On the Stability of Closed-loop Inverse Kinematics Algorithms for Redundant Robots

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Abstract—The purpose of this paper is to provide a convergence analysis of classical inverse kinematics algorithms for redundant robots, whose stability is usually proved only in the continuous-time domain, thus neglecting limits of the actual implementation in the discrete-time. Whereas, the convergence analysis carried out in this paper in the discrete-time domain provides a method to find bounds on the gain of the closed-loop inverse kinematics algorithms in relation to the sampling time. It also provides an estimation of the region of attraction (without resorting to Lyapunov arguments) i.e. upper bounds on the initial task space error. Simulations on a 11-DOF manipulator are performed to show how the found bounds on the gain are not too restrictive.

Index Terms—Inverse kinematics, Redundant robots, Stability proof.

I. INTRODUCTION

Inverse kinematics (IK) of robotic systems is one of the basic problems in robotics. It has been addressed and solved in a variety of manners both for non-redundant and redundant robots. At the beginning it was addressed by resorting to classical numerical methods, such as Newton-Raphson algorithm for finding zeros of nonlinear functions [1], [2], or to more general optimization algorithms [3], [4]. Later on it has been solved by inverting differential kinematics in a closed-loop fashion by viewing the IK problem as a feedback control problem [5], [6], leading to the so-called CLIK (Closed-Loop Inverse Kinematics) algorithm. More recently, mixed numerical-analytical approaches have been proposed leading to approximate solutions [7]. One of the main problems related to IK is the handling of multiple tasks. Task-priority redundancy resolution techniques [8], [9] were proposed to allow the specification of a primary task that is fulfilled with higher priority with respect to a secondary task. The same objective has been reached by resorting to a null-space based solution in [10], whose extension to any number of tasks has been recently presented in [11]. For a comprehensive review, the reader is referred to [12] and references therein.

The main difficulty in the study of algorithmic solutions to the IK problem, is related to the discrete-time nature of the dynamic system at hand, combined with its strong nonlinearity, deriving from the nonlinearity of the kinematics. Very few papers cope with this problem, e.g. [6],[13] proposed Lyapunov-based arguments to prove stability. This paper, which does not intend to propose any novel IK algorithm, presents novel proofs of convergence of the cited CLIK algorithms, which are used to solve the IK problem of redundant robots also in the multiple task case (for a complete discussion see [11]). Therefore, the study of their stability is relevant also when multiple tasks are considered.

The proof is obtained directly in the discrete-time domain, thus leading to useful guidelines for gain selection in relation to the sampling time. The availability of such a relation is useful for selecting the correct hardware for the actual implementation depending on the desired performance (i.e. convergence time of the IK algorithm). The adopted methodology is not based on Lyapunov arguments, since proving that the origin of the task space error space is asymptotically stable is not so trivial, because the Lyapunov function candidate depends not only on the task space but on the configuration variables too. Therefore, it cannot be shown to be positive definite without including terms depending on the configuration variables. The alternative approach followed here completely avoids the use of Lyapunov functions, nevertheless it rigorously proves the stability of the algorithm according to the comparison principle for discrete-time systems. The presented simulation, carried out on a 11-DOF manipulator with redundancy resolution, shows how the bounds on the gain of the jacobian pseudo-inverse algorithm are not restrictive at all.

II. THE INVERSE KINEMATICS PROBLEM

Let \( x \in \mathbb{R}^n \) be the vector of task variables of a robotic system with \( X \) a domain of \( \mathbb{R}^n \) and let \( q \in \mathbb{R}^m \) be the vector of the robotic system configuration with \( Q \) a domain of \( \mathbb{R}^m \), with \( m \leq n \). For example, in a robotic manipulator \( x \) is the pose of the end effector and \( q \) is the vector of joint positions, whereas in a platoon of mobile robots, \( q \) is the vector of coordinates representing the location of each robot and \( x \) is the vector of suitable task variables depending on the mission. The direct kinematics equation can be always written in the form

\[
k : Q \subseteq \mathbb{R}^m \to X \subseteq \mathbb{R}^n, \quad x = k(q).
\]

