Recursive utility and optimal growth with bounded or unbounded returns

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Abstract

In this paper we propose a unifying approach to the study of recursive economic problems. Postulating an aggregator function as the fundamental expression of tastes, we explore conditions under which a utility function can be constructed. We also modify the usual dynamic programming arguments to include this class of models. We show that Bellman’s equation still holds, so many results known for the additively separable case can be generalized for this general description of preferences. Our approach is general, allowing for both bounded and unbounded (above/below) returns. Many recursive economic models, including the standard examples studied in the literature, are particular cases of our setting.

Keywords: Recursive Utility, Dynamic Programming, Bellman Equation, Unbounded Returns.

JEL Classification: C61, D90

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1 Introduction

Many dynamic economic models rely on the assumption that preferences are represented by a functional which is additive over time and discounts future rewards at a constant rate. The additively separable hypothesis has long been recognised as special, but has dominated research in economics mainly because of its technical convenience. In the last few years, some criticism has been directed towards this assumption challenging its economic plausibility.

An important implication that stems from the axiomatic structure of time-additively separable preferences is that the rate of impatience is a specified constant, independent of the consumption stream. Obviously, this is very restrictive since it excludes plausible situations in which changes in consumption in the initial periods or an overall increase in consumption profile would affect impatience. A second and more severe complication arises in heterogeneous agents models. In this case, additive separability leads to a rather strong conclusion: unless all the agents have the same discount factor, the most patient agent ends up with a positive consumption in the long run while all other agents consume nothing. A third complication arises when uncertainty is introduced. Aversion to risk and aversion to intertemporal substitution are two conceptually distinct attributes of a consumer’s tastes and should be treated independently of each other. Time-additively separable preferences under expected utility do not allow this separation.

Recursive utility provides a way to escape these problems by allowing a flexible rate of time preference determined endogenously by the underlying consumption stream. Roughly speaking, there are two approaches to the construction of recursive utility functions. The first follows Koopman’s (1960) early work and is concerned with the axiomatics of preferences leading to a recursive representation of utilities. The second takes an aggregator function as the fundamental expression of tastes and then tries to recover the utility function from the assumed properties of the aggregator. The first paper that made the aggregator approach feasible was Lucas and Stokey (1984). Becker and Boyd (1997) provide an excellent exposition of these two approaches.

This paper rests on the second approach. Its primary aim is to explore conditions under which an aggregator could determine a unique utility function. However, since optimal growth models provide the basic framework for applied dynamic analysis in a wide range of areas in economics, and since recursive utility has important implications for standard results in these areas, it is important to
extend the study of optimal growth to a general class of recursive utility preferences. This is the second objective of the paper.

When the aggregator function is assumed to be bounded, it is always possible to associate with it a unique utility function. The proof of this result is well-known and relies on exploiting a contraction argument (see Lucas and Stokey (1984) and Dana and Le Van (1990)). Unbounded aggregators create some difficulties since the contraction method cannot be applied directly. However, this problem can partially be resolved if one introduces a weaker notion of boundedness. This approach has been followed by Boyd (1990) and Duran (2000) to deal with aggregators which are bounded from below but unbounded from above. Their argument relies on relaxing boundedness by considering functions that obey a growth condition. It is in this new space of functions that they obtain the contraction property for the recursion operator. Another very interesting approach has been proposed by Streufert (1990). His idea is based on the notion of bicovergence, a limiting condition ensuring that returns of any feasible path are sufficiently discounted from above (upper convergence) and sufficiently discounted from below (lower convergence).

For aggregators that allow $-\infty$ as a value, no general existence and uniqueness theorem has been established. Several authors have followed different methods to overcome this problem with the one proposed by Boyd (1990) being, to our knowledge, the most general one. Boyd combined the weighted contraction theorem with a “partial sum” technique to construct the utility functions.

Our paper introduces a unified treatment of the aggregator approach, covering situations in which the recursion operator is bounded or unbounded (below/above). Our assumptions are easy to check and apply to a wide class of aggregators including the standard examples mentioned in the literature. Following our approach, the recoverability of a utility function does not rely on a contraction mapping technique. Instead, our proof rests on important insights derived directly from the assumptions made on the aggregator function. Intuitively, one expects to obtain the utility function by recursive substitution as the pointwise limit of partial sums of returns. Under our assumptions, a utility function constructed in that way is always well-defined and recursive. Moreover, it is upper semicontinuous and satisfies a kind of transversality condition. It will turn out that these two properties of the return function are fundamental to establish its uniqueness as a reasonable solution.

Contraction methods have also been applied to obtain a fixed point of the maximizing operator in models with recursive preferences. Dynaming program-
ming arguments are well-established in models with bounded returns (see Lucas and Stokey (1984), Dana and Le Van (1991)). Moreover, when the returns are unbounded from above (but bounded from below), it is always possible to use a weighted contraction argument to escape the problem. This is the case in Becker and Boyd (1997) and Duran (2000).

For returns unbounded from below, the best-known results have been established for the case of time-additively separable preferences. In addressing the problem of unbounded returns, Alvarez and Stokey (1998) study a wide class of homogeneous problems. They show that the Principle of Optimality applies to problems of this sort. Their approach is based on finding restrictions that bound the growth rates of state variables from above along any feasible path, and, for the case where the utility may attain $-\infty$, from below along at least one feasible path. In their approach, both decreasing and increasing returns technologies are excluded. Another recent approach has been proposed by Rincon-Zapatero and Rodriguez-Palmero (2002, 2003). Their argument is mainly based on the contraction technique. But instead of considering the usual normed space of functions in which the Bellman operator fails to be a contraction, they focus on metric spaces, which are different depending on the characteristics of the problem. They also apply their method to deal with recursive preferences, but they only consider the case where returns are unbounded from above but bounded from below.

Le Van and Morhaim (2002) introduce a synthetic frame to the study of dynamic programming problems with time-additively separable objectives and bounded or unbounded (below/above) returns. Their argument does not depend on a contraction mapping technique but rather builds on important insights derived from the assumptions imposed on the return functions. In this paper, we go a step further and develop an analogous argument for recursive models. Our approach is general, allowing for both bounded and unbounded (below/above) returns without imposing additional restrictions on the technology apart from the usual ones. Many recursive economic models, including the standard examples studied in the literature, are particular cases of our setting.

The paper is organized as follows. In section 4.2, we introduce the aggregator and show how a utility function can be recovered from it. We also present some applications of our technique by considering specific forms for the aggregator function. In section 4.3, we prove existence of optimal paths and establish the properties of the value function. It turns out that the value function is the unique solution to the Bellman equation within the class of functions that satisfy these properties. Furthermore, we show that the operator defined by the Bellman
equation provides an algorithm to find the value function.

2 Recursive Preferences

An infinite sequence of elements in $\mathbb{R}^n$ will be denoted by $\mathbf{x}$, i.e. $\mathbf{x} = (x_t)_{t=0}^{\infty}$, with $x_t \in \mathbb{R}^n$, $\forall t$. The space of these sequences is denoted by $(\mathbb{R}^n)^\infty$.

Define $X(\beta) = \{ \mathbf{x} \in (\mathbb{R}_+^n) : \sup_t \frac{||x_t||}{\beta_t} < \infty \}$ for $\beta \geq 1$ and $X(\beta, M) = \{ \mathbf{x} \in (\mathbb{R}_+^n) : \sup_t \frac{||x_t||}{\beta_t} < M \}$ for $\beta \geq 1$ and $M > 0$. Note that $X(\beta) = \bigcup_{M > 0} X(\beta, M)$.

Let $\pi$ denote the first coordinate projection function and $S$ be the shift operator, i.e. $\pi \mathbf{x} = x_0$ and $S \mathbf{x} = (x_1, x_2, ... x_t, ...)$.

**Definition 1** A function $W : X \times Y \rightarrow Y \cup \{-\infty\}$, $X \subseteq \mathbb{R}_+^n$, $Y \subseteq \mathbb{R}$ with $0 \in Y$, is an aggregator if it satisfies the following properties:

(\textbf{W1}) There exists a set $D \subset \mathbb{R}_+^n$ such that:
- a) $D \neq X$
- b) $\forall (x, z) \in D \times Y$, $W(x, z) = -\infty$
- c) $\lim_{z \to -\infty} W(x, z) = -\infty$, $\forall x \in X$
- d) $\forall (x, z) \notin D \times Y$, $W(x, z) \in \mathbb{R}$.

(\textbf{W2}) The function $W$ is nondecreasing in its second argument.

(\textbf{W3}) The function $W$ is continuous at any $(x, z) \notin D \times Y$. If $(x, z) \in D \times Y$ and if $\lim_n (x^n, z^n) = (x, z)$, then $\lim_n W(x^n, z^n) = -\infty$.

(\textbf{W4}) There exists $\delta > 0$ such that

$$|W(x, z) - W(x, z')| \leq \delta |z - z'|$$

for all $x \notin D$ and all $z, z' \in Y$.

