On a production-inventory system of deteriorating items subject to random machine breakdowns with a fixed repair time

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Abstract

This paper considers the impact of random machine breakdowns on the classical Economic Production Quantity (EPQ) model for a product subject to exponential decay and under a no-resumption (NR) inventory control policy. A product is manufactured in batches on a machine that is subject to random breakdowns in order to meet a constant demand over an infinite planning horizon. The product is assumed to have a significant rate of deterioration and time to deterioration is described by an exponential distribution. Also, the time-to-breakdown is a random variable following an exponential distribution. Under the NR policy, when a breakdown occurs during a production run, the run is immediately aborted. A new run will not be started until all available inventories are depleted. Corrective maintenance of the production system is carried out immediately after a breakdown and it takes a fixed period of time to complete such an activity. The objective is to determine the optimal production uptime that minimizes the expected total cost per unit time consisting of setup, corrective maintenance, inventory carrying, deterioration, and lost sales costs. A near optimal production uptime is derived under conditions of continuous review, deterministic demand, and no shortages.

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1. Introduction

The economic production quantity (EPQ) model is a simple mathematical model to deal with inventory management issues in a production-inventory system. The EPQ model can be applied to a problem where a product is to be manufactured on a single machine so as to meet a deterministic demand over an infinite planning horizon. It is considered to be one of the most popular inventory control models used in industry (see Osteryoung et al. \cite{1}). However, this model is developed under many restricted assumptions. Many results have been reported in the literature where these assumptions are relaxed.

One limited assumption of these models is that a machine breakdown will never occur during a production run. Another assumption is that items produced have indefinitely long lives. In general, almost all production systems are subject to breakdowns and items produced will deteriorate over time. Sometimes the rate of deterioration is too low...
Nomenclature

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
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<tbody>
<tr>
<td>$P$</td>
<td>production rate of the product given in number of units per unit time</td>
</tr>
<tr>
<td>$D$</td>
<td>demand of the product given in number of units per unit time</td>
</tr>
<tr>
<td>$v$</td>
<td>unit production cost</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>$1 - D/P$, the traffic intensity</td>
</tr>
<tr>
<td>$\pi$</td>
<td>cost of a deteriorated unit</td>
</tr>
<tr>
<td>$S$</td>
<td>cost of setting up a production run</td>
</tr>
<tr>
<td>$R$</td>
<td>a fixed period of time for a maintenance activity to be executed on the machine</td>
</tr>
<tr>
<td>$h$</td>
<td>inventory carrying cost/unit/unit time</td>
</tr>
<tr>
<td>$M$</td>
<td>corrective maintenance cost of the machine</td>
</tr>
<tr>
<td>$\delta$</td>
<td>unit cost of lost sales</td>
</tr>
<tr>
<td>$X$</td>
<td>time-to-breakdown, a random variable following an exponential distribution</td>
</tr>
<tr>
<td>$\mu$</td>
<td>the parameter of an exponential probability density function of $X$</td>
</tr>
<tr>
<td>$\tau$</td>
<td>a production uptime of a production cycle when no machine breakdown occurs, a decision variable</td>
</tr>
<tr>
<td>$T_2$</td>
<td>production downtime during which there is no production in a cycle</td>
</tr>
<tr>
<td>$I_1(t_1)$</td>
<td>time-varying inventory level for the product in the cycle segment $t_1$, $0 \leq t_1 \leq \tau$</td>
</tr>
<tr>
<td>$I_2(t_2)$</td>
<td>time-varying inventory level for the product in the cycle segment $t_2$, $0 \leq t_2 \leq T_2$</td>
</tr>
<tr>
<td>$I(\tau)$</td>
<td>maximum inventory level at time $\tau$</td>
</tr>
<tr>
<td>$Z(\tau)$</td>
<td>total cost of multiple products produced by single machine/unit time</td>
</tr>
<tr>
<td>$\theta$</td>
<td>a constant deterioration rate (unit/unit time) of product $j$</td>
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Many models for dealing with deteriorating items have been proposed over the past several decades. Raafat [2] supplied a rather comprehensive literature survey of models up to 1990. Goyal and Giri [3] presented another review of the literature of deteriorating inventory since the early 1990s. As many models considering deteriorating items reported in the literature are not directly relevant to this topic, we review only the papers that considered various forms of the EPQ model. Misra [4] considered an EPQ model for deteriorating items with both a varying and a constant rate of deterioration. Shah and Jaiswal [5] expanded Misra’s model to include backlogging. Mak [6] studied a case with backordering for unfilled orders. Hwang and Hwang [7] investigated the effects of two inventory-issuing policies, first-in-first-out and last-in-first-out, for items with Weibull deterioration. Chowdhury and Chaudhuri [8] provided a model with deterministic and stochastic demand. Elsayed and Teresi [9] presented two EPQ models, one for deterministic demand and backlogging, and the other assumed the demand is a random variable having a normal distribution with known mean and variance. A two-parameter Weibull distribution is used to model the rate of deterioration for items. Kang and Kim [10] generated a modified EPQ model for deteriorating items to determine both price and production level so as to maximize profit. Park [11] studied an integrated production-inventory model where decaying raw materials are purchased from outside suppliers for the production of a finished product. But the finished product is not subject to deterioration. Hwang and Choi [12] provided models under FIFO and LIFO inventory policies with Weibull decay and complete backorders for shortages. Raafat [13] reconsidered the model given in Park [11] to cope with a case where the finished product is also subject to a constant rate of decay. Choi and Hwang [14] developed a model using control theory to determine the production rate for deteriorating items to minimize a total cost function over a finite planning period. Deb and Chaudhuri [15] proposed a model where shortages are backordered and the deterioration rate is a linear function of time. Aggarwal and Bahari-Hashani [16] looked into a model assuming a constant deterioration rate, a demand rate decreasing negative exponentially, allowing no shortages, and a known production rate which can vary from one period to another over a finite planning period. Heng et al. [17] derived a generalized production-inventory model extending Misra’s model to a situation where backorders are allowed. Raafat et al. [18] proposed an alternative approach for obtaining an optimal solution of a model given in Mak [6]. A numerical search method was provided for finding an optimal production cycle time and an optimal total inventory cycle time. Pakkala and Achary [19] considered a model with two storage facilities and a constant demand rate; shortages are allowed. Wee [20] investigated an EPQ model with partial backordering. Goyal and Gunasekaran [21] developed...
an integrated model for determining an optimal production quantity of a finished product and an optimal ordering quantity for raw materials in a multi-stage production system. Kogan and Khmelnitsky [22] studied continuous-time aggregate production planning problems for deteriorating items. An optimal control approach was proposed using differential equations to deal with the production and capacity decisions under a situation where regular production, subcontracting, and capacity change rates are controllable on one hierarchical level.