This function will be hereafter called “direct kinematics function” or “task function” without distinction. Under the assumption of absence of kinematic singularities within the configuration space \( Q \), the jacobian

\[
J(q) = \frac{\partial k(q)}{\partial q} \in \mathbb{R}^{m \times n}
\]

is full-rank \( \forall q \in Q \). This matrix will be hereafter called “robot jacobian” or “task jacobian” without distinction. Both the direct kinematics function and the robot jacobian are assumed to fulfill the following assumptions:

i) \( \exists \delta > 0 : \|J(q)\| \leq \delta, \forall q \in Q \)

ii) \( \exists \beta > 0 : \|J(q) J^T(q)\| \geq \beta, \forall q \in Q \)

iii) the function \( k(q) \) is smooth enough such that

\[
k(q + \bar{q}) = k(q) + J(q) \bar{q} + r_k(q),
\]

where the reminder \( r_k(q) \) is such that

\[
\exists \nu_k > 0 : \|r_k(q)\| \leq \nu_k \|\bar{q}\|^2, \quad \forall \bar{q} : q + \bar{q} \in Q,
\]

where, as matrix norm, the spectral norm, i.e. the largest singular value, has been assumed and the symbol \( \sigma(X) \) denotes the smallest singular value of the matrix \( X \). Assumptions i) and ii) are aimed at quantifying the distance from singularities, while assumption iii) is aimed at taking into account the degree of
smoothness of $k(q)$. Basically, the nonlinear functions have to possess second-order derivatives bounded, and fortunately in many applications such a requirement is verified. For example, every robot with revolute joints has a direct kinematics function constituted by polynomial combinations of trigonometric functions of the joint variables, therefore their second-order derivatives are certainly bounded. Such a degree of smoothness can be easily quantified by applying the following lemma.

**Lemma 1**: Given a vector function defined in a domain $D \subseteq \mathbb{R}^n$, $f : x \in D \rightarrow f(x) \in \mathbb{R}^m$ with the Hessian matrices $H(f_i)$ of all its components $f_i(x)$, $i = 1, \ldots, m$ norm-bounded uniformly in $D$, i.e.

$$\exists \nu_i > 0 : \|H(f_i(x))\| \leq \nu_i, \forall x \in D, \quad i = 1, \ldots, m \quad (3)$$

then, $\forall \tilde{x} \in D : x + \tilde{x} \in D$ and the whole line from $x$ to $\tilde{x}$ belongs to $D$, it is

$$f(x + \tilde{x}) = f(x) + \frac{\partial f(x)}{\partial x} \tilde{x} + r_f(x) \quad (4)$$

and the reminder $r_f(x)$ is such that

$$\exists \nu_f > 0 : \|r_f(x)\| \leq \nu_f \|\tilde{x}\|^2 \quad (5)$$

**Proof**: The proof is a direct consequence of Taylor’s theorem for functions of several variables [14] written with a second order bound estimator. This theorem ensures that it always exists a point $\xi$ on the line connecting $x$ and $\tilde{x}$, if this entirely belongs to the domain $D$, such that

$$f(x + \tilde{x}) = f(x) + \frac{\partial f(x)}{\partial x} \tilde{x} + \frac{1}{2} \begin{bmatrix} \tilde{x}^T H(f_1(\xi))\tilde{x} \\ \vdots \\ \tilde{x}^T H(f_m(\xi))\tilde{x} \end{bmatrix} \quad (6)$$

Denoting the last term of the right-hand side with the symbol $r_f(x)$, this can be easily upper bounded as

$$\|r_f(x)\| \leq \sqrt{m} \|r_f(x)\|_{\infty} = \sqrt{m} / 2 \max_i \|\tilde{x}^T H(f_i(\xi))\| \|\tilde{x}\| \leq \sqrt{m} / 2 \max_i \|H(f_i(\xi))\| \|\tilde{x}\|^2 \leq \sqrt{m} / 2 \max_i \nu_i \|\tilde{x}\|^2 \leq \nu_f \|\tilde{x}\|^2 \quad (7)$$

Let $x_{dh} \in X$, being $h \in \mathbb{Z}$ the discrete-time variable, be a desired task space position, the objective of any IK algorithm is to find one of the, in general many, configurations $q_{dh}$ such that

$$x_{dh} = k(q_{dh}) \quad (6)$$

which is a system of $m$ nonlinear equations, therefore the first ideas to tackle the problem resorted to iterative algorithms devoted to find zeros of nonlinear functions, e.g. the Newton-Raphson method in [1], [2]. In the following, two versions of the so-called CLIK algorithm are recalled [5], whose stability will be later proved in the discrete-time domain together with an estimation of the region of attraction.