(\textbf{W5}) For any $\beta \geq 1$ and any $M > 0$, there exists a function $\varphi(\beta) \in (0, 1)$ and a positive function $\psi(M, \beta)$, such that, $\forall x \in X(\beta, M)$, $\forall N \geq 0$, we have

$$\delta^N W(x_N, 0) \leq \varphi(\beta)^N \psi(M, \beta).$$

(\textbf{W6}) For any $\beta \geq 1$, there exists $\mathbf{x} \in X(\beta)$, such that

$$\lim_{N \to \infty} W(x_0, W(x_1, ..., W(x_N, 0), ...)) > -\infty.$$
2.1 Comments on the assumptions

1) Starting from an aggregator function, one can define an operator $R$ over all extended real-valued functions $U$ on $X(\beta)$ as follows:

$$RU(x) = W(\pi x, U(Sx)).$$

We are interested in utility functions that are fixed points of $R$. Intuitively, we expect to recover the utility function by recursive substitution, as the limit of

$$R^N 0(x) = W(x_0, W(x_1, ..., W(x_N, 0))...).$$

Therefore, $\forall x \in X(\beta)$ define

$$U^0(x) = \lim_{N \to \infty} R^N 0(x) = \lim_{N \to \infty} W(x_0, W(x_1, ..., W(x_N, 0))...).$$ (1)

2) Because of (W1), our set of admissible aggregators is allowed to contain aggregators that are unbounded from below. For such aggregators, $U^0$ may not be the only solution of $R$ since $U(x) = -\infty$ satisfies the recursion too.

In exploring conditions under which an unbounded aggregator could determine a unique utility function, Boyd (1990) and Duran (2000) face a similar problem. For aggregators bounded from below they are able to establish a general existence and uniqueness result. Their approach is based on finding a positive and continuous real-valued function $\Psi$, defined on a set $A \subset (\mathbb{R}_n^+)^\infty$, such that $W(\pi x, 0)$ is $\Psi$-bounded, i.e. $\sup_{x \in A} \frac{|W(\pi x, 0)|}{\Psi(x)} < +\infty$, and $\delta \|\Psi \circ S\|_{\Psi} < 1$, i.e. $\delta \sup_{x \in A} \frac{\Psi(Sx)}{\Psi(x)} < 1$. In Boyd, $A = X(\beta)$, while in Duran, $A$ is the set of feasible action plans endowed with the product topology. If such a function exists, it can be shown that the recursive operator $R$ is a contraction on $C_\Psi(A)$ (the space of $\Psi$-bounded and continuous functions on $A$). Since $C_\Psi(A)$ endowed with the norm $\|f\|_\Psi = \sup_{x \in A} \frac{|f(x)|}{\Psi(x)}$ is a Banach space, $R$ has a unique fixed point $U^\ast$. Since $\lim_{N \to \infty} R^N 0 = U^\ast$, it follows that $U^\ast = U^0$.

Boyd (1990) deals also with aggregators that are unbounded from below. To circumvent the problem posed by paths that permit $U(x) = -\infty$, he considers first a region that excludes them from the set of admissible paths. The utility function is initially defined on the set of sequences with growth rates bounded from above and from below. As before, a contraction argument applies on this set, yielding a unique $\Psi$-bounded utility function $U^\ast$. Next, by using a limiting argument analogous to partial summation, he extends $U^\ast$ to the utility function $U^0$ defined on all $X(\beta)$. It turns out that $U^0$ is the only recursive upper semicontinuous extension of $U^\ast$ to $X(\beta)$. 


3) Assumption \((W2)\) is weaker than that imposed in Boyd (1990) (he requires the aggregator to be increasing with respect to both variables). Boyd makes use of the monotonicity of the aggregator with respect to its first argument to obtain the upper semicontinuity of \(U^0\).

4) Assumption \((W3)\) is standard and means that \(W\) is continuous in a generalised sense.

5) Note that we do not impose \(\delta < 1\) in \((W4)\). That is, we do not exclude undiscounted or upcounting models. We will see below that for a large class of aggregators we can allow for \(\delta \geq 1\), but at the expense of imposing a stronger requirement in \((W5)\).

6) Assumption \((W5)\) is a crucial one. It is essential to establish the meaningfulness of the limit in (1), as well as, its uniqueness as a reasonable solution of \(R\). Note that Boyd (1990) and Duran (2000) implicitly impose a stronger requirement than \((W5)\). Requiring the recursion operator at zero to be \(\Psi\)-bounded, actually implies that for every \(x \in A\), there exists \(m > 0\), such that
\[
|W(x_N, 0)| < m, \quad \forall N \geq 0.
\]
The following relation is therefore true:
\[
\delta^N |W(x_N, 0)| \leq \delta \frac{\Psi(S^N x)}{\Psi(S^{N-1} x)} \times \delta \frac{\Psi(S^{N-1} x)}{\Psi(S^{N-2} x)} \times \ldots \times \delta \frac{\Psi(S x)}{\Psi(x)} \Psi(x) m.
\]
Provided that \(\max\{\Psi(x) : x \in A\} \in \mathbb{R}\) and \(\Psi\) satisfies \(\alpha = \delta \|\Psi \circ S\|_\Psi < 1\), it follows that for any \(N \geq 0\),
\[
\delta^N |W(x_N, 0)| \leq \varphi(\beta)^N \psi(M, \beta),
\]
where \(\varphi(\beta) = \alpha\) and \(\psi(M, \beta) = m \max\{\Psi(x) : x \in A\}\). Obviously, the above requirement is stronger than \((W5)\).

7) As we mentioned above, our analysis encompasses undiscounted and upcounted models \((\delta \geq 1)\), provided that one imposes a stronger requirement in \((W5)\). We introduce this assumption below:

\((W5')\) \(W\) satisfies assumption \((W5)\) and there exists at least one \(x \in X(\beta, M)\), such that, for any \(N \geq 0\),
\[
\delta^N |W(x_N, 0)| \leq \varphi(\beta)^N \psi(M, \beta).
\]

There is a broad class of aggregators that verify this assumption. Clearly, \((W5)\) implies \((W5')\) for aggregators that satisfy \(W(x, 0) \geq 0\). Another application is to aggregators that satisfy \(-bx^\rho \leq W(x, 0) \leq 0\), where \(b > 0\), \(\rho < 0\) and \(\delta^\beta \rho < 1\). Consider the sequence \(x = (1, \beta, \ldots, \beta^t, \ldots)\). For any \(N \geq 0\), one has:
\[ \delta^N |W(x_N, 0)| = \delta^N |W(\beta^N, 0)| \leq (\delta\beta^N)^N b. \]

8. Assumption (W6) implies the existence of an action plan \( x \in X(\beta) \), such that \( U^0(x) > -\infty \).

2.2 The Existence of Recursive Utility

Lemma 1 Let \( \beta \geq 1 \) and \( M > 0 \). Assume (W1), (W2), (W4) and (W5). Then, for every \( x \in X(\beta, M) \):

i) \( \lim_{N \to \infty} R_N^0(x) \) exists in \( \mathbb{R} \cup \{-\infty\} \) and is uniformly bounded from above.

ii) If \( \delta < 1 \), then for any \( y \in \mathbb{R} \),

\[ \lim_{N \to \infty} W(x_0, W(x_1, ..., W(x_N, y)) ...) = \lim_{N \to \infty} R_N^0(x). \]

Proof: i) Let \( x \in X(\beta, M) \). Assume first that \( x_t \in \mathbb{D} \) for some \( t \). In this case, (W1) implies that \( W(x_0, W(x_1, ..., W(x_N, 0)) ...) = -\infty, \ \forall N \geq t \). Therefore,

\[ \lim_{N \to \infty} R_N^0(x) = -\infty. \]

Consider now the case where \( x_t \notin \mathbb{D}, \ \forall t \). Define

\[ S_N = W(x_0, W(x_1, ..., W(x_N, 0)) ...) - W(x_0, W(x_1, ..., W(x_{N-1}, 0)) ...), \]

and note that

\[ W(x_0, W(x_1, ..., W(x_N, 0)) ...) = S_N + S_{N-1} + ... + S_1 + W(x_0, 0) \]

\[ = (S_N^+ - S_{N-1}^-) + (S_{N-1}^+ - S_{N-2}^-) + ... + (S_1^+ - S_0^-) + W(x_0, 0). \]

From (W2), (W4) and (W5) we have:

\[ S_N \leq W(x_0, W(x_1, ..., W(x_N, 0)) ...) - W(x_0, W(x_1, ..., W(x_{N-1}, 0)) ...) \leq \delta^N W^+(x_N, 0) \leq \varphi(\beta)^N \psi(M, \beta). \]

This implies, \( S_N^+ \leq \varphi(\beta)^N \psi(M, \beta) \) and \( \sum_{N=1}^{\infty} S_N^+ \leq \sum_{N=1}^{\infty} S_N^- + W(x_0, 0) \). It follows that

\[ \lim_{N \to \infty} R_N^0(x) = \sum_{N=1}^{\infty} S_N^+ - \sum_{N=1}^{\infty} S_N^- + W(x_0, 0) \]

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exists in $\mathbb{R} \cup \{-\infty\}$.

