On the Existence of $n$-Tuple Magic Rectangles
Phaisatcha Inpoonjai and Thiradet Jiarasuksakun

Abstract—Magic rectangles are a classical generalization of the well-known magic squares, and they are related to graphs. A graph $G$ is called degree-magic if there is a labelling of the edges by integers $1, 2, \ldots, |E(G)|$ such that the sum of the labels of the edges incident with any vertex $v$ is equal to $(1+|E(G)|)\deg(v)/2$. In this paper we generalize magic rectangles to be $n$-tuple magic rectangles, and prove the necessary and sufficient conditions for the existence of even $n$-tuple magic rectangles. Using this existence we identify the sufficient condition for degree-magic labellings of the $n$-fold self-union of complete bipartite graphs to exist.

Keywords—magic squares, magic rectangles, degree-magic graphs

1. Introduction

Magic rectangles are a natural generalization of the magic squares which have widely intrigued mathematicians and the general public. A magic $(p,q)$-rectangle $R$ is a $p \times q$ array in which the first $pq$ positive integers are placed such that the sum over each row of $R$ is constant and the sum over each column of $R$ is another (different if $p \neq q$) constant. Harmuth [1, 2] studied magic rectangles over a century ago and proved that

**Theorem 1** ([1, 2]) For $p, q > 1$, there is a magic $(p, q)$-rectangle $R$ if and only if $p \equiv q \pmod{2}$ and $(p, q) \neq (2, 2)$.

In 1990, Sun [3] studied the existence of magic rectangles. Later, Bier and Rogers [4] studied balanced magic rectangles, and Bier and Kleinschmidt [5] studied centrally symmetric and magic rectangles. Then Hagedorn [6] presented a simplified modern proof of the necessary and sufficient conditions for a magic rectangle to exist. The concept of magic rectangles was generalized to $n$-dimensions and several existence theorems were proven by Hagedorn [7].

For simple graphs without isolated vertices, if $G$ is a graph, then $V(G)$ and $E(G)$ stand for the vertex set and the edge set of $G$, respectively. Cardinalities of these sets are called the order and size of $G$.

Let a graph $G$ and a mapping $f$ from $E(G)$ into positive integers be given. The index mapping of $f$ is the mapping $f^*$ from $V(G)$ into positive integers defined by

$$f^*(v) = \sum_{e \in E(G)} \eta(v, e)f(e) \text{ for every } v \in V(G),$$

where $\eta(v, e)$ is equal to 1 when $e$ is an edge incident with a vertex $v$, and 0 otherwise. An injective mapping $f$ from $E(G)$ into positive integers is called a magic labelling of $G$ for an index $\lambda$ if its index mapping $f^*$ satisfies

$$f^*(v) = \lambda \text{ for all } v \in V(G).$$

A magic labelling $f$ of a graph $G$ is called a supermagic labelling if the set $\{f(e) : e \in E(G)\}$ consists of consecutive positive integers. A graph $G$ is supermagic (magic) whenever a supermagic (magic) labelling of $G$ exists.

A bijective mapping $f$ from $E(G)$ into $\{1, 2, \ldots, |E(G)|\}$ is called a degree-magic labelling (or only $d$-magic labelling) of a graph $G$ if its index mapping $f^*$ satisfies

$$f^*(v) = \frac{1+|E(G)|}{2}\deg(v) \text{ for all } v \in V(G).$$

A $d$-magic labelling $f$ of $G$ is called balanced if for all $v \in V(G)$, the following equation is satisfied

$$\left|\left\{e \in E(G) : \eta(v, e) = 1, f(e) \leq \left\lfloor \frac{|E(G)|}{2} \right\rfloor \right\} \right| = \left|\left\{e \in E(G) : \eta(v, e) = 1, f(e) > \left\lfloor \frac{|E(G)|}{2} \right\rfloor \right\} \right|.$$  

A graph $G$ is degree-magic (balanced degree-magic) or only $d$-magic when a $d$-magic (balanced $d$-magic) labelling of $G$ exists.

The concept of magic graphs was introduced by Sedláček [8]. Later, supermagic graphs were introduced by Stewart [9]. There are now many papers published on magic and supermagic graphs; we refer the reader to Gallian [10] for more comprehensive references. Recently, the concept of degree-magic graphs was introduced by Bezegová and Ivančo [11] as an extension of supermagic regular graphs. They also established the basic properties of degree-magic graphs and proved that

**Proposition 1** ([11]) For $p, q > 1$, the complete bipartite graph $K_{p,q}$ is $d$-magic if and only if $p \equiv q \pmod{2}$ and $(p, q) \neq (2, 2)$.