Balkhi and Benkherouf [23] proposed a method for obtaining an optimal production cycle time of deteriorating items in a model where demand and production rates are functions of time. Mandal et al. [24] derived a profit-maximization multi-item production-inventory model having limited storage space and stock-dependent demand in a fuzzy environment. Su et al. [25] considered a case with a constant rate of decay over a finite time horizon. The production rate is dependent on demand and shortages are allowed. Wee and Law [26] applied the discounted cash-flow approach to a production-inventory system where the rate of decay is modeled by a two-parameter Weibull distribution. In addition, the demand rate is a decreasing linear function of the selling price. Yang and Wee [27] investigated a multi-lot-size production-inventory system with constant production and demand rates.

Note that the factor of machine breakdown is not considered in all of the aforementioned papers. Hence, we study in this paper an EPQ model for a single product with exponential deterioration that is subject to random machine breakdowns. In particular, we consider a no-resumption (NR) inventory control rule. Under this rule, when a breakdown occurs during a production run, the production is immediately aborted. A new production run will be started only when all available inventories are depleted. Corrective maintenance of the production system is carried out immediately after a breakdown. The production system is restored to the same initial working condition after each maintenance action. The time interval between the beginning of a production run and the time epoch at which a breakdown occurs (time-to-breakdown) is assumed to be an exponentially distributed random variable. The corrective maintenance activity takes a fixed amount of time. Consider the case where a breakdown has occurred before the end of a production time. If the inventory build-up until the occurrence of a breakdown is not enough to cover the demand during the repair period, then shortages will occur. In this case, we assume that demands that are not filled become lost sales. As an example, unfilled demand may have to be met by units purchased from outside suppliers which will incur additional costs.