Differently from analytic solutions (available only in special cases) and the iterative algorithms mentioned above, the CLIK algorithms rely on the inversion of the differential kinematics in the continuous-time domain, i.e.

$$\dot{x}(t) = J(q(t)) \dot{q}(t) \quad (7)$$

done in a closed-loop fashion as originally proposed in [5]

$$\ddot{q}(t) = J^T(q(t))(\dot{x}_{dh}(t) + \gamma(x_{dh}(t) - k(q(t)))) \quad (8)$$

so as to avoid drift of the tracking error when implemented in discrete-time, e.g. by resorting to the Euler integration method with sampling time $T$, namely

$$q_{h+1} = q_h + T J^T(q_h) (\dot{x}_{dh} + \gamma(x_{dh} - k(q_h))) \quad (9)$$

and to improve convergence rate acting on the positive gain $\gamma$. In the two equations above the symbol $X^T$ denotes the Moore-Penrose pseudo-inverse of the lower-rectangular matrix $X$. Therefore, in view of assumptions $i,ii$ and of standard properties of matrix norm, it is

$$\|J^T(q)\| \leq \delta / \beta \leq \delta^*, \forall q \in Q. \quad (10)$$

Alternatively to the jacobian pseudo-inverse algorithm, the jacobian transpose method can be applied when a constant task space desired position is considered, i.e.

$$q_{h+1} = q_h + \gamma T J^T(q_h)(x_d - k(q_h)) \quad (11)$$

The stability analysis of discrete-time systems similar to those in (9) and (11) has been tackled in a very few papers (e.g. [6],[13]) based on Lyapunov methods. In Section III an alternative convergence analysis of these algorithms will be given, providing criteria to select the gain $\gamma$ in relation to the sampling time, as well as an estimation of the region of attraction will be given.

III. STABILITY ANALYSIS

In the discrete-time version of the CLIK algorithm using the pseudo-inverse of the jacobian (9), with a constant $x_d$, the dynamics of the task space error $e_h = x_d - k(q_h)$ is governed by the equation

$$e_{h+1} = e_h - k(q_h) + \gamma T J^T_h e_h = e_h - k(q_h) - \gamma T J^T_h e_h - r_k(q_h) = (1 - \gamma T) e_h - r_k(q_h), \quad (12)$$

where the notation $J_h = J(q_h)$ has been introduced for brevity, and the expression of $k(q)$ in iii) has been exploited.

The proof of the algorithm convergence and the estimation of the region of attraction depending on $\gamma$ are addressed by the following theorem. The proof will not make use of any Lyapunov argument as most of the papers dealing with IK of redundant robots (the most recent one is [11]) do. However, when redundant robots are considered, proving that the origin of the task space error space is asymptotically stable requires a Lyapunov function candidate which includes terms depending not only on the task space variables, but also on the configuration variables. The alternative approach followed here completely avoids the use of Lyapunov functions, nevertheless it rigorously proofs the stability of the algorithm. It is based on the following Lemma that is a consequence of the comparison principle for discrete-time systems and is a special case of classical results on recurrence inequalities that can be found in [15].
Lemma 2: Let \( b_h \) be a non-negative sequence satisfying
\[
b_{h+1} \leq \alpha b_h + c, \tag{13}\]
where \( \alpha \) and \( c \) are non-negative real numbers. If \( b_0 \leq a_0 \), being \( a_0 \) the initial condition of the dynamic system
\[
a_{h+1} = \alpha a_h + c, \tag{14}\]
then
\[
b_h \leq a_h, \quad \forall \ h \geq 0. \tag{15}\]
Proof: The proof is by induction. The claim is true for \( h = 0 \). Suppose it is true for \( h \), then for \( h + 1 \) it is
\[
b_{h+1} \leq \alpha b_h + c \leq \alpha a_h + c = a_{h+1}. \tag{16}\]