**Theorem 2** ([11]) The complete bipartite graph $K_{p,q}$ is balanced $d$-magic if and only if the following statements hold:

(i) $p \equiv q \equiv 0 \pmod{2}$;
(ii) if $p \equiv q \equiv 2 \pmod{4}$, then $\min\{p, q\} \geq 6$.

In this paper we introduce $n$-tuple magic rectangles. To show their existence, we introduce the closely related concept
of centrally \( n \)-tuple symmetric rectangles. Then we use the existence of centrally \( n \)-tuple symmetric rectangles to give a construction of even \( n \)-tuple magic rectangles. Finally, we identify the sufficient condition for \( d \)-magic labellings of the \( n \)-fold self-union of complete bipartite graphs to exist.

II. The \( n \)-Tuple Magic Rectangles

In this section we introduce \( n \)-tuple magic rectangles and prove the necessary and sufficient conditions for even \( n \)-tuple magic rectangles to exist.

**Definition 1** An \( n \)-tuple magic \((p, q)\)-rectangle \( R := (r_{i,j}^1) \ldots (r_{i,j}^p) \) is a class of \( n \) arrays in which each array has \( p \) rows and \( q \) columns, and the first \( npq \) positive integers are placed such that the sum over each row of any array of \( R \) is constant and the sum over each column of \( R \) is another (different if \( p \neq q \)) constant.

Let \( R := (r_{i,j}^1) \ldots (r_{i,j}^p) \) be an \( n \)-tuple magic \((p, q)\)-rectangle. As each row sum of any array of \( R \) is constant and the sum over each column of \( R \) is another constant, \( p \neq q \), we then have

**Proposition 2** If \( R \) is an \( n \)-tuple magic \((p, q)\)-rectangle, then the following statements hold:

(i) if \( n \) is odd, then \( p \equiv q \pmod{2} \);

(ii) if \( n \) is even, then \( p \equiv q \equiv 0 \pmod{2} \).

Proposition 2 allows the set of \( n \)-tuple magic rectangles to be divided into sets of odd and even rectangles. We quickly see that an \( n \)-tuple magic \((2, 2)\)-rectangle does not exist. To show the existence of other even \( n \)-tuple magic rectangles, we introduce the closely related concept of centrally \( n \)-tuple symmetric \((p, q)\)-rectangles as follows.

**Definition 2** Let \( x > -1 \) and let \( R \) be a class of \( n \) even rectangular arrays in which each array has \( p \) rows and \( q \) columns and the entries of \( R \) are numbers \( \pm(x+1), \ldots, \pm(x+npq/2) \). \( R \) is a centrally \( n \)-tuple symmetric \((p, q)\)-rectangle of type \( x \) if the sum over each row and column of any array is zero. Additionally, if \( R \) has an equal number of positive and negative numbers in each row and column of any array, we say that \( R \) is balanced.

If \( R \) is an even \( n \)-tuple magic \((p, q)\)-rectangle, then by subtracting \((npq+1)/2 \) from each entry of \( R \), we obtain a centrally \( n \)-tuple symmetric \((p, q)\)-rectangle of type \(-1/2\). Similarly, every centrally \( n \)-tuple symmetric \((p, q)\)-rectangle of type \(-1/2\) determines an even \( n \)-tuple magic \((p, q)\)-rectangle. Thus, we can use the existence of centrally \( n \)-tuple symmetric \((p, q)\)-rectangles to prove the existence of even \( n \)-tuple magic \((p, q)\)-rectangles.

**Lemma 1** For \( x, y > -1 \), if a balanced centrally \( n \)-tuple symmetric \((p, q)\)-rectangle of type \( x \) exists, then a balanced centrally \( n \)-tuple symmetric \((p, q)\)-rectangle of type \( y \) exists.

**Proof.** Suppose that \( R := (r_{i,j}^1) \ldots (r_{i,j}^p) \) is the given rectangle. Then we define a \((p, q)\)-rectangle \( S := (s_{i,j}^1) \ldots (s_{i,j}^q) \) by

\[
s_{i,j}^t = (y-x) \text{sgn}(r_{i,j}^t) + r_{i,j}^t, \quad \text{for every } t \in \{1, 2, \ldots, n\}.
\]

