Some results on odd factor of graphs

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Abstract
A \(\{1, 3, \cdots, 2n-1\}\)-factor of a graph \(G\) is defined to be a spanning subgraph of \(G\), each degree of whose vertices is one of \(\{1, 3, \cdots, 2n-1\}\), where \(n\) is a positive integer. In this paper, we give a sufficient condition for a graph to have a \(\{1, 3, \cdots, 2n-1\}\)-factor.

1 Main theorem

We consider finite graphs that have neither loops nor multiple edges. Let \(G\) be a graph with vertex set \(V(G)\). For a vertex \(v\) of \(G\), we write \(\text{deg}_G(v)\) for the degree of \(v\) in \(G\). For a subset \(S\) of \(V(G)\), the neighborhood \(\Gamma_G(S)\) of \(S\) is defined to be the set of vertices if \(G\) that are adjacent to at least one vertex of \(S\). Let \(I\) be a set of nonnegative integers. A graph \(G\) is called an \(I\)-graph if \(\text{deg}_G(x) \in I\) for all \(x \in V(G)\). We call a spanning \(I\)-subgraph of \(G\) an \(I\)-factor of \(G\). In particular, a \(\{1, 3, \cdots, 2n-1\}\)-factor of a graph \(G\) is an spanning subgraph \(F\) of \(G\) such that the degree of every vertex of \(F\) is contained in \(\{1, 3, \cdots, 2n-1\}\), where \(n\) is a positive integer. A \(\{k\}\)-factor will be called a \(k\)-factor.

The following proposition gives a sufficient condition for the existence of a 1-factor in a graph by using neighborhoods.

**Proposition 1 (Anderson\(^2\),[4p.115])** \(\) Let \(G\) be a graph with an even number of vertices. If

\[
\Gamma_G(X) = V(G) \quad \text{or} \quad |\Gamma_G(X)| \geq \frac{4}{3}|X| - \frac{2}{3}
\]

for all \(X \subset V(G)\), then \(G\) has a 1-factor.
Our next main theorem is an extension of this proposition, and its proof is analogous to that of Proposition 1.

**Theorem 1** Let $G$ be a graph with an even number of vertices, and let $n$ be a positive integer. If

$$\Gamma_G(X) = V(G) \quad \text{or} \quad |\Gamma_G(X)| > (1 + \frac{1}{3(2n-1)})|X| - \frac{1}{2n-1}$$

for all $X \subset V(G)$, then $G$ has a $\{1, 3, \cdots, 2n-1\}$-factor.

This theorem is best possible in the sense that the condition in Theorem 1 cannot be replaced by the condition that

$$\Gamma_G(X) = V(G) \quad \text{or} \quad |\Gamma_G(X)| \geq (1 + \frac{1}{3(2n-1)})|X| - \frac{1}{2n-1}$$

for all $X \subset V(G)$. This fact will be shown in Theorem 2.

We give some definitions before proving Theorem 1. For a subset $S$ of $V(G)$, we denote by $G - S$ the subgraph of $G$ obtained from $G$ by deleting the vertices in $S$ together with their incident edges. We write $o(G)$ for the number of odd components (components with odd order) of $G$. Our proof of Theorem 1 depends on the following theorem, which is a generalization of Tutte’s 1-factor Theorem and will be extended in Theorem 3.

**Proposition 2 (Amahashi[1])** Let $n$ be a positive integer. Then a graph $G$ has a $\{1, 3, \cdots, 2n-1\}$-factor if and only if

$$\text{odd}(G - X) \leq (2n - 1)|X| \quad \text{for all} \quad X \subset V(G).$$

**Proof of Theorem 1.** Suppose that $G$ satisfies the condition in the theorem but has no $\{1, 3, \cdots, 2n-1\}$-factor. Then there exist a subset $S \subset V(G)$ with $o(G - S) > (2n - 1)|S|$ by Proposition 2. Let $|V(G)| = p$. Since $p$ is even, by parity, we may assume $o(G - S) \geq (2n - 1)|S| + 2$. Let $m$ denote the number of isolated vertices of $G - S$, and put $t = 1 + (1/3(2n-1))$ and $r = 1/(2n-1)$. We consider two cases.