Examples of the EPQ models that deal with random breakdowns are surveyed below. Groeneveldt et al. [28] considered the effects of machine breakdowns and corrective maintenance on the economic lot sizing decisions. They proposed two production control policies to deal with machine breakdown. The first policy assumes that production will not resume after a breakdown. Instead, a new production cycle starts after the on-hand inventory is depleted. This is called the no-resumption (NR) policy. Under the second policy, production is immediately resumed after a breakdown if the on-hand inventory is below a certain threshold level. This policy is called the abort/resume (AR) policy. Both policies assume the repair time is negligible. Maintenance of the production unit is carried out after the occurrence of a machine breakdown or after the end of a predetermined time interval, whichever occurs first. Each maintenance action restores the system to the same initial working condition. A later study by Groeneveldt et al. [29] explored the problem of simultaneously determining lot sizing and safety stock policies when repair times are significant, which followed a general distribution, and the time to failure were exponential. Posner and Berg [30] investigated a model with a compound Poisson demand distribution. The time-to-machine-failure and repair times were both exponentially distributed. Some results in the steady-state condition were derived. Hopp et al. [31] proposed an inventory control policy for a two-machine production system with an intermediate buffer, but only the first machine is subject to random breakdowns.

Tse and Makis [32] developed a model with preventive replacement subject to minor and major machine failures. Kuhn [33] considered the impact of stochastic breakdowns on a dynamic lot sizing model. Kim and Hong [34] generalized the model obtained by Groeneveldt et al. [28] to deal with the case where the time interval between two successive failures was generally distributed. The repair time was assumed to be instantaneous. Chung [35] focused on the bounds of production lot sizing and safety stock when the failure distribution was exponential. With these bounds, a simple algorithm to obtain an optimal production lot size and safety stock level was developed. Moinzadeh and Aggarwal [36] investigated a production-inventory system that was subject to random disruptions. The restoration time was assumed to be constant. The time between two successive breakdowns is exponentially distributed. Shortages were completely backordered. A continuous review, order-up-to inventory policy was proposed. Based on the results from Poisson and regenerative processes, Abboud [37] presented a simple approximation to the EPQ problem with machine breakdowns. In a subsequent paper, Abboud [38] considered a problem of Markov chain production-inventory system
when the time-to-failure and the repair time were geometrically distributed. Lee and Rung [39] studied a two-stage production system where only the first machine was subject to random breakdowns. While the aforementioned papers dealt with the EPQ models with random machine breakdowns, the factor of process deterioration was ignored. It is conceivable that a process may concurrently deteriorate due to aging and be disrupted due to machine breakdowns.

Since the time-to-breakdown during a production run is a random variable, the objective is to determine an optimal target production uptime that minimizes the expected (long-run) average cost per unit of time including setup, corrective maintenance, inventory holding, and deterioration costs. Since a closed-form solution of the optimal production uptime is not readily available, a numerical procedure is developed for finding a near-optimum solution.

In the next section, we present assumptions and notation that are employed for the development of the model. Then the model is defined and derived in Sections 3 and 4. Several illustrative numerical examples and final concluding remarks are given in the subsequent sections.

2. Assumptions

The following assumptions are used in the development of the model.

1. The demand rate of each product is a known constant.
2. An infinite planning horizon is assumed.
3. The production rate of each product is finite. The machine has a large enough capacity to cover the demand of all products.
4. Once a unit of a product is produced, it is immediately available to meet demand.
5. A unit starts deteriorating the moment that it is received into inventory.
6. The time to deterioration of each product follows an exponential distribution.
7. If a breakdown occurs during a production run, then a fixed amount of time is needed to carry out the maintenance activity.
8. There is no replacement or repair for a deteriorated item.
9. The production uptime is unknown but it will not vary from one cycle to another.
10. Inventory holding cost is charged only on the amount of undecayed stock.
11. The cost of a deteriorated unit is known and includes any disposal cost or salvage value.

3. Model description

This paper deals with an EPQ model that considers the effect of random machine breakdowns on the decision of the optimal production uptime for deteriorating items under a NR policy. Under this operating policy, a production run is to be executed for a predetermined period of time (i.e., the production uptime) provided that no machine breakdown has occurred in this period. Otherwise, the production run is immediately aborted when a breakdown occurs. Since inventories are built up gradually during the production uptime, a new production run is to be started only when all on-hand inventories are depleted. The corrective maintenance activity is carried out immediately after a breakdown occurs. This activity takes a fixed period of time to complete. If the inventory build-up during the production uptime is not enough to meet the demand required for the entire period of the corrective maintenance, shortages will occur. We assume all shortages are lost sales. We also assume that each maintenance activity restores the production system to the same initial working conditions. A setup is required before each production run and a small preventive maintenance cost is always included in the setup cost. Examples of those situations with a constant restoration time were given in Moinzadeh and Aggarwal [36]. The corrective maintenance costs will be incurred only when a breakdown occurs during a production run.