Theorem 1: Under the assumptions i)—iii), if the initial task space error \( e_0 \) and the gain \( \gamma \) are such that
\[
0 < \gamma \leq 1/T \quad \text{and} \quad \|e_0\| < \frac{1}{\gamma T \nu h \delta^2} \tag{17}\]
or
\[
1/T < \gamma < 2/T \quad \text{and} \quad \|e_0\| < \frac{2 - \gamma T}{\gamma^2 T^2 \nu h \delta^2}, \tag{18}\]
then the CLIK algorithm in (9) ensures exponential convergence of the task space error dynamics and the configuration variables \( q_h \) are bounded and converge to a constant value, i.e.,
\[
\exists \alpha \in (0, 1), \phi > 0: \quad \|e_h\| \leq \phi \alpha^h, \quad \forall h \geq 0 \tag{19}\]
\[
\exists \rho > 0: \quad \|q_h\| \leq \rho, \quad \forall h \geq 0 \tag{20}\]
\[
\lim_{h \to \infty} \|q_{h+1} - q_h\| = 0. \tag{21}\]
Proof: From (12) the following inequalities are obtained
\[
\|e_{h+1}\| \leq \|1 - \gamma T\| \|e_h\| + \|r_k(q_h)\| \leq \|1 - \gamma T\| \|e_h\| + \nu h \|\gamma T J_h^e\| \|e_h\|^2 \leq \|1 - \gamma T\| + \gamma^2 T^2 \nu h \delta^2 \|e_h\| \|e_h\|, \tag{22}\]
where the bounds in assumption iii) and in Eq. (10) have been exploited together with standard norm properties. Now, assume that the task error norm is bounded, i.e.,
\[
\|e_h\| \leq \phi, \quad \forall h \geq 0. \tag{23}\]
In a moment it will be shown that such a condition is guaranteed by the sole hypothesis that the initial condition satisfies one of the two conditions (17) or (18). Equation (22) becomes
\[
\|e_{h+1}\| \leq \|1 - \gamma T\| + \gamma^2 T^2 \nu h \delta^2 \|e_h\| \leq \alpha \|e_h\|, \quad \forall h \geq 0. \tag{24}\]
Firstly, assume that the gain and the initial task space error satisfy (17), by choosing \( \phi = \|e_0\| \) it is
\[
\phi < \frac{1}{\gamma T \nu h \delta^2} \Rightarrow \alpha < 1, \tag{25}\]
hence the following scalar linear system
\[
\tilde{e}_{h+1} = \alpha \tilde{e}_h \tag{26}\]
is asymptotically stable and its response with initial condition \( \tilde{e}_0 = \phi \) is
\[
\tilde{e}_h = \phi \alpha^h, \quad h \geq 0. \tag{27}\]
Therefore, in view of (24) and recalling that \( \|e_0\| = \tilde{e}_0 = \phi \), from Lemma 2 it results
\[
\|e_h\| \leq \phi \alpha^h, \quad \forall h \geq 0, \tag{28}\]
which proves (19) and ensures also that (23) is verified. To conclude the proof, the boundedness of \( q_h \) can be easily shown by considering Eq. (9) and the following chain of inequalities
\[
\|q_{h+1}\| \leq \|q_h\| + \|\gamma T J_h^e\| \|e_h\| \leq \|q_h\| + \gamma T \delta^2 \phi \alpha^h, \tag{29}\]
where Eqs. (10) and (28) have been exploited. Now, consider the scalar linear system
\[
\tilde{q}_{h+1} = \tilde{q}_h + \gamma T \delta^2 \phi \alpha^h, \tag{30}\]
whose response with initial condition \( \tilde{q}_0 = \|q_i\| \) is
\[
\tilde{q}_h = \tilde{q}_0 + \gamma T \delta^2 \phi \frac{(1 - \alpha)}{1 - \alpha^h}. \tag{31}\]
Again, in view of (29),(31) and by applying Lemma 2, it results
\[
\|q_h\| \leq \|q_0\| + \gamma T \delta^2 \phi \frac{(1 - \alpha)}{1 - \alpha^h} = \rho, \quad \forall h \geq 0, \tag{32}\]
which proves the claim (20). To conclude the proof, observe that from (9) it follows that
\[
\|q_{h+1} - q_h\| \leq \gamma T H^{-1}\|e_h\| \leq \gamma T \delta^2 \|e_h\|, \tag{33}\]
and thus, in view of (28), the last claim (21) immediately follows. If the gain \( \gamma \) and the initial task error \( e_0 \) satisfy assumption (18), the proof is perfectly analogous. Theorem above clearly shows that the gain of the CLIK algorithm has to be chosen in relation to the sampling time and, as expected, the lower is the sampling time, the larger can be selected the gain. Moreover, the bound on the initial task space error allows to estimate the region of attraction of the origin of the task space error space, which can be enlarged only by reducing such gain or by staying far from singularities. Finally, the upper limit to the gain \( 2/T \) is certainly given by a sufficient condition, although it does not appear too restrictive as confirmed by the simulations presented in Section IV. Furthermore, the theorem ensures that no inner motion can occur even if no redundancy resolution scheme is adopted (see claim (21) of Theorem 1). Finally, the exponential convergence condition (19) implies the local exponential asymptotical stability of the origin of the task space error space.