**Case 1.** $m > o$. Since $|\Gamma_G(V(G) - S)| \neq V(G)$, we have

$$|\Gamma_G(V(G) - S)| > t|V(G) - S| - r = tp - t|S| - r.$$ 

It is clear that $|\Gamma_G(V(G) - S)| \geq p - m$. From these inequalities, we obtain

$$p < \frac{t|S| + r - m}{t - 1}. \quad (1)$$

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On the other hand, counting the vertices of the odd components of $G - S$, we have $m + 3((2n - 1)|S| + 2 - m) \leq p - |s|$, and thus

$$(3(2n - 1) + 1)|S| + 6 - 2m \leq p. \quad (2)$$

Combining inequalities (1) and (2), we obtain

$$(3(2n - 1) + 1)|S| + 6 - 2m < \frac{t|S| + r - m}{t - 1}. \quad (3)$$

Substituting the values of $t$ and $r$ into (3), we can get $3 + (6n - 5)m < 0$, a contradiction.

**Case 2.** $m = 0$. In this case, every odd component has at least three vertices. Let $X$ be the set of vertices of any $(2n - 1)|S| + 1$ odd components of $G - S$. Since $\Gamma_G(X) \neq V(G)$, we have $|\Gamma_G(X)| > t|S| - r$ and hence

$$|X| < \frac{|S| + r}{t - 1}. \quad (4)$$

On the other hand, $|X| \geq 3((2n - 1)|S| + 1)$ as well. So combining it with inequality (4), we obtain

$$3((2n - 1)|S| + 1) < \frac{|S| + r}{t - 1}.$$

Substituting the values of $t$ and $r$ into the above inequality, we get $0 < 0$, a contradiction.

Consequently, the proof is complete. \[\blacksquare\]

If a graph $G$ consists of $n(n \geq 2)$ disjoint copies of a graph $H$, then we write $G = nH$. The join $G = A + B$ has $V(G) = V(A) \cup V(B)$ and $E(G) = E(A) \cup E(B) \cup \{xy|x \in V(A)\text{ and } y \in V(B)\}$.

**Theorem 2** For every position integer $n$, there exists infinitely many graphs $G$ that have no $\{1, 3, \cdots, 2n - 1\}$-factor and satisfy

$$\Gamma_G(X) = V(G) \quad \text{or} \quad |\Gamma_G(X)| \geq (1 + \frac{1}{3(2n - 1)})|X| - \frac{1}{2n - 1}$$

for all $X \subset V(G)$.

**Proof** Let $m$ be a positive integer. We define a graph $G$ by $G = K_m + ((2n - 1)m + 2)K_3$, where $K_m$ and $K_3$ denote the complete graphs of order $m$ and $3$, respectively. It is trivial that $G$ is of even order. Put $S = V(K_m)$. Then $o(G - S)$ has $(2n - 1)m + 2$ odd components, and so $G$ has no
In this case, we have $S$ for all $x$. We denote such a function by $f$ defined on $V$. This was mentioned before by Knao. Let $G$ be any subset of $V(G)$.

In this section, we give an extension of Amahashi’s Theorem (proposition 2), which was mentioned before by Knao. Let $G$ be a graph and $f$ be a function defined on $V(G)$ such that $f(x)$ is a position odd integer for every $x \in V(G)$. We denote such a function by $f : V(G) \to \{1, 3, 5, \cdots \}$. Then a spanning subgraph $F$ of $G$ is called an $(1, f)$-odd-factor if $deg_F(x) \in \{1, 3, 5, \cdots \}$ for all $x \in V(G)$. It is obvious that if $f(x) = 2n - 1$ for all $x \in V(G)$, then a $(1, f)$-odd-factor and a $\{1, 3, \cdots , 2n - 1\}$-factor are the same. We prove the following theorem.

**Theorem 3** Let $G$ be a graph and $f : V(G) \to \{1, 3, 5, \cdots \}$. Then $G$ has a $(1, f)$-odd-factor if and only if

$$o(G - S) \leq \sum_{x \in S} f(x) \quad (5)$$

for all $S \subset V(G)$.

In order to prove Theorem 3, we need the following two lemmas.
Lemma 4 Let $G$ be a tree of even order and $f : V(G) \to \{1, 3, 5, \ldots\}$. Then $G$ has a $(1, f)$-odd-factor if and only if
\[ o(G-x) \leq f(x) \text{ for all } x \in V(G). \]

Proof The proof is similar to that of Theorem 1 of [1].