Although the time-to-breakdown is a random variable given in the model, we can consider the time epoch where each production run begins as a renewal epoch. A sample path representing the behavior of the inventory system at any time during a production cycle is depicted in Fig. 1. The sample path is explained below. Note that a detailed description of a deterministic EPQ model for deteriorating items that is not subject to random machine breakdowns was given in Misra [4].

In Fig. 1, when the inventory level declines to zero, a production run is set up at time $t = 0$ to run for a period of $\tau$ units of time provided that no machine breakdown has occurred. During this production uptime period, the inventory is gradually built up at a rate of $P - D$ because the production rate is greater than the demand rate. This rate is offset
by a constant deterioration rate $\theta$. The inventory level increases to its maximum, $I_1(\tau)$, at $t = \tau$ time units if no machine breakdowns occur. After the maximum inventory is reached, the run is terminated. From this point on, the on-hand inventory will be used to meet the demand plus the loss due to deterioration. Another production run will be started when all on-hand inventory are depleted. This period is called the production downtime. The entire cycle is represented by the dotted line

On the other hand, if a machine breakdown occurs at time $t = x$ with $x < \tau$, then the production run is immediately aborted. In this case, the inventory level will be raised only up to $I_1(x)$. A corrective maintenance is undertaken for a fixed period of time $R$. A new production run will be started only when all on-hand inventory are depleted. Let $T_2 = T - x$ denote the production downtime of a cycle. The inventory build-up $I_1(x)$ will be used to meet the demand and the loss due to deterioration during $T_2$. If $T_2 \geq R$, no shortages will occur; otherwise, lost sales occur during a period of $R - T_2$. Since the machine is subject to random breakdowns, the actual production uptime and the entire cycle time depend on the time point at which a breakdown occurs.

The time interval between two successive starting points of production is a production cycle time, denoted by $T$. This cycle time is composed of production uptime, downtime, and a possible repair time. Conditioning on $X = x$ and with the assumption that items follow an exponential deterioration process, the inventory level of the system at time $t$ can be described by the following differential equations:

$$\frac{dI_1(t_1)}{dt_1} + I_1(t_1)\theta = P - D, \quad \text{for } 0 \leq t_1 \leq x$$

$$\frac{dI_2(t_2)}{dt_2} + I_2(t_2)\theta = -D, \quad \text{for } 0 \leq t_2 \leq T_2.$$  

The boundary conditions associated with these equations are: at $t_1 = 0$, $I_1(t_1) = 0$, the initial inventory, and at $t_2 = 0$, $I_2(t_2) = I_1(x)$. Eqs. (1) and (2) can be solved to yield the following two expressions for inventory level at a particular time point. These two expressions are used in the derivation of our model.

$$I_1(t_1) = \frac{P - D}{\theta} \left(1 - e^{-\theta t_1}\right), \quad \text{for } 0 \leq t_1 \leq x$$

$$I_2(t_2) = \frac{D}{\theta} \left(e^{\theta t_2} - e^{\theta t_1}\right), \quad \text{for } 0 \leq t_2 \leq T_2.$$  

Let $u$ denote the time point at which a machine breakdown occurs and the inventory built up is just enough to meet the demand and the loss due to deterioration during the fixed repair period. This implies that $u$ is a function of the repair time $R$. Since $R$ is given, $u$ becomes a constant. When the machine breaks down on $u$, it follows that $T_2 = R$. Using Eqs. (3) and (4) and the condition $I_1(u) = I_2(0)$, we obtain the following equation:

$$\frac{P - D}{\theta} \left(1 - e^{-\theta u}\right) = \frac{D}{\theta} \left(e^{\theta R} - 1\right).$$
Thus, the expression of $u$ can be found as
\[
  u = -\frac{1}{\theta} \ln \left( \frac{P}{P - D} - \frac{D}{P - D} e^{\theta R} \right).
\] 
(5)

This implies that if the machine breaks down before the time point $u$, a cost of lost sales will incur. The objective
of the model is to determine an optimal production uptime $\tau^*$ that minimizes the expected (long-run) average
cost per unit time. As stated before, we consider the time epoch where each production run begins as a renewal
epoch. Therefore, based on the renewal reward theorem (see Ross [40]) the expected total cost per unit time can be
obtained by dividing the expected total cost per renewal cycle to the expected duration of a renewal cycle as the
following:
\[
  Z(\tau) = \frac{E[\text{cost per cycle}]}{E[\text{duration of a cycle}]} = \frac{E[C(\tau)]}{E[T(\tau)]} = \frac{g(\tau)}{h(\tau)}.
\] 
(6)