The entries of \( S \) are the numbers \( \pm(y+1), \ldots, \pm(y+npq/2) \). For any \( t \in \{1, 2, \ldots, n\} \) and \( 1 \leq i \leq p \), the sum of each row is

\[
\sum_{j=1}^{q} s_{i,j}^t = \sum_{j=1}^{q} ((y-x) \text{sgn}(r_{i,j}^t) + r_{i,j}^t) = (y-x) \sum_{j=1}^{q} \text{sgn}(r_{i,j}^t) + \sum_{j=1}^{q} r_{i,j}^t = 0,
\]

and for all \( 1 \leq j \leq q \), the sum of each column is

\[
\sum_{i=1}^{p} s_{i,j}^t = \sum_{i=1}^{p} ((y-x) \text{sgn}(r_{i,j}^t) + r_{i,j}^t) = (y-x) \sum_{i=1}^{p} \text{sgn}(r_{i,j}^t) + \sum_{i=1}^{p} r_{i,j}^t = 0.
\]

Thus, \( S \) is a centrally \( n \)-tuple symmetric \((p, q)\)-rectangle of type \( y \). For any \( t \in \{1, 2, \ldots, n\} \), if \( r_{i,j}^t \) is positive, then \( r_{i,j}^t = x+m \) for some \( m \geq 1 \). Hence, \( s_{i,j}^t = y+m \) is also positive. Similarly, \( r_{i,j}^t \) negative implies \( s_{i,j}^t \) negative. Therefore, \( S \) is balanced.

**Proposition 3** If a balanced centrally \( n \)-tuple symmetric \((p, q)\)-rectangle exists, then an \( n \)-tuple magic \((p, q)\)-rectangle exists.

**Proof.** Suppose \( R \) is the given rectangle. If \( R \) has type \( x \), then by Lemma 1, there exists a balanced centrally \( n \)-tuple symmetric \((p, q)\)-rectangle of type \(-1/2\). Therefore, an \( n \)-tuple magic \((p, q)\)-rectangle exists.

**Example 1** We consider a balanced centrally 5-tuple symmetric \((4, 2)\)-rectangle \( R := (r_{i,j}^1) \ldots (r_{i,j}^5) \) of type 1 as follows.

\[
R := \begin{pmatrix}
2 & -2 & 6 & -6 & 10 & -10 \\
-3 & 3 & -7 & 7 & 11 & -11 \\
-4 & 4 & -8 & 8 & 12 & -12 \\
5 & -5 & -9 & 9 & 13 & -13 \\
\end{pmatrix}
\]

Then we define a 5-tuple \((4, 2)\)-rectangle \( S := (s_{i,j}^1) \ldots (s_{i,j}^5) \) related to \( R \) by

\[
s_{i,j}^t = -\frac{3}{2} \text{sgn}(r_{i,j}^t) + r_{i,j}^t, \quad \text{for every } t \in \{1, 2, 3, 4, 5\}.
\]
Thus, $S$ is a balanced centrally 5-tuple symmetric $(4, 2)$-rectangle $S$ of type $-1/2$ as follows.

$$ S := \begin{pmatrix} 1 & 1 & 9 & 9 & 17 & 17 & 25 & 25 & 33 & 33 \\ 2 & 2 & \frac{2}{2} & \frac{2}{2} & \frac{2}{2} & \frac{2}{2} & \frac{2}{2} & \frac{2}{2} & \frac{2}{2} & \frac{2}{2} \\ 3 & 3 & 11 & 11 & 19 & 19 & 27 & 27 & 35 & 35 \\ 2 & 2 & \frac{2}{2} & \frac{2}{2} & \frac{2}{2} & \frac{2}{2} & \frac{2}{2} & \frac{2}{2} & \frac{2}{2} & \frac{2}{2} \\ 5 & 5 & 13 & 13 & 21 & 21 & 29 & 29 & 37 & 37 \\ 2 & 2 & \frac{2}{2} & \frac{2}{2} & \frac{2}{2} & \frac{2}{2} & \frac{2}{2} & \frac{2}{2} & \frac{2}{2} & \frac{2}{2} \\ 7 & 7 & 15 & 15 & 23 & 23 & 31 & 31 & 39 & 39 \\ 2 & 2 & \frac{2}{2} & \frac{2}{2} & \frac{2}{2} & \frac{2}{2} & \frac{2}{2} & \frac{2}{2} & \frac{2}{2} & \frac{2}{2} \end{pmatrix}. $$

By adding $41/2$ to each entry of $S$, we obtain a 5-tuple magic $(4, 2)$-rectangle $T$ as below.


Clearly, the sum over each row of any array is 41 and the sum over each column is 82.

**Proposition 4** If a balanced centrally $n$-tuple symmetric $(p_i, q)$-rectangle $R$ and a centrally $n$-tuple symmetric $(p^+_q, q)$-rectangle $S$ exist, then a centrally $n$-tuple symmetric $(p_i + p^+_q, q)$-rectangle $T$ exists. If $S$ is a balanced rectangle, then $T$ can also be chosen to be balanced.

