}$ and $N(A) \cup A$, respectively.

A set $R \subseteq V(G)$ is called irredundant in $G$, if for each vertex $v \in R$, $N[v] - N[R - \{v\}] \neq \emptyset$.

This definition fits intuitive ideas of redundancy, for in the context of communication network, any vertex that may receive a communication from some vertex $x$ in $R$, may also be informed from some vertex in $R - \{x\}$, i.e., $x$ may be removed from $R$ without affecting the totality of accessible vertices. It is apparent that irredundance is a hereditary property and that any independent set of vertices is also an irredundant set.

The minimum (maximum) of the cardinalities of the maximal irredundant sets of $G$ is called the lower (upper) irredundance number and it is denoted by $ir(G)$, ($IR(G)$).

The study of domination in graphs has been initiated by Ore [6], for a survey see a special volume of the Discrete Mathematics 86 (1990). Applications of minimum dominating sets have been suggested by many authors. The determination of the domination number is an NP-complete problem (see [4]). It should be noted that bounds on $\gamma(G)$ do exist through the parameters which are also difficult to determine.

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Let $\mathcal{I}$ denote the class of all finite simple graphs. A graph property is a non-empty isomorphism-closed subclass of $\mathcal{I}$. (We also say that a graph has the property $\mathcal{P}$ if $G \in \mathcal{P}$).

A property $\mathcal{P}$ of graphs is said to be induced hereditary if whenever $G \in \mathcal{P}$ and $H \leq G$, then also $H \in \mathcal{P}$. For hereditary properties with respect to other partial order on $\mathcal{I}$ we refer the reader to [1].

Any induced hereditary property $\mathcal{P}$ of graphs is uniquely determined by the set of all its forbidden induced subgraphs

$$C(\mathcal{P}) = \{ H \in \mathcal{I} : H \notin \mathcal{P} \text{ but } (H - v) \in \mathcal{P} \text{ for any } v \in V(H) \}.$$
Let us denote by $\mathcal{M}$ the set of all induced hereditary properties of graphs. According to [1] we list below some of the induced hereditary properties.

Let $N \subseteq X$ such that $v \in N$ containing $v$. Especially, $N(v) = N_G(v)$.

Next, for a vertex $v \in V(G)$ we denote the set of all forbidden subgraphs containing $v$ by $C_{G,P}(v) = \{H' \leq G : v \in V(H'), H' \simeq H \in C(P)\}$.

The number $|C_{G,P}(v)|$ is called $P$-degree of $v$ in $G$ and is denoted by $\deg_{G,P}(v)$.

If $\deg_{G,P}(v) = 1$, then $v$ is said to be $P$-pendant in $G$ and if $\deg_{G,P}(v) = 0$, then $v$ is said to be $P$-isolated in $G$.

A set $D \subseteq V$ is said to be $P$-dominating in $G$ if $N_P(v) \cap D \neq \emptyset$ for any $v \in V - D$.

A set $D \subseteq V$ is said to be strongly $P$-dominating in $G$ if for each $v \in V - D$ there is $H' \leq G$ containing $v$ such that $H' \simeq H \in C(P)$ and $V(H') - \{v\} \subseteq D$.

The minimum (maximum) of the cardinalities of the minimal $P$-dominating sets in $G$ is called the lower (upper) $P$-domination number of $G$ and it is denoted by $\gamma_P(G)$, $(\Gamma_P(G))$, respectively.

The minimum (maximum) of the cardinalities of the minimal strongly $P$-dominating sets in $G$ is called the lower (upper) strong $P$-domination number and it is denoted by $\gamma'_P(G)$, $(\Gamma'_P(G))$, respectively.

If $P = I_{n-2}$, then the $I_{n-2}$-dominating sets are also called $K_n$-dominating sets in $G$ (see [5]).

A set $R \subseteq V$ is called $P$-irredundant if for every vertex $v \in R$, $N_P[v] - N_P[R - \{v\}] \neq \emptyset$.

The minimum (maximum) of the cardinalities of the maximal $P$-irredundant sets is called the lower (upper) $P$-irredundance number of $G$ and is denoted by $ir_P(G)$, $(IR_P(G))$, respectively.
A set $S \subseteq V(G)$ is $\mathcal{P}$-independent in $G$ if $G[S] \in \mathcal{P}$. A set $S \subseteq V(G)$ is said to be strongly $\mathcal{P}$-independent in $G$ if for every $v \in S$, $N_{\mathcal{P}}(v) \cap S = \emptyset$.

The minimum (maximum) of the cardinalities of the maximal strongly $\mathcal{P}$-independent sets in $G$, is called the strong $\mathcal{P}$-independence number of $G$ and it is denoted by $\iota'_\mathcal{P}(G)$, $(\alpha'_{\mathcal{P}}(G))$.