Define $J = \sup\{W(x_0, 0): ||x_0|| \leq M\}$. Since $\mathbb{D} \neq \mathbb{R}_+^n$ and $W$ is continuous, $J \in \mathbb{R}$. We obtain:

$$\lim_{N \to \infty} R^N 0(x) \leq \frac{\psi(M, \beta)}{1 - \varphi(\beta)} + J.$$  

ii) Let $x \in X(\beta, M)$. If $x_t \in \mathbb{D}$ for some $t$, then for any $y \in \mathbb{R}$,

$$\lim_{N \to \infty} W(x_0, W(x_1, ..., W(x_N, y))) = \lim_{N \to \infty} R^N 0(x) = -\infty.$$  

Assume that $x_t \notin \mathbb{D}, \forall t$. Assumption (W4) implies that

$$|W(x_0, W(x_1, ..., W(x_N, y))) - W(x_0, W(x_1, ..., W(x_N, 0)))| \leq \delta^{N+1} |y|.$$  

Since $\delta < 1$, the result is obvious. ■

Remark 1: We know that for every $x \in X(\beta)$, there exists $M > 0$, such that $x \in X(\beta, M)$. Based on the previous lemma, we conclude that $U^0(x) = \lim_{N \to \infty} R^N 0(x)$ is always meaningful on $X(\beta)$.

Proposition 1 Let $\beta \geq 1$. Assume (W1)-(W5). Then:

i) The return function $U^0$ is a fixed point of the recursive operator $R$.

ii) For any $x \in X(\beta)$, $\lim_{N \to \infty} \delta^N [U^0(S^N x)]^+ = 0$. More precisely, for any $M > 0$, for any $\varepsilon > 0$, there exists an integer $N_0$, such that, for any $x \in X(\beta, M)$, for any $N \geq N_0$, we have $\delta^N [U^0(S^N x)]^+ \leq \varepsilon$.

iii) The return function $U^0$ is upper semicontinuous on $X(\beta, M)$ for the product topology.

iv) The return function $U^0$ is $\beta$-upper semicontinuous on $X(\beta)$.

Proof: i) Let $x \in X(\beta)$. Observe that if $x_0 \in \mathbb{D}$ or $U^0(Sx) = -\infty$, (W1) implies $W(\pi x, U^0(Sx)) = -\infty = U^0(x)$. Otherwise, (W3) implies,

$$U^0(x) = \lim_{N \to \infty} W(x_0, W(x_1, ..., W(x_N, 0))) = W(x_0, \lim_{N \to \infty} W(x_1, ..., W(x_N, 0))) = W(\pi x, U^0(Sx)).$$

ii) Let $x \in X(\beta)$. If $x_t \in \mathbb{D}$ for some $t$, then $[U^0(S^N x)]^+ = 0, \forall N \geq t$ and the claim is true. Assume that $x_t \notin \mathbb{D}, \forall t$. Assumptions (W2), (W4) imply that

$$W(x_N, W(x_{N+1}, ..., W(x_{N+t}, 0))) \leq W^+(x_N, 0) + \delta W^+(x_{N+1}, 0) + ... + \delta^t W^+(x_{N+t}, 0).$$

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Since $x \in X(\beta)$, there exists $M > 0$, such that $x \in X(\beta, M)$. Using (W5), we get:

$$\delta^n W(x_N, W(x_{N+1}, \ldots, W(x_{N+t}, 0))) \leq \delta^n W^+(x_N, 0) + \delta^{n+1} W^+(x_{N+1}, 0) + \ldots + \delta^{n+t} W^+(x_{N+t}, 0) \leq \varphi(\beta)^n (1 + \varphi(\beta) + \ldots + \varphi(\beta)^t) \psi(M, \beta).$$

Fix $N$ and let $t \to \infty$. We obtain:

$$\delta^n U^0(S^N x) \leq \frac{\varphi(\beta)^N}{1 - \varphi(\beta)} \psi(M, \beta).$$

Since $\varphi(\beta) < 1$, for every $\varepsilon > 0$, one can always find $N_0$ such that, if $N \geq N_0$, then $\frac{\varphi(\beta)^N}{1 - \varphi(\beta)} \psi(M, \beta) \leq \varepsilon$. It follows that $\lim_{N \to \infty} \delta^n [U^0(S^N x)]^+ = 0$.

iii) Let $(x^n) \in X(\beta, M)$ be a sequence such that $x^n \to x \in X(\beta, M)$ for the product topology. We distinguish three cases.

**Case 1:** Assume that $x^n \not\in \mathbb{D}$, $\forall n$, $\forall t$.

Fix $N$ and choose $t > N$. From (W2), (W4) and (W5) we obtain:

$$W(x^n_0, W(x^n_1, \ldots, W(x^n_N, 0))) = W(x^n_0, W(x^n_1, \ldots, W(x^n_{N-1}, 0))) - W(x^n_0, W(x^n_1, \ldots, W(x^n_{N-1}, 0))) + W(x^n_0, W(x^n_1, \ldots, W(x^n_{N-2}, 0))) - W(x^n_0, W(x^n_1, \ldots, W(x^n_{N-2}, 0))) + W(x^n_0, W(x^n_1, \ldots, W(x^n_N, 0))) \leq \delta^n W^+(x^n_0, 0) + \ldots + \delta^{n+t} W^+(x^n_{N+1}, 0) + W(x^n_0, W(x^n_1, \ldots, W(x^n_N, 0))) \leq \varphi(\beta)^t \psi(M, \beta) + \ldots + \varphi(\beta)^{N+1} \psi(M, \beta) + W(x^n_0, W(x^n_1, \ldots, W(x^n_N, 0))) \leq \frac{\varphi(\beta)^{N+1}}{1 - \varphi(\beta)} \psi(M, \beta) + W(x^n_0, W(x^n_1, \ldots, W(x^n_N, 0))).$$

Fix $n$ and $N$ and let $t \to \infty$. We get:

$$U^0(x^n) \leq \frac{\varphi(\beta)^{N+1}}{1 - \varphi(\beta)} \psi(M, \beta) + W(x^n_0, W(x^n_1, \ldots, W(x^n_N, 0))).$$

Let $n \to \infty$:

$$\limsup_n U^0(x^n) \leq \frac{\varphi(\beta)^{N+1}}{1 - \varphi(\beta)} \psi(M, \beta) + W(x_0, W(x_1, \ldots, W(x_N, 0))).$$
Let $N \to \infty$:

$$\limsup_{n} U^{0}(x^{n}) \leq U^{0}(x).$$

**Case 2:** Assume that for any $n$, there exists $t$, such that $x^{n}_{t} \in D$.

In this case,

$$\limsup_{n} U^{0}(x^{n}) = -\infty \leq U^{0}(x).$$

**Case 3:** Assume that there exists a subsequence $(n_{k})$, such that for any $k$, there exists $t$ with $x^{n_{k}}_{t} \in D$.

Observe that $U^{0}(x^{n_{k}}) = -\infty$. Let $(n_{k}')$ be a subsequence such that $x^{n_{k}'}_{t} \notin D$, $\forall k'$, $\forall t$. We have seen (Case 1) that $\limsup_{k'} U^{0}(x^{n_{k}'}) \leq U^{0}(x)$. It follows that

$$\limsup_{n} U^{0}(x^{n}) = \limsup_{k'} U^{0}(x^{n_{k}'}) \leq U^{0}(x).$$

iv) Recall that $U^{0}$ is meaningful on $X(\beta)$. Let $(x^{n}) \in X(\beta)$ be a sequence such that $x^{n} \to x \in X(\beta)$ for the $\beta$-topology. Note that, for $n$ large enough, there exists $M > 0$, such that $x^{n} \in X(\beta, M)$ and $x \in X(\beta, M)$. Because of (iii), the claim is true.

We have the following theorem.

**Theorem 1** Let $\beta \geq 1$ and denote by $H$ the set of upper semicontinuous functions on $X(\beta)$ that satisfy: If $U \in H$,

(a) there exists at least one $x \in X(\beta)$ such that $U(x) > -\infty$

(b) $\lim_{N \to \infty} \delta^{N}[U(S^{N}x)]^{+} = 0$, $\forall x \in X(\beta)$.

Assume (W1)-(W6) and impose $\delta < 1$ in (W4). Then, the return function $U^{0}$ defined in (1) is the only fixed point of the recursion operator $R$ in $H$.

**Proof:** Let $U$ be another $\beta$-upper semicontinuous recursive utility function that satisfies (a) and (b). Let $x \in X(\beta)$. If $x_{t} \in D$ for some $t$, then $U(x) = U^{0}(x) = -\infty$ and the claim is true. Assume that $x_{t} \notin D$, $\forall t$. In this case (W2) and (W2) imply,

$$W(x_{0}, ..., W(x_{N-1}, U(S^{N}x))...)) - W(x_{0}, ..., W(x_{N-1}, 0)...)) \leq \delta^{N}[U(S^{N}x)]^{+}.$$ 

Since $U$ satisfies property (b), taking the limits on both sides when $N \to \infty$ gives, $U(x) \leq U^{0}(x)$. We next show that $U(x) \geq U^{0}(x)$.  