**Proof.** Suppose $S$ has type $x$. By Lemma 1, we know that there exists a balanced centrally $n$-tuple symmetric $(p_i, q)$-rectangle $R'$ of type $x + np_x q/2$. Then by stacking $R'$ and $S$ together, we obtain a rectangle $T$ whose rows' and columns' sum is zero. Thus, $T$ is a centrally $n$-tuple symmetric $(p_i + p^+_q, q)$-rectangle of type $x$. If $S$ is balanced, then it is easy to see that $T$ is also balanced.

Since $n$-tuple magic $(p_i, q)$-rectangles correspond to centrally $n$-tuple symmetric $(p_i, q)$-rectangles of type $-1/2$, we have the following corollary.

**Corollary 1** Suppose an $n$-tuple magic $(p_i, q)$-rectangle and a balanced centrally $n$-tuple symmetric $(p^+_q, q)$-rectangle exist. Then an $n$-tuple magic $(p_i + p^+_q, q)$-rectangle exists.

Using the concept of a centrally $n$-tuple symmetric rectangle, we can prove the existence of even $n$-tuple magic rectangles. Our tools are the balanced centrally $n$-tuple symmetric $(2, 4)$-rectangle $A := (a^1_{i,j}, a^2_{i,j}, ..., a^n_{i,j})$ given by

$$ (a^t_{i,j}) = \begin{pmatrix} 4t - 3 & -4t + 2 & -4t + 1 & 4t \\ -4t + 3 & 4t - 2 & 4t - 1 & -4t \end{pmatrix}, $$

and the $n$-tuple magic $(2, 6)$-rectangle $B := (b^1_{i,j}, b^2_{i,j}, ..., b^n_{i,j})$ given by

$$ b^t_{i,j} = \begin{pmatrix} 1 + 12(n - t) & 11 + 12(n - t) & 3 + 12(n - t) \\ 9 + 12(n - t) & 8 + 12(n - t) & 7 + 12(n - t) \end{pmatrix} $$

for all $t \in \{1, 2, ..., n\}$.

**Proposition 5** Let $q > 2$ be an even integer. Then an $n$-tuple magic $(2, q')$-rectangle exists.

**Proof.** We induct on $q$. The existence of $n$-tuple rectangles $A$ and $B$ shows that we need only prove the proposition for $q \geq 8$. Assume we know that an $n$-tuple magic $(2, q')$-rectangle exists for all even $q' < q$. Then we know an $n$-tuple magic $(2, q - 4)$-rectangle $R$ exists. By Corollary 1, we can add $R$ and $A$ together to form an $n$-tuple magic $(2, q)$-rectangle.

**Proposition 6** Let $p$ and $q$ be even positive integers with $(p, q) \neq (2, 2)$. Then an $n$-tuple magic $(p, q)$-rectangle exists.

**Proof.** By Proposition 5, we can assume that $q > 2$. Using $A$ and Proposition 4, induction shows that a balanced centrally $n$-tuple symmetric $(p, 4)$-rectangle $R$ exists. Thus, an $n$-tuple magic $(p, 4)$-rectangle exists and we can assume that $q > 4$. Now assume that an $n$-tuple magic $(p, q')$-rectangle exists for all even $q' < q$. We then know that an $n$-tuple magic $(p, q - 4)$-rectangle $S$ exists. By Corollary 1, we can add $R$ and $S$ together to give an $n$-tuple magic $(p, q)$-rectangle.

**Example 2** The following arrays are examples of even $n$-tuple magic rectangles.

A triple magic $(6, 4)$-rectangle

$$ \begin{pmatrix} 1 & 19 & 66 & 60 & 25 & 43 & 42 & 36 \\ 50 & 68 & 17 & 11 & 26 & 44 & 41 & 35 \\ 3 & 21 & 64 & 58 & 27 & 45 & 40 & 34 \\ 70 & 52 & 9 & 15 & 46 & 28 & 33 & 39 \\ 23 & 5 & 56 & 62 & 47 & 29 & 32 & 38 \\ 72 & 54 & 7 & 13 & 48 & 30 & 31 & 37 \end{pmatrix} $$

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Then each row sum of any array is 146 and each column sum of any array is 219.