The minimum (maximum) of the cardinalities of the maximal $\mathcal{P}$-independent sets in $G$, is called the $\mathcal{P}$-independence number of $G$ and it is denoted by $\iota_{\mathcal{P}}(G)$, $(\alpha_{\mathcal{P}}(G))$.

Notice, that if $\mathcal{P} = \mathcal{O}$, then $\mathcal{P}$-dominating and strongly $\mathcal{P}$-dominating sets in $G$ are dominating sets, $\mathcal{P}$-independent and strongly $\mathcal{P}$-independent sets are independent sets, also $\mathcal{P}$-irredundant sets are irredundant sets in an ordinary sense.

The following theorem generalizes a clasical result of Ore [6].

**Theorem 1.** Let $D$ be a $\mathcal{P}$-dominating set of a graph $G$. Then $D$ is a minimal $\mathcal{P}$-dominating set of $G$ if and only if for each vertex $d \in D$, $d$ has one of the following properties:

(i) there exists a vertex $v \in V - D$ such that $N_{\mathcal{P}}(v) \cap D = \{d\}$,

(ii) $N_{\mathcal{P}}(d) \cap D = \emptyset$.

**Proof.** Suppose that $D$ is a minimal $\mathcal{P}$-dominating set of $G$. Then for each vertex $d \in D$, the set $D - \{d\}$ is not a $\mathcal{P}$-dominating set of $G$. Hence, there is a vertex $v \in V - (D - \{d\})$ that is $\mathcal{P}$-adjacent to no vertex of $D - \{d\}$. If $v = d$, $d$ is $\mathcal{P}$-adjacent to no vertex of $D$, while if $v \in V - D$, then since $D$ is a $\mathcal{P}$-dominating set of $G$, $N_{\mathcal{P}}(v) \cap D = \{d\}$.

Conversely, if every vertex $d \in D$ has at least one of the properties (i) or (ii), then $D - \{d\}$ is not a $\mathcal{P}$-dominating set of $G$. 

**Theorem 2.** If $G$ is a graph without $\mathcal{P}$-isolated vertices, then there exists a minimum $\mathcal{P}$-dominating set of vertices of $G$ in which every vertex has property (i).

**Proof.** Among all the $\mathcal{P}$-dominating sets of $G$ with cardinality equal to $\gamma_{\mathcal{P}}(G)$, let $D$ be chosen so that $D$ contains the maximum possible numbers of vertices which are $\mathcal{P}$-adjacent to some vertex of $D$ in $G$. Suppose there exists a vertex $d \in D$, that $d$ has no property (i). However, by Theorem 1, $d$ has the property (ii). This implies that $d$ is $\mathcal{P}$-adjacent to no vertex
of \( D \). Since \( G \) is a graph without isolated vertices, then there exists a vertex \( w \in N_P(d) \) and \( w \in V(G) - (D - \{d\}) \). The vertex \( w \) is \( P \)-adjacent to some vertex of \( D - \{d\} \). Let \( D' = (D - \{d\}) \cup \{w\} \). Necessarily \( D' \) is a \( P \)-dominating set of \( G \) with \( |D'| = \gamma_P(G) \) and the set \( D' \) contains more vertices than the set \( D \) which are \( P \)-adjacent to some vertices of \( D' \). This contradicts our choice of \( D \).

Now we shall establish some properties of \( P \)-dominating, strongly \( P \)-dominating, \( P \)-independent and strongly \( P \)-independent sets, and \( P \)-irredundant sets.

**Proposition 3.** If \( D \subseteq V(G) \) is a minimal strongly \( P \)-dominating set in \( G \), then \( D \) is \( P \)-dominating in \( G \).

Proposition 3 implies the following inequality.

For any graph \( G \),

\[
\gamma_P(G) \leq \gamma'_P(G).
\]

**Proposition 4.** Let \( G \) be a graph. If \( X \) is a maximal \( P \)-independent set in \( G \), then \( X \) is a minimal strongly \( P \)-dominating set in \( G \).

**Proof.** For each vertex \( v \in V - X \) a subgraph \( G[X \cup \{v\}] \) has no property \( P \). Hence, there exists an induced subgraph \( H' \) of \( G \), \( H' \simeq H, H \in C(P) \), such that \( V(H') \cap X = V(H') - \{v\} \). It implies that \( X \) is the strongly \( P \)-dominating set. Moreover, for each vertex \( x \in X \) the set \( X - \{x\} \) is not strongly \( P \)-dominating. It follows from the fact that there is no induced subgraph \( H' \simeq H \in C(P) \) containig the vertex \( x \) and \( V(H') \subseteq X \). Thus, \( X \) is a minimal strongly \( P \)-dominating set.