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Since $U$ satisfies property $(a)$, there exists $x' \in X(\beta)$ and $y \in \mathbb{R}$ such that $U(x') = y$. Let $M$ be such that both $x$ and $x'$ belong to $X(\beta, M)$. Consider the sequence $(x^N)$ defined as follows: $x^N_t = x_t$, for $t = 0, ..., N - 1$ and $x^N_t = x'_{t-N}$, for $t \geq N$. Obviously, $(x^N) \in X(\beta, M)$ and $x^N \to x$ for the product topology. We have:

$$U(x^N) = W(x^N_0, U(Sx^N)) = W(x^N_0, W(x^N_1, ... , W(x^N_{N-1}, U(Sx^N))) ... ) = W(x_0, W(x_1, ... , W(x_{N-1}, U(x')))) = W(x_0, W(x_1, ... , W(x_{N-1}, y))).$$

The upper semicontinuity of $U$ together with Lemma 1(ii) imply:

$$U(x) \geq \limsup_N U(x^N) = \lim_{N \to \infty} W(x_0, W(x_1, ... , W(x_{N-1}, y))) = \lim_{N \to \infty} W(x_0, W(x_1, ... , W(x_{N-1}, 0))) = U^0(x).$$

We conclude the proof. ■

Remark 2: Since Theorem 1 requires $\delta < 1$, it is not applicable to undiscounted or upcounted models. We show below that our uniqueness result is still valid in cases where $\delta \geq 1$, provided that $(W5)$ is replaced by $(W5')$. It is easy to see that $(W5')$ implies $(W6)$. Indeed, under $(W5')$, there exists $x \in X(\beta, M)$, such that for any $N \geq 0$, the following relation is true:

$$\delta^N |W(x_N, W(x_{N+1}, ... , W(x_{N+t}, 0)))| \leq \delta^N |W(x_N, 0)| + \delta^{N+1} |W(x_{N+1}, 0)| + ... + \delta^{N+t} |W(x_{N+t}, 0)| \leq \varphi(\beta)^N (1 + \varphi(\beta) + ... + \varphi(\beta)^t) \psi(M, \beta)$$

Fix $N$ and let $t \to \infty$. We obtain:

$$\delta^N |U^0(S^N x)| \leq \frac{\varphi(\beta)^N}{1 - \varphi(\beta)} \psi(M, \beta).$$

If we take $N = 0$, then $|U^0(x)| \leq \frac{1}{1 - \varphi(\beta)} \psi(M, \beta)$, in which case $(W6)$ is verified. In addition, we have $\lim_{N \to \infty} \delta^N |U^0(S^N x)| = 0$. We get the following result.
Theorem 2 Let $H'$ denote the set of functions $U$ that belong to $H$ and in addition satisfy:

$$\exists \, x \in X(\beta) \text{ such that } \lim_{N \to \infty} \delta^N |U(S^N x)| = 0.$$  

Assume $(W_1)$-$(W_5')$. Then, the return function $U^0$ defined in (1) is the only fixed point of the recursion operator $R$ in $H'$.

**Proof:** Let $U$ be another $\beta$-upper semicontinuous recursive utility function in $H'$. Let $x \in X(\beta)$. If $x_t \in D$ for some $t$, then $U(x) = U^0(x) = -\infty$ and the claim is true. Assume that $x_t \notin D, \forall t$. The proof of $U(x) \leq U^0(x)$ is identical to the one presented in Theorem 1. We next show that $U(x) \geq U^0(x)$.

Since $U$ belongs to $H'$, there exists $x' \in X(\beta)$, such that $\lim_{N \to \infty} \delta^N |U(S^N x')| = 0$. Let $M$ be such that both $x$ and $x'$ belong to $X(\beta, M)$. Consider the sequence $(x^N)$ defined as follows: $x^N_t = x_t$, for $t = 0, ..., N - 1$ and $x^N_t = x't$, for $t \geq N$. Note that $(x^N) \in X(\beta, M)$ and $x^N \to x$ for the product topology. We have:

$$U(x^N) = W(x_0^N, W(x_1^N, ..., W(x_{N-1}^N, U(S^N x^N)))...)$$

$$= W(x_0, W(x_1, ..., W(x_{N-1}, U(S^N x')))...).$$

It follows that

$$\left| W(x_0^N, ..., W(x_{N-1}^N, U(S^N x^N))) - W(x_0^N, ..., W(x_{N-1}, 0)) \right|$$

$$= \left| W(x_0, ..., W(x_{N-1}, U(S^N x'))) - W(x_0, ..., W(x_{N-1}, 0)) \right|$$

$$\leq \delta^N |U(S^N x')|.$$  

Taking the limits as $N \to \infty$ we get:

$$\lim_{N \to \infty} W(x_0, W(x_1, ..., W(x_{N-1}, U(S^N x')))...)) = U^0(x)$$

The upper semicontinuity of $U$ implies:

$$U(x) \geq \limsup_{N} U(x^N)$$

$$= \lim_{N \to \infty} W(x_0, W(x_1, ..., W(x_{N-1}, U(S^N x')))...))$$

$$= U^0(x).$$

We conclude the proof. \(\blacksquare\)
2.3 Examples

Example 1: Consider the class of aggregators which satisfy the following assumptions:

i) \( W : \mathbb{R}_+^n \times \mathbb{R}_+ \to \mathbb{R}_+ \) is continuous and nondecreasing in both arguments.

ii) \( W(0,0) = 0 \).

iii) There exists \( \delta \in ]0,1[ \) such that

\[
\left| W(x,z) - W(x,z') \right| \leq \delta \left| z - z' \right|
\]

for all \( x \in \mathbb{R}_+^n \) and all \( z, z' \in \mathbb{R}_+ \).

iv) For every \( x \in \mathbb{R}_+^n \), \( 0 \leq W(x,0) \leq A \|x\|^\rho + B \), where \( A \) and \( B \) are nonnegative constants and \( \rho > 0 \).

It is obvious that for this class of aggregators assumptions (W1)-(W4) and (W6) are satisfied. We show that (W5) is also verified.

Note that, \( \forall x \in X(\beta,M), \forall N \geq 0 \), one has:

\[
\delta^N W(x_N,0) \leq A \delta^N \|x_N\|^\rho + \delta^N B \leq (\delta^\beta)^N (AM^\rho + B).
\]

Provided that \( \delta^\beta < 1 \), (W5) is true.

Consider the following specification for \( W \), borrowed from Koopmans et al. (1964)

\[
W(c,z) = \frac{\zeta}{\gamma} \log (1 + ax^\rho + \gamma z), \text{ with } a, \rho, \gamma, \zeta > 0, \rho, \zeta < 1.
\]

It is obvious that assumptions (W1)-(W3) and (W6) are satisfied.

Observe that (W4) is also satisfied since

\[
\frac{\partial W}{\partial z} = \frac{\zeta}{(1 + ax^\rho + \gamma z)} \leq \zeta < 1.
\]
We next show that (W5) is verified. Note that \( \forall x \in X(\beta, M), \forall N \geq 0 \), one has:

\[
\delta^N W(x_N, 0) \leq \delta^N \frac{\zeta \log(1 + ax_N^\rho)}{\gamma} \\
\leq \delta^N \log\left(1 + a(\beta^N M)^\rho\right) \\
= \delta^N \left(1 + a(\beta^N M)^\rho\right)^\gamma \frac{\log\left(1 + a(\beta^N M)^\rho\right)}{(1 + a(\beta^N M)^\rho)} \\
\leq \delta^N \left(1 + a(\beta^N M)^\rho\right)^\gamma \\
\leq (\delta^\gamma)^N \left(1 + aM^\rho\right)^\gamma.
\]

Provided that \( \delta^\gamma < 1 \), (W5) is true.

**Example 2:** Consider the class of aggregators \( W : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+ \) defined by:

\[
W(x, z) = u(x)w(z/u(x)) = u(x)w(y).
\]

\( W \) is assumed to be continuous. We also assume that the function \( u \) is continuous, strictly increasing and homogeneous of degree \( \gamma > 0 \), while the function \( w \) is \( C^1 \), strictly increasing and concave. Furthermore, \( u \) and \( w \) satisfy, \( u(0) = 0 \), \( w(0) \neq 0 \) and \( 0 < \sup_{y \geq 0} w'(y) < 1 \). As an example consider the following specification for \( u \) and \( w \):

\[
u(x) = x^\gamma \quad \text{and} \quad w(y) = \frac{[1 + \delta y]^\rho}{\rho}, \quad \rho, \delta \in ]0, 1[.
\]

The importance of this class of aggregators stems from the fact that they give rise to recursive utilities (homogeneous of degree \( \gamma \)) that are consistent with balanced growth (see Dolmas (1996)).