Now, the convergence of the discrete-time IK algorithm, which makes use of the jacobian transpose, i.e., Eq. (11), will be proved. The analysis will be carried out by resorting to the same methodology used in the previous theorem and to the following Lemma.

Lemma 3: Let \( H \) be a full rank \( m \times n \) matrix, with \( m \leq n \), then
\[
\|I - \eta H H^T\| = 1 - \eta \|H H^T\|, \quad \forall \eta \in [0, 1/\|H H^T\|], \tag{34}\]
Proof: Consider the singular value decomposition $H = U \Sigma V^T$, the assumption that $H$ is full rank implies that $\Sigma = \text{diag}\{\sigma_1, \ldots, \sigma_m\}$ is such that $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_m > 0$ and of course it is $\Sigma^2 = \text{diag}\{\sigma_1^2, \ldots, \sigma_m^2\}$, where $\sigma_1^2 \geq \sigma_2^2 \geq \ldots \sigma_m^2 > 0$ are the singular values of $HH^T$, whose norm is therefore $\sigma_1^2$. Furthermore, it is

$$I - \eta HH^T = I - \eta U \Sigma V^T V \Sigma U^T = U(I - \eta \Sigma^2)U^T,$$

where the matrix $I - \eta \Sigma^2 = \text{diag}\{1 - \eta \sigma_1^2, \ldots, 1 - \eta \sigma_m^2\}$ is such that $1 - \eta \sigma_m^2 > 1 - \eta \sigma_m > 0$ and $\eta \in [0, 1]$. Thus, the decomposition (35), except for a simple reordering, is the singular value decomposition of $I - \eta HH^T$, whose norm is therefore $1 - \eta \sigma_m = 1 - \eta \Sigma^T \Sigma$. The task space error dynamics of the algorithm (11) is governed by the equation

$$e_{h+1} = x_d - k(q_h + \gamma T J_h^T e_h) - (I - \gamma T J_h J_h^T) e_h - r_k(q_h),$$

where $k(q)$ in (iii) has been exploited again. The following theorem carries out the convergence analysis of this algorithm and provides a bound to the gain $\gamma$.

**Theorem 2:** Under the assumptions i), ii), iii), if the initial task space error $e_0$ and the gain $\gamma$ are such that

$$0 < \gamma < \frac{1}{T \sigma^2}$$

and $\|e_0\| < \frac{\beta}{\gamma T \nu_k \sigma^2}$,

then the CLIK algorithm in (11) ensures exponential convergence of the task space error dynamics and the configuration variables $q_h$ are bounded and converge to a constant value, i.e.

$$\exists \alpha \in (0, 1), \phi > 0: \|e_h\| \leq \phi \alpha^h, \forall h \geq 0$$

$$\exists \rho > 0: \|q_h\| \leq \rho, \forall h \geq 0$$

$$\lim_{h \to \infty} \|q_{h+1} - q_h\| = 0.$$

Proof: From (36) and, owing to assumption (37), by applying Lemma 3 to compute the norm of the matrix $I - \gamma T J_h J_h^T$, the following inequalities are obtained

$$\|e_{h+1}\| \leq \|I - \gamma T J_h J_h^T\| \|e_h\| + \|r_k(q_h)\| \leq (\|I - \gamma T J_h J_h^T\| + \gamma^2 T^2 \nu_k \sigma^2 \|e_h\|) \|e_h\| = (1 - \gamma T \nu_k J_h J_h^T + \gamma^2 T^2 \nu_k \sigma^2 \|e_h\|) \|e_h\| \leq (1 - \gamma T \beta + \gamma^2 T^2 \nu_k \sigma^2 \|e_h\|) \|e_h\|,$$

where the bounds in assumptions i), ii), iii) have been exploited together with standard norm properties. Now assume that the task error norm is bounded, i.e.

$$\|e_h\| \leq \phi, \forall h \geq 0.$$

(42)

In a moment it will be shown that such a condition is guaranteed by the sole hypothesis that the initial condition satisfies condition (37). Equation (41) becomes

$$\|e_{h+1}\| \leq (1 - \gamma T \beta + \gamma^2 T^2 \nu_k \sigma^2 \phi) \|e_h\| \leq \alpha \|e_h\|, \forall h \geq 0.$$

(43)

Since, by assumption, the gain and the initial task space error satisfy (37), by choosing $\phi = \|e_0\|$ it is

$$\phi < \frac{\beta}{\gamma T \nu_k \sigma^2} \Rightarrow \alpha < 1.$$

(44)

The rest of the proof is perfectly analogous to the one of Theorem 1 and thus it is omitted for brevity.