A 4-tuple magic \((4, 4)\)-rectangle

\[
\begin{pmatrix}
7 & 12 & 49 & 62 \\
50 & 61 & 8 & 11 \\
16 & 3 & 58 & 53 \\
57 & 54 & 15 & 4
\end{pmatrix}
\begin{pmatrix}
23 & 28 & 33 & 46 \\
34 & 45 & 24 & 27 \\
32 & 19 & 42 & 37 \\
41 & 38 & 31 & 20
\end{pmatrix}
\begin{pmatrix}
39 & 44 & 17 & 30 \\
18 & 29 & 40 & 43 \\
48 & 35 & 26 & 21 \\
25 & 22 & 47 & 36
\end{pmatrix}
\begin{pmatrix}
55 & 60 & 1 & 14 \\
2 & 13 & 56 & 59 \\
64 & 51 & 10 & 5 \\
9 & 6 & 63 & 52
\end{pmatrix}
\]

Then each row sum and each column sum of any array in a rectangle equals 130.

### III. The n-Fold Self-Union of Complete Bipartite Graphs

For any integer \(n \geq 1\), the \(n\)-fold self-union of a graph \(G\), denoted by \(nG\), is the union of \(n\) disjoint copies of \(G\). In this section we identify the sufficient condition for degree-magic labellings of the \(n\)-fold self-union of complete bipartite graphs \(nK_{p,q}=K^1_{p,q} \cup K^2_{p,q} \cup \ldots \cup K^n_{p,q}\) to exist.

**Theorem 3** For any integer \(n \geq 1\) and even integers \(p, q > 1\), let \(K^{\ast}_{p,q}\) be the \(t^\ast\) copy of \(K_{p,q}\) for all \(t \in \{1, 2, \ldots, n\}\). A mapping \(f\) from \(E(nK_{p,q})\) into positive integers given by

\[
f(u'_{i,j}) = r'_{i,j} \quad \text{for every} \quad u'_{i,j} \in E(K^{\ast}_{p,q}),
\]

is a d-magic labelling of \(nK_{p,q}\) if and only if \(R := (r'_{i,j}) (r'_{i,j}) \ldots (r'_{i,j})\) is an \(n\)-tuple magic \((p, q)\)-rectangle.

**Proof.** Let \(U' = \{u'_{1,1}, u'_{1,2}, \ldots, u'_{1,p}\}\) and \(V' = \{v'_{1}, v'_{2}, \ldots, v'_{q}\}\) be partite sets of \(K^{\ast}_{p,q}\). Suppose that \(R\) is an \(n\)-tuple magic \((p, q)\)-rectangle. Then \(f\) is a bijection from \(E(nK_{p,q})\) onto \(\{1, 2, \ldots, npq\}\). For any \(u'_{i} \in U'\), we have

\[
f^\ast(u') = \sum_{j=1}^{q} f(u'_{i,j}) = \sum_{j=1}^{q} r'_{i,j} = q(npq+1) = \frac{npq+1}{2} \deg(u'),
\]

and for any \(v'_{j} \in V'\), we have

\[
f^\ast(v') = \sum_{i=1}^{p} f(u'_{i,j}) = \sum_{i=1}^{p} r'_{i,j} = p(npq+1) = \frac{npq+1}{2} \deg(v'),
\]

i.e., \(f\) is a d-magic labelling of \(nK_{p,q}\).

Now suppose that \(f\) is a d-magic labelling of \(nK_{p,q}\). For all \(1 \leq i \neq s \leq p\), we have

\[
\sum_{j} r'_{i,j} = \sum_{j} f(u'_{i,j}) = f^\ast(u') = f^\ast(u') = \sum_{j} r'_{i,j}.
\]

For all \(1 \leq j \neq z \leq q\), we have

\[
\sum_{i} r'_{i,j} = \sum_{i} f(u'_{i,j}) = f^\ast(v') = f^\ast(v') = \sum_{i} r'_{i,j}.
\]

By (5), we have

\[
\sum_{j} r'_{i,j} = \sum_{j} q(npq+1) = \frac{npq+1}{2}.
\]

By (6), we have

\[
\sum_{i} r'_{i,j} = \sum_{i} p(npq+1) = \frac{npq+1}{2}.
\]

Therefore, \(R\) is an \(n\)-tuple magic \((p, q)\)-rectangle.

According to Theorem 3 and Proposition 6, we obtain the following result.

**Proposition 7** Let \(p\) and \(q\) be even positive integers with \((p, q) \neq (2, 2)\). Then \(nK_{p,q}\) is a d-magic graph for all integers \(n \geq 1\).

**Example 3** We can construct a d-magic graph \(3K_{4,8}\) (see Figure 1) with the labels on edges \(u'_{i,j}\) of \(3K_{4,8}\), where \(1 \leq i \leq 4, 1 \leq j \leq 8\) and \(1 \leq t \leq 3\), in TABLE I.

![Figure 1](image-url)
TABLE I. THE LABELS ON EDGES OF D-MAGIC GRAPH $3K_{4,8}$

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References