Assumptions (W1)-(W3) and (W6) are trivially satisfied. Assumption (W4) is also verified for \( \delta = \sup_{y \geq 0} w'(y) \). Let us check (W5). Observe that \( \forall x \in X(\beta, M), \forall N \geq 0 \), one has:

\[
\delta^N W(x_N, 0) = \delta^N u(x_N)w(0) \\
\leq \delta^N (\beta^N M)^\gamma u(1)w(0) \\
\leq (\delta^\gamma)^N M^\gamma u(1)w(0).
\]

Provided that \( \delta^\gamma < 1 \), (W5) is satisfied.
Example 3 (CRRA): \( W(x, z) = \frac{x^\theta}{\theta} + \delta z, \ x \in \mathbb{R}_+, \ z \in \mathbb{R}, \) with \( \theta < 1 \) and \( \delta > 0. \) Note that for \( u(x) = \frac{x^\theta}{\theta} \) and \( w(y) = 1 + \delta y \) this aggregator is of the form,

\[
W(x, z) = u(x)w(z/u(x)) = u(x)w(y).
\]

When \( \theta, \delta \in [0, 1[, \ W \) belongs to the class of aggregators considered in Example 2. We examine the case where \( \theta < 0. \)

Obviously, assumptions (W1)-(W4) are satisfied. Since \( W(x, 0) \leq 0, \) (W5) is fulfilled. Observe that

\[-bx^\theta \leq W(x, 0) \leq 0,
\]

for some \( b \geq -\frac{1}{\theta}. \) Provided that \( \delta \beta^\theta < 1, \) (W5)' is also verified for \( x = (1, \beta, ..., \beta^t, ...). \)

Furthermore, since \( \beta^\theta < 1, \) one can permit \( \delta \geq 1. \)

Example 4 (Logarithmic Case): \( W(x, z) = \log x + \delta z, \ x \in \mathbb{R}_+, \ z \in \mathbb{R}, \) with \( \delta \in [0, 1[. \)

(W1)-(W4) are obvious. (W6) is also verified for \( x = (1, 1, ..., 1, ...). \) Let us check (W5).

Observe that \( \forall x \in X(\beta, M), \forall N \geq 0, \) one has:

\[
\delta^N W(x_N, 0) \leq N\delta^N \log \beta + \delta^N \log M.
\]

Choose \( \alpha \in [\delta, 1[ \) such that \( \log \left( \frac{\alpha}{\delta} \right) > \frac{\log 3}{3}. \) Note that \( \frac{\log 3}{3} > \frac{\log 2}{2} \) and that the function \( \frac{\log x}{x} \) is decreasing for \( x \geq 3. \) It follows that for every \( N \geq 0, \) \( \log \left( \frac{\alpha}{\delta} \right) > \frac{\log N}{N} \) or equivalently, \( \alpha^N > N\delta^N. \) Hence, \( \forall x \in X(\beta, M), \forall N \geq 0 \) one has:

\[
\delta^N W(x_N, 0) \leq \alpha^N \log \beta + \delta^N \log M
\]

Take \( \varphi(\beta) = \alpha \) and \( \varphi(M, \beta) = \log \beta + M. \)

Example 5 (Uzawa-Epstein-Hynes Aggregator): \( W(x, z) = (u(x) + z)e^{-v(x)}, \ x \in \mathbb{R}_+, \ z \in \mathbb{R}. \) The functions \( u \) and \( v \) are continuous and satisfy \( v \geq 0, \) \( v' > 0 \) and \( u < 0, \) \( u' > 0. \)

Assumptions (W1)-(W3) and (W5) are obvious. When \( v(0) > 0, \) (W4) is satisfied with \( \delta = e^{-v(0)} < 1. \) Assumption (W6) is also satisfied since \( \forall x \in X(\beta), \)

\[
\lim_{N \to \infty} R^N 0(x) = \sum_{t=0}^{\infty} u(x_t) \exp\left[-\sum_{\tau=0}^{t} v(x_{\tau})\right] > \sum_{t=0}^{\infty} u(0) \exp\left[-\sum_{\tau=0}^{t} v(0)\right].
\]

\[
= \frac{u(0)}{1 - e^{-v(0)}}.
\]
A special case appears when \( u(x) = -1 \) and \( v(x) = x \). Observe that in this case \( v(0) = 0 \), so (W4) is satisfied with \( \delta = 1 \). It is easy to see that (W5') is verified. Indeed, for \( x = (1, \beta, \ldots, \beta^t, \ldots) \) one has:
\[
\delta^N |W(x_N, 0)| = (e^{-\beta})^N
\]

Take \( \varphi(\beta) = (e^{-\beta}) \) and \( \psi(M, \beta) = 1 \).

**Example 6 (Rawls Criterion):** Let \( W(x, z) = \inf\{x, z\}, x \in \mathbb{R}_+, z \in \mathbb{R}_+ \). (W1)-(W3) and (W6) are trivially satisfied. One can easily check that (W4) is verified with \( \delta = 1 \). Since \( W(x_N, 0) = \inf\{x_N, 0\} = 0 \), (W5') is true.

**Remark 3:** In all of the examples presented above the aggregator function satisfies the assumptions (W1)-(W6) or (W1)-(W5'). Our existence and uniqueness theorem applies directly in these cases, yielding a unique recursive utility function \( U^0 \) defined as in (1). Consider now a monotonic transformation \( \phi \) to this recursive utility function. There are three important questions that naturally arise. First, does the new utility function \( bU = \phi \circ U^0 \) have a recursive representation? Or equivalently, does exist an aggregator function \( \hat{W} \) such that \( \forall x \in X(\beta) \) one has \( \hat{U}(x) = \hat{W}(x, \hat{U}(Sx)) \)? Second, given that \( U^0 \) is unique, does \( \hat{U} \) is the only recursive utility function associated with the aggregator \( \hat{W} \)? Finally, given that an aggregator \( \hat{W} \) exists, does it satisfy our sufficient conditions (assumptions (W1)-(W6) or (W1)-(W5'))?

The answer to the first question is immediate. Since \( \hat{U} = \phi \circ U^0 \), it follows that
\[
\hat{U}(x) = \phi \circ W(x, \phi^{-1} \circ \hat{U}(Sx)).
\]

Hence, the new aggregator is given by:
\[
\hat{W}(x, z) = \phi \circ W(x, \phi^{-1} \circ z).
\]

The answer to the second question is also obvious. Since \( \phi \) is one-to-one and \( U^0 \) is the unique recursive utility associated with \( W \), it follows that \( \hat{U} \) is unique.

The answer to the third question is not always an affirmative one. Below we display examples where the function \( \hat{U} = \phi \circ U^0 \) is associated with an aggregator \( \hat{W} \) which may or may not satisfy our sufficient conditions.

**Example 7:** Consider the aggregator \( W(x, z) = x^\theta + \delta z, x \in \mathbb{R}_+, z \in \mathbb{R}_+ \), with \( \theta, \delta \in ]0, 1[ \).
Provided that \( \delta \beta^\theta < 1 \), this aggregator function satisfies assumptions (W1)-(W6). The associated recursive utility function is given by \( U^0(x) = \sum_{t=0}^{\infty} \delta^t x_t^\theta \). Consider the following monotonic transformation for this recursive utility function:

\[
\phi(z) = z^{\frac{1}{\gamma}}, \quad \gamma > 0.
\]

In this case, \( \hat{U}(x) = U^0(x) \frac{1}{e^x} \) and

\[
\hat{W}(x, z) = \phi \circ W(x, \phi^{-1} \circ z) = [x^\theta + \delta^\gamma]^{\frac{1}{\gamma}}.
\]

When \( \gamma = \theta \), \( \hat{W} \) has the CES form. Aggregators of this form have been used in stochastic models to generate a form of non-expected utility (see Epstein and Zin (1989)). One can easily check that for this aggregator \( \delta = +\infty \), so assumption (W4) is clearly violated.

**Example 8:** Consider the aggregator \( W(x, z) = x^\theta + z, \ x \in \mathbb{R}^+, \ z \in \mathbb{R}, \ \theta < 0 \).

One can easily check that \( W \) satisfies assumptions (W1)-(W5'). The associated recursive utility function is given by \( U^0(x) = \sum_{t=0}^{\infty} x_t^{\frac{\theta}{\theta'}} \). Consider the following monotonic transformation for this recursive utility function:

\[
\phi(z) = e^z + 1.
\]

Observe that \( \phi \circ U^0 = e^{U^0(x)} + 1 \) is associated with the aggregator,

\[
\hat{W}(x, z) = \phi \circ W(x, \phi^{-1} \circ z) = (1 + z)e^{x^\theta} + 1.
\]

One can easily check that \( \hat{W} \) satisfies assumptions (W1)-(W5').

### 3 Optimal Growth with Recursive Utility

Consider an economy where at each period \( t \) there exists a vector of capital stocks on hand, denoted by \( k_t \in \mathbb{R}_p^+ \). There are \( m \) consumption goods, all consumed or freely disposed of in the period they are produced. Let \( c_t \in \mathbb{R}_m^+ \) denote the vector of these consumption goods in every period. Technological possibilities are described by a correspondence \( \Gamma : \mathbb{R}_+^p \rightarrow \mathbb{R}_+^p \). An accumulation path is a
sequence $k = (k_t)_{t=0}^\infty$ and is said to be feasible from some initial condition $k_0$ when $k_{t+1} \in \Gamma(k_t)$ for all $t \geq 0$. Let $\Pi(k_0)$ denote the collection of all feasible from $k_0$ capital sequences. Feasible consumption sequences are described by a correspondence $\Omega : \mathbb{R}^p_+ \times \mathbb{R}^p_+ \to \mathbb{R}^m_+$. A consumption sequence $c = (c_t)_{t=0}^\infty$ is said to be feasible from some $k_0$ when there is at least one $k \in \Pi(k_0)$ such that $c_t \in \Omega(k_t, k_{t+1})$ for all $t \geq 0$. Let $\Sigma(k_0)$ denote the collection of all feasible consumption sequences from $k_0$.