Note that, differently from the stability analysis performed in [6], there is no necessity to include any additional matrix to avoid inner motions. In fact, no inner motion can occur since the configuration variables are shown to reach a steady-state (see claim (40) in Theorem 2). Moreover, also in this case the limit on the gain $\gamma$ is related to the sampling time and to the robot kinematics. In particular, the closer is any kinematic singularity to the configuration space $\mathcal{Q}$, the lower is the bound $\beta$ and thus the lower can be selected the gain $\gamma$ if the same amplitude of the region of attraction is desired. On the other hand, the gain can be increased if a smaller region of attraction can be tolerated.

IV. SIMULATIONS

The presented case study is the classical IK problem for the eleven-link humanoid manipulator described in Section 2.9.9 of [16]. It is constituted by a four-link torso and a seven-link arm. The simulations are intended to show how the sufficient conditions of Theorem 1 are not restrictive at all. The simulations were carried out in MATLAB/Simulink using the Robotics Toolbox [17]. The forward kinematics of the robot has been obtained by solving the exercise 2.14 of [16]. In order to perform the simulation, a subset $\mathcal{Q}$ of the joint space has been considered, in which the jacobian matrix of the manipulator is full-rank. Within this joint space, it is easy to evaluate, through a standard nonlinear optimization algorithm, the constants $\delta = 4.19$ and $\beta = 0.076$, in assumptions i), ii), respectively. To verify that the direct kinematics function have norm-bounded Hessian matrices, namely that assumption iii) holds, the constant $\nu_k = 9.17$ is easily estimated recalling its definition in Lemma 1, i.e. $\nu_k = \sqrt{2/\max \{\nu_i\}}$, being $\nu_i$ the upper bound of the $i$-th Hessian matrix norm of the function $k(q)$. These constants are useful to estimate the regions of attraction of the two algorithms, i.e. the maximum allowed initial task space error norm, according to Eqs. (17),(18),(37).

The desired constant position in the task space is $x_d = (-0.071 -0.639 0.507 1.48 0.202 -2.22 \pi/2 \pi/3 \pi/3 0 \pi/4 \pi/4 -3\pi/4 2/3\pi 3/4\pi 3/4\pi 0)^T$, in which the orientation is expressed by using the ZYX Euler angles. The considered sampling time is $T = 1$ms, while the initial robot configuration is $q_i = (\pi/2 \pi/3 \pi/3 0 \pi/4 \pi/4 -3\pi/4 2/3\pi 3/4\pi 0)^T$. The robot is used in the following simulations to show that the bounds on the algorithm gain $\gamma$ of Theorems 1 are not too restrictive. The algorithm based on the jacobian pseudo-inverse (Eq. (9)) has been tested with three values of the gain, i.e. $\gamma = 0.5/T, \gamma = 1.9/T$ and $\gamma = 2.01/T$, the last one just a little bit beyond the bound in (18). In this simulation, the algorithm includes the redundancy resolution, performed by resorting to the classical technique of the projector into the jacobian null space, i.e.

$$q_{h+1} = q_h + \gamma T J_h^T e_h + T(I - J_h^T J_h)q_{0_h},$$

(45)
where $\dot{q}_{0m} = -\nabla(w(q_m))^T$ and $w(q)$ is a cost function to minimize [16]. The distance in the joint space from the center of joint ranges has been chosen as cost function. The resulting task space error norm is reported in Fig. 1 for all gain values, showing how the algorithm converges only with the gain values less than $2/T$. Figure 2 reports the time history of the cost function with and without the redundancy resolution, and, in the former case, better values are obtained. The results obtained by using the IK algorithm with the Jacobian transpose are reported in Fig. 3, which shows that the gain bound found in Theorem 2 is more restrictive than those found in Theorem 1, since with a gain $\gamma$ equal to $1/(\delta^2 T)$ the algorithm still converges, whereas the minimum value of the gain $\gamma$ leading to an unstable behavior of the algorithm is $20/(\delta^2 T)$.

V. CONCLUSIONS

The paper provided novel proofs of convergence of two closed-loop IK algorithms in the discrete-time domain for redundant robots. Despite the nonlinearity of the algorithms the proof did not make use of Lyapunov arguments, nevertheless sufficient conditions for the stability and an estimation of the region of convergence were provided. A simulation case study showed that the found gain bounds are not too restrictive.

REFERENCES