From Proposition 4, we obtain the following inequalities.

For any graph \( G \),

\[
\gamma'_P(G) \leq i_P(G) \leq \alpha_P(G) \leq \Gamma'_P(G).
\]

**Proposition 5.** Let \( G \) be a graph. If \( X \) is a maximal strongly \( P \)-independent set, then \( X \) is a minimal \( P \)-dominating set.

**Proof.** Let \( X \) be a maximal strongly \( P \)-independent set in \( G \). Suppose there exists a vertex \( v \in V - X \) such that each induced subgraph \( H' \) of \( G \) such that \( v \in V(H') \), \( H' \simeq H \in C(P) \) has no common vertices with the set \( X \),
thus $X \cup \{v\}$ is strongly $\mathcal{P}$-independent, a contradiction. Hence, for each vertex $v \in V - X$ there is $H' \leq G, H' \simeq H, v \in V(H'), H \in C(P)$ such that $N_{\mathcal{P}}(v) \cap X \neq \emptyset$. Hence, $X$ is $\mathcal{P}$-dominating. Moreover, by the definition of a strongly $\mathcal{P}$-independent set, for each $x \in X, N_{\mathcal{P}}(x) \cap (X - \{x\}) = \emptyset$, thus, $X$ is a minimal $\mathcal{P}$-dominating set in $G$. 

Proposition 5 implies the following property.

For any graph $G$,

$$\gamma_{\mathcal{P}}(G) \leq \gamma'_{\mathcal{P}}(G) \leq \alpha'_{\mathcal{P}}(G) \leq \Gamma_{\mathcal{P}}(G). \quad (3)$$

Proposition 6. Let $G$ be a graph without $\mathcal{P}$-isolated vertices. If $S$ is a maximal strongly $\mathcal{P}$-independent set in $G$, then $V - S$ is strongly $\mathcal{P}$-dominating.

Proof. By the definition of the strongly $\mathcal{P}$-independent set, for each vertex $v \in S$ there is a subgraph $H', H' \leq G$ such that $v \in V(H'), H' \simeq H \in C(\mathcal{P})$ and $V(H') \cap (V - S) = V(H') - \{v\}$. Therefore, we obtain.

Let $G$ be a graph without $\mathcal{P}$-isolated vertices. Then

$$\gamma'_{\mathcal{P}}(G) \leq |V(G)| - \gamma_{\mathcal{P}}(G). \quad (4)$$

Proposition 7. Let $G$ be a graph. If $D$ is a minimal $\mathcal{P}$-dominating set, then $D$ is maximal $\mathcal{P}$-irredundant.

Proof. Let $D$ be a minimal $\mathcal{P}$-dominating. By Theorem 1, every vertex $d \in D$ has one of the properties (i) or (ii).

Assume $d$ has the property (i). Thus there exists vertex $v \in V - D$ such that $N_{\mathcal{P}}(v) \cap D = \{d\}$, then $v \in N_{\mathcal{P}}[d]$ and $v \notin N_{\mathcal{P}}[D - \{d\}]$. It implies that $v \in (N_{\mathcal{P}}[d] - N_{\mathcal{P}}[D - \{d\}]).$

Suppose that $d$ has the property (ii) and $d$ has no property (i). Therefore, $d \notin N_{\mathcal{P}}[D - \{d\}]$ and $d \in (N_{\mathcal{P}}[d] - N_{\mathcal{P}}[D - \{d\}]).$ Thus, $D$ is an irredundant set in $G$. Moreover, $N_{\mathcal{P}}(D) = V(G)$ and hence for each $v \in V - D$, the set $D \cup \{v\}$ is not $\mathcal{P}$-irredundant. Hence, $D$ is a maximal $\mathcal{P}$-irredundant set. 

From this theorem we have.

For any graph $G$,

$$ir_{\mathcal{P}}(G) \leq \gamma_{\mathcal{P}}(G) \leq \Gamma_{\mathcal{P}}(G) \leq IR_{\mathcal{P}}(G). \quad (5)$$
Theorem 8. For any graph $G$ we have the following inequalities:

\begin{align*}
\text{ir}_P(G) & \leq \gamma_P(G) \leq \iota_P'(G) \leq \Gamma_P(G) \leq 1R_P(G). \\
\text{ir}_P''(G) & \leq \gamma_P''(G) \leq \iota_P''(G) \leq \alpha_P''(G) \leq \Gamma_P''(G).
\end{align*}

Proof. (6) is obtained from (3) and (5) and (7) from (1), (2), (5).

Remark 1. Notice that the inequalities (6) are generalizations of results of Cokayne and Hedetniemi [3].

Remark 2. We know that some of the inequalities are strict for some properties and some graphs.

References


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