We next specify our assumptions for the preferences and the technology.

Preferences are assumed to be represented by a utility function $U$ that is assumed to be recursive, generated by an aggregator $W$ that satisfies the assumptions ($W1$)-($W6$) of the previous section (in the case of upcounted or undiscounted models assumption ($W5$) has to be replaced by assumption ($W5'$)). Define

$$U : c \in \Sigma(k_0) \to U(c) = \lim_{N \to \infty} W(c_0, W(c_1, ..., W(c_N, 0), ...)).$$

Regarding technology, the following assumptions are typically made in this context:

(T1): The correspondence $\Gamma$ is non-empty, compact-valued and continuous for any $k \in \mathbb{R}^p_+$.

(T2): The correspondence $\Omega$ is non-empty, compact-valued and continuous for any $(k, y) \in \mathbb{R}^p_+ \times \mathbb{R}^p_+$.

(T3): There exists $\beta \geq 1$ and a continuous, positive function $\Psi(k_0)$, such that, for any $k_0 \in \mathbb{R}^p_+$,

$$k \in \Pi(k_0) \implies \sup_t \frac{||k_t||}{\beta^t} \leq \Psi(k_0)$$

$$c \in \Sigma(k_0) \implies \sup_t \frac{||c_t||}{\beta^t} \leq \Psi(k_0).$$

Observe that if $c \in \Sigma(k_0)$, then $c \in X(\beta, M)$ with $M = \Psi(k_0)$.

Lemma 2 Assume ($T1$)-($T3$). Then:

i) For every $k_0 \in \mathbb{R}^p_+$, $\Pi(k_0)$ and $\Sigma(k_0)$ are compact for the product topology.

ii) The correspondences $\Pi$ and $\Sigma$ are upper hemicontinuous for the product topology.
Proof: i) It is standard.

ii) The proof of the upper hemicontinuity of $\Pi$ is standard. Let us prove that $\Sigma$ is upper hemicontinuous.

Let $(k_0^n)$ be a sequence that converges to $k_0$ and $(c^n)$ be a corresponding feasible consumption sequence. (T3) implies that, for $n$ large enough, $(c^n)$ belongs to a compact set for the product topology. Without loss of generality, we can assume that $c^n \to c$. The upper hemicontinuity of $\Pi(k_0)$ implies that there exists a subsequence $(k_{nm})$, such that $k_{nm} \in \Pi(k_0^n)$ and $k_{nm} \to k \in \Pi(k_0)$. Since $c_{0m} \in \Omega(k_0^n, k_1^{nm})$ and since $\Omega$ is continuous, we have $c_0 \in \Omega(k_0, k_1)$. By induction, $c_t \in \Omega(k_t, k_{t+1})$, $\forall t$. ■

3.1 Value function-Bellman equation

Given some initial condition $k_0 \in \mathbb{R}_+^p$, we define the value function as:

$$V(k_0) = \max_{c \in \Sigma(k_0)} U(c).$$

A plan $c$ is said to be optimal from $k_0$ when $c \in \Sigma(k_0)$ and $V(k_0) = U(c)$. A solution to the Bellman equation can be seen as a fixed point of the maximizing operator

$$Tf(k) = \max\{W(c, f(y)) : c \in \Omega(k, y), \ y \in \Gamma(k)\},$$

over all real-valued functions $f$ on $\mathbb{R}_+^p$.

There is a close connection between the solution of the Bellman equation and the value function. In what follows we study this connection. In particular, we would like the value function to solve the Bellman equation and a feasible program $(k, c)$ to be optimal if and only if it verifies $V(k_t) = W(c_t, V(k_{t+1}))$, $\forall t$.

Proposition 2 Assume (T1)-(T3) and (W1)-(W6). Then:

i) There exists a solution to the optimal growth model.

ii) The value function $V$ is upper semicontinuous.

iii) The value function $V$ satisfies the Bellman equation, i.e.

$$\forall k \in \mathbb{R}_+^p, \ V(k) = \max\{W(c, V(y)) : c \in \Omega(k, y), \ y \in \Gamma(k)\}.$$  

Proof: i) Recall that $U$ is upper semicontinuous for the product topology on $X(\beta, M)$, where $M = \Psi(k_0)$. Since $\Sigma(k_0)$ is compact, the result is obvious.

ii) Since $U(c)$ may be equal to $-\infty$ for some $c$, we cannot use the Maximum Theorem to prove the statement. For that reason we give a direct proof.
Let \((k_0^n)\) be a sequence that converges to \(k_0\). Let \((k_0^{nm})\) be a subsequence such that \(\limsup_n V(k_0^n) = \lim_{m} V(k_0^{nm})\). From (i), it follows that for every \(k_0^{nm}\), there exists a consumption plan \(c^{nm} \in \Sigma(k_0^{nm})\), such that
\[
V(k_0^{nm}) = U(c_0^{nm}, c_1^{nm}, c_2^{nm}, \ldots) = W(c_0^{nm}, U(Sc^{nm})).
\]
Since \(W\) satisfies (W2) and (W4), we have:
\[
W(c_0^{nm}, U(Sc^{nm}))
\]
\[
= W(c_0^{nm}, W(c_1^{nm}, \ldots, W(c_{N-1}^{nm}, U(SNc^{nm})) \ldots))
\]
\[
\leq W(c_0^{nm}, W(c_1^{nm}, \ldots, W(c_{N-1}^{nm}, U^+(SNc^{nm})) \ldots))
\]
\[
\leq \delta^N U^+(SNc^{nm}) + W(c_0^{nm}, W(c_1^{nm}, \ldots, W(c_{N-1}^{nm}, 0) \ldots)).
\]
Since \(\Sigma\) is upper hemi-continuous, the subsequence \((k_0^{nm})\) can be chosen such that the corresponding consumption sequence \((c^{nm})\) converges to some \(c \in \Sigma(k_0)\). In this case, we obtain:
\[
\limsup_n V(k_0^n) = \lim_{m} V(k_0^{nm})
\]
\[
\leq \delta^N U^+(SNc) + W(c_0, W(c_1, \ldots, W(c_{N-1}, 0) \ldots)).
\]
Letting \(N \to \infty\), and using Proposition 1(ii) we get:
\[
\limsup_n V(k_0^n) \leq U(c) \leq V(k_0).
\]
iii) From (i), it follows that given some initial condition \(k_0\), there exists \(c \in \Sigma(k_0)\), such that
\[
V(k_0) = U(c_0, c_1, \ldots, c_t, \ldots)
\]
\[
= W(c_0, U(Sc)).
\]
Since \(c \in \Sigma(k_0)\), there exists \(k \in \Pi(k_0)\), such that \(c_t \in \Omega(k_t, k_{t+1})\), \(\forall t\). It follows that \(Sc \in \Sigma(k_1)\) for some \(k_1 \in \Gamma(k_0)\). We have:
\[
V(k_0) \leq W(c_0, V(k_1))
\]
\[
\leq \max\{W(c, V(y)) : c \in \Omega(k_0, y), y \in \Gamma(k_0)\}.\]

Conversely, since the set \(\{(c, y) : c \in \Omega(k_0, y), y \in \Gamma(k_0)\}\) is compact, there exists \((c_0, k_1)\) with \(c_0 \in \Omega(k_0, k_1), k_1 \in \Gamma(k_0)\), such that
\[
\max\{W(c, V(y)) : c \in \Omega(k_0, y), y \in \Gamma(k_0)\} = W(c_0, V(k_1)).
\]
Given $k_1 \in \Gamma(k_0)$, there exists $S c \in \Sigma(k_1)$, such that $V(k_1) = U(S c)$. Hence,

\[ W(c_0, V(k_1)) = W(c_0, U(S c)) = U(c) \leq V(k_0). \]

\[ \square \]

Let us consider the following set:

\[ \Pi^{'}(k_0) = \{ k \in \Pi(k_0) : \exists c \in \Sigma(k_0) \text{ such that } U(c) > -\infty \}. \]

Observe that if $\Pi^{'}(k_0) \neq \emptyset$, then $V(k_0) > -\infty$. Moreover, $k \in \Pi^{'}(k_0)$ implies $S^{t+1}k \in \Pi^{'}(k_t)$, $\forall t \geq 1$.