4. Approximation solutions

In this section, we develop the expression for $Z(\tau)$, $g(\tau)$, and $h(\tau)$. First we derive the expected average inventory
carried per cycle below. From Eqs. (3) and (4), the expected average inventory carried per cycle, $\bar{I}$, given that $X = x$
is
\[
  E[\bar{I}|X = x] = \int_0^x I_1(t_1) dt_1 + \int_0^{T_2} I_2(t_2) dt_2 = \int_0^x \frac{P - D}{\theta} (1 - e^{-\theta t_1}) dt_1 + \int_0^{T_2} \frac{D}{\theta} \left( e^{\theta t_2} - e^{\theta T_2} \right) dt_2
\]
\[
  = \frac{P - D}{\theta} \left( x + \frac{1}{\theta} e^{-\theta x} - \frac{1}{\theta} \right) + \frac{D}{\theta} \left( -T_2 + \frac{1}{\theta} e^{\theta T_2} - \frac{1}{\theta} \right).
\] 
(7)

Since $T_2$ is dependent on the value $x$, we adopt the idea used in Misra [4] and Mak [6] to express $T_2$ in terms of $x$
in Eq. (7) so that there is only one variable $x$ in the equation. At the time point when a production run is terminated
during a cycle, we can set up an equation from (3) and (4) as follows:
\[
  (P - D) \left( 1 - e^{-\theta x} \right) = I_1(x) = I_2(0) = D \left( e^{\theta T_2} - 1 \right).
\] 
(8)

This implies that
\[
  T_2 = \frac{1}{\theta} \ln \left[ \frac{P}{D} - \frac{P - D}{D} e^{-\theta x} \right].
\] 
(9)

If we substitute for $T_2$ in (7) with the expression of (9) for further analysis, the derivation will be very complex.
Instead, we derive an approximate solution. We assume $\theta x$ and $\theta T_2$ are relatively small, and use the Taylor series
approximations to all exponential terms. This approach is commonly used in most research projects in order to derive
a closed-form solution. An example can be seen in Misra [4]. Expanding exponential functions gives
\[
e^{-\theta x} = 1 - \theta x + \frac{\theta^2}{2} x^2
\]
\[
e^{\theta T_2} = 1 + \theta T_2 + \frac{\theta^2}{2} T_2^2.
\]

Using the above expressions for the exponential functions in (8), we obtain
\[
  (P - D) \left( x - \frac{1}{2} \theta x^2 \right) = D \left( T_2 + \frac{1}{2} \theta T_2^2 \right).
\] 
(10)

It follows that an approximate expression of $T_2$ in terms of $x$ is given as
\[
  T_2 \approx \frac{P - D}{D} \left( x - \frac{1}{2} \theta x^2 \right).
\] 
(11)
Now using Taylor series approximation to those exponential terms in (7) and combining with the expression in (11), we obtain

\[
E[\bar{I}|X = x] \approx \begin{cases} 
\frac{P^2}{2D} x^2, & \text{for } x \leq \tau \\
\frac{P^2}{2D} \tau^2, & \text{for } x > \tau. 
\end{cases}
\] (12)

Note that this approximation is commonly used in obtaining the average inventory carried in a production cycle (see Misra [4]). Next, we examine the expected deterioration cost per cycle. The number of deteriorating units per cycle is equal to the difference between the number of units produced in a cycle and the number of units used to meet the demand in a cycle. This implies that the expected number of deteriorating units, \( \bar{B} \), given \( X = x \) can be expressed as

\[
E[\bar{B}|X = x] \approx \begin{cases} 
P \frac{1}{2} \lambda \theta x^2, & \text{for } x \leq \tau \\
P \frac{1}{2} \lambda \theta \tau^2, & \text{for } x > \tau. 
\end{cases}
\] (13)

Since we assume the probability density function of \( X \) is given as \( f(x) = \mu e^{-\mu x}, \) for \( x > 0, \) we obtain, after some algebra, the following expression for the expected inventory carried per cycle:

\[
E[\bar{I}] = E_X[E[\bar{I}|X = x]] = \int_0^\tau \frac{P^2}{2D} \lambda x^2 \mu e^{-\mu x} \, dx + \int_\tau^\infty \frac{P^2}{2D} \lambda \tau^2 \mu e^{-\mu x} \, dx = \frac{P^2}{D} \lambda \left[ \frac{1}{\mu^2} (1 - e^{-\mu \tau}) - \frac{1}{\mu} \tau e^{-\mu \tau} \right].
\] (14)