Lemma 5 Let $G$ be a bipartite graph with partite sets $X$ and $Y$, and let $g$ be an integer valued function defined on $X$. Then $G$ has a spanning subgraph $H$ such that
\[ \deg_H(x) = g(x) \text{ for all } x \in X \text{ and } \deg_H(y) = 1 \text{ for all } y \in Y \]
if and only if
\[ |Y| = \sum_{x \in X} g(x) \text{ and } |\Gamma_G(S)| \geq \sum_{x \in S} g(x) \text{ for all } S \subset X. \]

Proof The lemma is an immediate consequence of Hall’s Marriage Theorem[3].

Proof of Theorem 3. This theorem can be proved similar as proposition 2. Assume that $G$ has a $(1, f)$-factor $F$. Then we have
\[ o(G-S) \leq \sum_{x \in S} \deg_F(x) \leq \sum_{x \in S} f(x) \]
since there exists at least one edge of $F$ between every odd component of $G-S$ and $S$.

We next prove the sufficiency by induction on $|V(G)| + |E(G)|$. Without loss of generality, we may assume that $G$ is connected. Moreover, we have that $|V(G)|$ is even by setting $S = \emptyset$ in (5). It is immediate that
\[ o(G-S) \equiv |S| \equiv \sum_{x \in S} f(x)(mod 2). \quad (6) \]

By Lemma 1, if $G$ is a tree, then $G$ has a $(1, f)$-odd-factor. Hence we may assume that $G$ is not a tree. We consider two cases.

Case 1. $o(G-S) < \sum_{x \in S} f(x)$ whenever $\emptyset \neq S \subset V(G)$.

There exists an edge $e$ such that $G-e$ is connected, where $G - e$ denotes the subgraph of $G$ obtained from $G$ by deleting only the edge $e$. For every $S \subset V(G)$, it follows from (6) that
\[ o((G-e)-S) \leq o(G-S) + 2 \leq \sum_{x \in S} f(x). \]
Thus $G - e$ has a $(1, f)$-odd-factor by the induction hypothesis, and hence $G$ has a $(1, f)$-odd-factor.

**case 2.** $o(G - S) = \sum_{x \in S} f(x)$ for some nonempty $S \subset V(G)$.

Choose such a subset $S_0$ so that $|S_0|$ is maximum. Then every even component $D$ of $G - S_0$ has a $(1, f)$-odd-factor $F(D)$ since $D$ satisfies condition (5). Let $X$ be the set of all odd components of $G - S_0$ and let $B$ be a bipartite graph with partite sets $X$ and $S_0$, in which $C \in X$ and $s \in S_0$ are joined by an edge if and only if $G$ contains an edge joining $s$ to a vertex of $C$. Then we can show that $B$ has a spanning subgraph $H$ such that

$$d_H(C) = 1 \text{ for all } C \in X \text{ and } d_H(s) = f(s) \text{ for all } s \in S_0$$

by Lemma 2 and by the choice of $S_0$. For every edge $e' = Cs$ of $H$, there exists an edge $e$ of $G$ such that $e$ joins a vertex of $C$ to $s$. We can show that the subgraph $C + e$ of $G$, which is obtained from $C$ by adding an edge $e$ together with its end vertex $s$, has a $(1, f')$-odd-factor $F'(C + e)$ by the induction hypothesis, where $f'(x) = f(x)$ if $x \neq s$ and $f'(s) = 1$. Consequently, we obtain a desired $(1, f)$-odd-factor $F$ of $G$ given by

$$F = \{F(D) | D \text{ are even components of } G - S_0\}$$

$$\cup \{F'(C + e) | C \text{ are odd components of } G - S_0 \text{ and } e' \in E(H)\}. \ ■$$

Note that it seems to be difficult to give a sufficient condition for a graph to have a $(1, f)$-odd-factor by using neighborhoods. The following natural question is open: Is it possible to characterize graphs $G$ that satisfy

$$\text{odd}(G - X) \leq 2n|X| \text{ for all } X \subset V(G)$$

in terms of factors?

**Acknowledgement**

The authors wish to thank referees for their suggestions and corrections.

**References**