**Proposition 3** Assume (T1)-(T3) and (W1)-(W6). Then, the value function satisfies:

i) $\forall k_0 \in \mathbb{R}_{+}^p, \forall k \in \Pi(k_0), \lim_{t \to \infty} \delta^t V^+(k_t) = 0$. More precisely, $V$ satisfies the following property: For any $k_0 \in \mathbb{R}_{+}^p$, for any $\varepsilon > 0$, there exists an integer $T_0$, such that, for any $k \in \Pi(k_0)$, for any $t \geq T_0$, we have $\delta^t V^+(k_t) \leq \varepsilon$.

ii) $\forall k_0 \in \mathbb{R}_{+}^p, \forall k \in \Pi^{'}(k_0), \forall c \in \Sigma(k_0)$ with $c_t \notin \mathbb{D}, \forall t$,

\[ \lim_{T \to \infty} [W(c_0, ..., W(c_T, V(k_{T+1})))...) - W(c_0, ..., W(c_T, 0)...)] = 0. \]

**Proof:** i) Take $(k, c)$ such that $k \in \Pi(k_0)$, $c \in \Sigma(k_0)$. From Proposition 2(i), it follows that for any $T$, there exists $(c'_{T+1}, c'_{T+2}, ...) \in \Sigma(k_{T+1})$, such that $V(k_{T+1}) = U(c'_{T+1}, c'_{T+2}, ...)$. Let $c''$ denote a consumption path defined as follows: $c''_t = c_t$, for $t \leq T$ and $c''_t = c'_t$, for $t \geq T + 1$. Observe that $c'' \in \Sigma(k_0)$ and that $V(k_{T+1}) = U(S^{T+1} c'')$. Using Proposition 1(ii) we have:

\[ \lim_{T \to \infty} \delta^{T+1} V^+(k_{T+1}) = \lim_{T \to \infty} \delta^{T+1} U^+(S^{T+1} c'') = 0. \]

It follows that for any $k_0 \in \mathbb{R}_{+}^p$, for any $\varepsilon > 0$, there exists $T_0$, such that, for any $k \in \Pi(k_0)$, for any $t \geq T_0$, we have $\delta^t V^+(k_t) \leq \varepsilon$.

ii) Take $(k, c)$ such that $k \in \Pi^{'}(k_0)$, $c \in \Sigma(k_0)$ with $c_t \notin \mathbb{D}, \forall t$. Assumptions (W2) and (W4) imply that

\[ W(c_0, W(c_1, ..., W(c_T, V(k_{T+1})))...) - W(c_0, W(c_1, ..., W(c_T, 0)...)) \leq \delta^{T+1} V^+(k_{T+1}). \]
Using (i) we get:
\[
\limsup_T [W(c_0, ..., W(c_T, V(k_{T+1}))) - W(c_0, ..., W(c_T, 0)))] \leq 0.
\]

Observe that
\[
U(c) = W(c_0, ..., W(c_T, U(S^{T+1}_c))) 
\leq W(c_0, ..., W(c_T, V(k_{T+1}))).
\]

One has:
\[
U(c) - W(c_0, ..., W(c_T, 0)) 
\leq W(c_0, ..., W(c_T, V(k_{T+1}))) - W(c_0, ..., W(c_T, 0)).
\]

Taking the limits on both sides we get:
\[
\liminf_T [W(c_0, ..., W(c_T, V(k_{T+1}))) - W(c_0, ..., W(c_T, 0)))] \geq 0.
\]

We conclude the proof.  

Let \( F \) denote the set of functions \( f : \mathbb{R}^p_+ \to \mathbb{R} \cup \{ -\infty \} \) which are upper semicontinuous and satisfy:

(a) \( \forall k_0 \in \mathbb{R}^p_+, \forall \varepsilon > 0, \exists T_0, \text{ such that, } \forall k \in \Pi(k_0), \forall t \geq T_0, \delta^t f^+(k_t) \leq \varepsilon. \)

(b) \( \forall k_0 \in \mathbb{R}^p_+, \forall k \in \Pi'(k_0), \forall c \in \Sigma(k_0) \text{ with } c_t \notin \mathbb{D}, \forall t, \)
\[
\lim_{T \to \infty} [W(c_0, ..., W(c_T, f(k_{T+1}))) - W(c_0, ..., W(c_T, 0))] = 0.
\]

**Theorem 3** Assume (T1)-(T3) and (W1)-(W6). Then, the value function \( V \) is the unique solution in \( F \) to the Bellman equation.

**Proof:** Let \( \hat{V} \in F \) be another solution to Bellman equation.

We first show that \( \hat{V}(k_0) \leq V(k_0), \forall k_0 \in \mathbb{R}^p_+ \). Assume first that \( \Pi'(k_0) = \emptyset \). In this case, \( V(k_0) = -\infty \) or equivalently, \( U(c) = -\infty, \forall c \in \Sigma(k_0) \). We claim that \( \hat{V}(k_0) = -\infty \). Assume the contrary. Then, there exists \( k \in \Pi(k_0) \) and \( c \in \Sigma(k_0) \) with \( c_t \notin \mathbb{D}, \forall t, \) such that
\[
\hat{V}(k_0) = W(c_0, W(c_1, ..., W(c_t, \hat{V}(k_{t+1}))) ...).
Observe that
\[
\hat{V}(k_0) - W(c_0, W(c_1, \ldots, W(c_t, 0))) = W(c_0, W(c_1, \ldots, W(c_t, \hat{V}(k_{t+1})) \ldots) - W(c_0, W(c_1, \ldots, W(c_t, 0))) \\
\leq \delta^{t+1}\hat{V}^+(k_{t+1}).
\]

Since \(\hat{V}\) satisfies property (a), it follows that \(-\infty < \hat{V}(k_0) \leq U(c)\) which is a contradiction to \(U(c) = -\infty\).

Now assume that \(\Pi'(k_0) \neq \emptyset\). In this case, there exists \(k \in \Pi'(k_0)\) and \(c \in \Sigma(k_0)\), such that
\[
\hat{V}(k_0) = W(c_0, W(c_1, \ldots, W(c_t, \hat{V}(k_{t+1})) \ldots)).
\]

If \(c_t \in \mathbb{D}\) for some \(t\), then \(-\infty = \hat{V}(k_0) \leq V(k_0)\). If \(c_t \notin \mathbb{D}\), \(\forall t\), we have:
\[
\hat{V}(k_0) - W(c_0, W(c_1, \ldots, W(c_t, 0))) \leq \delta^{t+1}\hat{V}^+(k_{t+1}).
\]

Since \(\hat{V}\) satisfies property (a), it follows that \(-\infty < \hat{V}(k_0) \leq U(c) \leq V(k_0)\).

We next show that \(\hat{V}^+(k_0) \geq V(k_0), \forall k_0 \in \mathbb{R}_+^p\). Assume that \(\Pi'(k_0) = \emptyset\). In this case, \(V(k_0) = -\infty\) and the claim is true. Assume that \(\Pi'(k_0) \neq \emptyset\). In this case, there exists \(k \in \Pi'(k_0)\) and \(c \in \Sigma(k_0)\) with \(c_t \notin \mathbb{D}\), \(\forall t\), such that
\[
V(k_0) = W(c_0, W(c_1, \ldots, W(c_t, V(k_{t+1}))) \ldots)).
\]

Observe that for any \(t \geq 0\),
\[
\hat{V}(k_0) \geq W(c_0, W(c_1, \ldots, W(c_t, \hat{V}(k_{t+1})) \ldots)).
\]

One has:
\[
\hat{V}(k_0) - W(c_0, W(c_1, \ldots, W(c_t, 0))) \geq W(c_0, W(c_1, \ldots, W(c_t, \hat{V}(k_{t+1})) \ldots) - W(c_0, W(c_1, \ldots, W(c_t, 0))).
\]

Since \(\hat{V}\) satisfies property (b), it follows that \(\hat{V}(k_0) \geq U(c) = V(k_0)\). We conclude the proof. \(\blacksquare\)

**Theorem 4** Assume \((T1)-(T3)\) and \((W1)-(W6)\). Then, a feasible program \((k, c)\) is optimal, if and only if, it verifies \(V(k_t) = W(c_t, V(k_{t+1}))\), \(\forall t\).
Proof: Let \( k_0 \in \mathbb{R}^n_+ \) and assume that \((k, c)\) is a feasible program satisfying \( V(k_t) = W(c_t, V(k_{t+1})), \forall t. \) We have

\[
V(k_0) - W(c_0, W(c_1, ..., W(c_T, 0)))
\]

\[
= W(c_0, W(c_1, ..., W(c_T, V(k_{T+1})))...) - W(c_0, W(c_1, ..., W(c_T, 0))...).
\]

\[
\leq \delta^{T+1} V^+(k_{T+1}).
\]

Taking the limits on both sides as \( T \to \infty \), gives \( V(k_0) \leq U(c) \). Since the definition of \( V \) implies \( V(k_0) \geq U(c) \), the plan is optimal.

Conversely, let \( k_0 \in \mathbb{R}^n_+ \) and assume that \((k, c)\) is optimal. We claim that \( V(k_0) = W(c_0, V(k_1)) \). Since \( V(k_0) = U(c) = W(c_0, U(Sc)) \) and since the sequence \((c_1, c_2, ...)\) is feasible from \( k_1 \), it follows that \( V(k_0) \leq W(c_0, V(k_1)) \). Let \((c_1', c_2', ..., c_t', ...)\) denote an optimal path from \( k_1 \). We have

\[
W(c_0, V(k_1)) = W(c_0, U(Sc')) \leq V(k_0),
\]

because the sequence \((c_0, c_1', c_2', ...)\) is feasible from \( k_0 \). Continuing by induction establishes that \( V(k_t) = W(c_t, V(k_{t+1})), \forall t. \)

### 3.2 Algorithm to find the Value function

Let \( T \) be the mapping which associates with any u.s.c function \( f \) from \( \mathbb{R}^n_+ \) into \( \mathbb{R} \cup \{-\infty\} \) the u.s.c function \( Tf \) defined as follows:

\[
Tf(k) = \max\{W(c, f(y)) : c \in \Omega(k, y), y \in \Gamma(k)\}.
\]

**Theorem 5** Assume (T1)-(T3) and (W1)-(W6). Then:

i) \( T \) maps \( F \) into \( F \).

ii) For every \( k_0 \in \mathbb{R}^n_+ \), \( V(k_0) = \lim_n T^n f(k_0), \) where \( f \) is any function in \( F \). In particular, \( V(k_0) = \lim_n T^n 0(k_0). \)

**Proof:** i) It is clear that \( Tf \) is upper semicontinuous. Let us show that \( T \) maps \( F \) into \( F \). We first show that \( T \) satisfies property (a) in the definition of \( F \).

Since \( f \) is upper semicontinuous, for every \( t \geq 0 \), there exists \( c_t \in \Omega(k_t, k_{t+1}), k_{t+1} \in \Gamma(k_t), \) such that \( Tf(k_t) = W(c_t, f(k_{t+1})). \) Observe that

\[
W(c_t, f(k_{t+1})) \leq W^+(c_t, 0) + \delta f^+(k_{t+1}),
\]

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One has:
\[
\begin{align*}
\delta^i(Tf)^+(k_t) &\leq \delta^i W^+(c_t,0) + \delta^{t+1} f^+(k_{t+1}) \\
&\leq \varphi(\beta)^i \psi(M,\beta) + \delta^{t+1} f^+(k_{t+1}).
\end{align*}
\]

Since \( f \) satisfies property (a) and \( \varphi(\beta) < 1 \), it follows that, given \( \varepsilon > 0 \), \( \varphi(\beta)^i \psi(M,\beta) + \delta^{t+1} f^+(k_{t+1}) \leq \varepsilon \), for any \( t \) large enough.

We next show that \( Tf \) satisfies property (b) in the definition of \( F \). Take \( (k, c) \), such that \( k \in \Pi'(k_0) \) and \( c \in \Sigma(k_0) \) with \( c_t \notin \mathbb{D}, \forall t \). We have:
\[
\begin{align*}
W(c_0,\ldots,W(c_{t-1},W(c_t,f(k_{t+1}))\ldots)) - W(c_0,\ldots,W(c_{t-1},0)\ldots)) \\
&\leq W(c_0,\ldots,W(c_{t-1},Tf(k_t))\ldots)) - W(c_0,\ldots,W(c_{t-1},0)\ldots)) \\
&\leq \delta^i(Tf)^+(k_t).
\end{align*}
\]

Taking the limits on both sides we get:
\[
\limsup_t [W(c_0,\ldots,W(c_{t-1},Tf(k_t))\ldots)) - W(c_0,\ldots,W(c_{t-1},0)\ldots)) \leq 0.
\]

Conversely, we have:
\[
\begin{align*}
W(c_0,\ldots,W(c_{t-1},Tf(k_t))\ldots)) - W(c_0,\ldots,W(c_{t-1},0)\ldots)) \\
&\geq W(c_0,\ldots,W(c_{t-1},W(c_t,f(k_{t+1}))\ldots)) - W(c_0,\ldots,W(c_{t-1},0)\ldots)) \\
&= W(c_0,\ldots,W(c_{t-1},W(c_t,f(k_{t+1}))\ldots)) - W(c_0,\ldots,W(c_t,0)\ldots)) \\
&+ W(c_0,\ldots,W(c_t,0)\ldots)) - W(c_0,\ldots,W(c_{t-1},0)\ldots)).
\end{align*}
\]

Observe that
\[
U(c) = \lim_{t \to \infty} W(c_0,W(c_1,\ldots,W(c_{t-1},0)\ldots)) = \lim_{t \to \infty} W(c_0,W(c_1,\ldots,W(c_2,0)\ldots)).
\]

Since \( f \) satisfies property (b) we get:
\[
\liminf_t [W(c_0,\ldots,W(c_{t-1},Tf(k_t))\ldots)) - W(c_0,\ldots,W(c_{t-1},0)\ldots)) \geq 0.
\]

ii) Let \( k_0 \in \mathbb{R}^n_+ \) and denote by \( l \) a cluster point of \( \{T^n f(k_0)\} : l = \lim_\nu T^n f(k_0) \).

We first show that \( l \geq V(k_0) \).

If \( \Pi(k_0) = \emptyset \), then \( V(k_0) = -\infty \), in which case \( l \geq V(k_0) \). Assume \( \Pi'(k_0) \neq \emptyset \).

Take \( (k, c) \), such that \( k \in \Pi'(k_0) \) and \( c \in \Sigma(k_0) \) with \( c_t \notin \mathbb{D}, \forall t \). Observe that
\[
T^n f(k_0) \geq W(c_0,W(c_1,\ldots,W(c_{t-1},f(k_{t+1})))\ldots)).
\]

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It follows that

\[ T^\nu f(k_0) - W(c_0, \ldots, W(c_{\nu-1}, 0)) \]
\[ \geq W(c_0, \ldots, W(c_{\nu-1}, f(k_\nu))) - W(c_0, \ldots, W(c_{\nu-1}, 0)). \]

Since \( f \) satisfies property (b), we get \( l \geq U(c) \). Thus, \( l \geq V(k_0) \).

Let us show that \( l \leq V(k_0) \). Since \( T^\nu f \) is u.s.c. for every \( \nu \), there exists \( k_1^\nu \in \Gamma(k_0), k_2^\nu \in \Gamma(k_1^\nu), \ldots, k_{\nu-1}^\nu \in \Gamma(k_{\nu-1}^\nu) \) and \( c_0^\nu \in \Omega(k_0, k_1^\nu), \ldots, c_{\nu-1}^\nu \in \Omega(k_{\nu-2}^\nu, k_{\nu-1}^\nu) \), such that

\[ T^\nu f(k_0) = W(c_0^\nu, W(c_1^\nu, \ldots, W(c_{\nu-1}^\nu, f(k_\nu))) \].

For every \( \nu \), take a sequence \( (\tilde{k}_{\nu+1}^\nu, \ldots, \tilde{k}_{\nu+1}^\nu) \in \Pi(k_\nu^\nu) \). Observe that the sequence \( (\tilde{k}^\nu) \) with \( \tilde{k}^\nu = (k_0^\nu, k_1^\nu, \ldots, k_\nu^\nu) \), \( \forall \nu \) belongs to \( \Pi(k_0) \) and for that reason it can be assumed that converges to some \( k \in \Pi(k_0) \). The corresponding consumption sequence \( (\tilde{c}^\nu) \) with \( \tilde{c}^\nu = (c_0^\nu, \ldots, c_{\nu-1}^\nu, \tilde{c}_\nu^\nu) \), \( \forall \nu \) will also belong to \( \Sigma(k_0) \), and therefore it can be assumed that converges to some \( c \in \Sigma(k_0) \). Note that

\[ T^\nu f(k_0) - W(c_0^\nu, W(c_1^\nu, \ldots, W(c_{\nu-1}^\nu, 0)) \]
\[ = W(c_0^\nu, \ldots, W(c_{\nu-1}^\nu, f(k_\nu))) - W(c_0^\nu, \ldots, W(c_{\nu-1}^\nu, 0)) \]
\[ \leq W(c_0^\nu, \ldots, W(c_{\nu-1}^\nu, f^+(k_\nu))) - W(c_0^\nu, \ldots, W(c_{\nu-1}^\nu, 0)) \]
\[ \leq \delta^\nu f^+(k_\nu^\nu). \]

We obtain:

\[ T^\nu f(k_0) \leq W(c_0^\nu, \ldots, W(c_{\nu-1}^\nu, 0)) + \delta^\nu f^+(k_\nu^\nu). \]

Fix \( N \) and choose \( \nu - 1 > N \). A similar argument as in the proof of Proposition 1(iii) (see Case 1) implies that

\[ W(c_0^\nu, \ldots, W(c_{\nu-1}^\nu, 0)) \]
\[ \leq W(c_0^\nu, \ldots, W(c_{N-1}^\nu, 0)) + \delta^{\nu-1} W^+(c_{\nu-1}^\nu, 0) + \ldots + \delta^N W^+(c_N^\nu, 0) \]
\[ \leq W(c_0^\nu, \ldots, W(c_{N-1}^\nu, 0)) + \frac{\phi(\beta)^N}{1 - \phi(\beta)} \psi(M, \beta). \]

Since \( f \in F \), given \( \varepsilon > 0 \), for any \( \nu \) large enough, we have \( \delta^\nu f^+(k_\nu^\nu) \leq \varepsilon \).

If we let \( \nu \to +\infty \), we get:

\[ l = \lim_{\nu} T^\nu f(k_0) \leq W(c_0, \ldots, W(c_{N-1}, 0)) + \frac{\phi(\beta)^N}{1 - \phi(\beta)} \psi(M, \beta) + \varepsilon. \]
Letting $N \to +\infty$, we obtain:

$$l \leq U(c) + \varepsilon.$$  

Since the above inequality holds for any $\varepsilon > 0$, we have $l \leq U(c) \leq V(k_0)$. ■

References