The expected number of deteriorating units is given by

\[
E[\bar{B}] = E_X[E[\bar{B}|X = x]] = \int_0^\tau \frac{1}{2} P \lambda \theta x^2 (\mu e^{-\mu x}) \, dx + \int_\tau^\infty \frac{1}{2} P \lambda \theta \tau^2 (\mu e^{-\mu x}) \, dx = P \lambda \theta \left[ \frac{1}{\mu^2} (1 - e^{-\mu \tau}) - \frac{1}{\mu} \tau e^{-\mu \tau} \right].
\] (15)

The lost sales occur only when the inventory build-up at the moment of a breakdown is not enough to meet the demand required during the period for carrying out a corrective maintenance activity. Hence, the expected number of lost sales is given by

\[
L = D \int_0^u (R - T_2) \mu e^{-\mu x} \, dx = D \int_0^u \left[ R - \frac{P - D}{D} \left( x - \frac{1}{2} \theta x^2 \right) \right] \mu e^{-\mu x} \, dx = \left[ DR - P \lambda \frac{1}{\mu} \left( 1 - \frac{\theta}{\mu} \right) \right] (1 - e^{-\mu u}) + P \lambda \left( 1 - \frac{1}{2} \theta u - \frac{\theta}{\mu} \right) u e^{-\mu u}.
\] (16)

Note that the function for \( L \) is independent of \( \tau. \) Next, we use (5) to obtain

\[
R = \frac{1}{\theta} \ln \left[ \frac{P}{P - D} - \frac{D}{P - D} e^{\theta R} \right] \quad \text{or} \quad u = \frac{1}{\theta} \ln \left[ \frac{P}{P - D} - \frac{D}{P - D} \right].
\]

Since \( u \) must be nonnegative, it follows that

\[
R \leq \frac{1}{\theta} \ln \left( \frac{P - D}{D} \right).
\]

Summing the setup cost, corrective maintenance cost, inventory carrying cost, deteriorating cost, and the cost of lost sales, the expected total cost per cycle can be obtained as:
To simplify the expression of $g(\tau)$, we define two functions $f_1(\tau)$ and $f_2(\tau)$. The cycle length may be greater than $x$, as the unique nonnegative solution of the following equation:

$$Z = S + M (1 - e^{-\mu x}) + \delta L + h E[\hat{I}] + \pi E[\hat{B}].$$

The expected duration of a cycle is obtained as

$$h(\tau) = \int_0^x (x + R) \mu e^{-\mu x} \, dx + \int_0^{\tau} P(D - \frac{1}{2} P) e^{-\mu x} \, dx$$

$$+ \int_0^{\infty} P(D - \frac{1}{2} P) e^{-\mu x} \, dx$$

$$= (R + \frac{1}{\mu}) (1 - e^{-\mu u}) - ue^{-\mu u} + \frac{P}{D} \left( u + \frac{1}{\mu} - \frac{1}{2} \theta \lambda u^2 - \frac{1}{\mu} \theta \lambda u - \frac{1}{\mu^2} \theta \lambda \right) e^{-\mu u}$$

$$+ \frac{P}{D} \left( \frac{1}{\mu^2} \theta \lambda - \frac{1}{\mu} + \frac{1}{\mu} \theta \lambda \tau \right) e^{-\mu \tau}.$$  \hspace{1cm} (17)

Note that in (17), if $x > u$, some inventory may be left after the corrective maintenance is completed. As a result, the cycle length may be greater than $x + R$. In such a case, the expected cycle length is computed at the second integration. In the next property, we establish the convexity of the function $Z(\tau)$.

**Property 1.** The function $Z(\tau)$ is convex for $\tau > 0$.

**Proof.** To simplify the expression of $h(\tau)$, we let

$$A = \left( R + \frac{1}{\mu} \right) (1 - e^{-\mu u}) - ue^{-\mu u} + \frac{P}{D} \left( u + \frac{1}{\mu} - \frac{1}{2} \theta \lambda u^2 - \frac{1}{\mu} \theta \lambda u - \frac{1}{\mu^2} \theta \lambda \right) e^{-\mu u}.$$

This implies that

$$h(\tau) = A + \frac{P}{D} \left( \frac{1}{\mu^2} \theta \lambda - \frac{1}{\mu} + \frac{1}{\mu} \theta \lambda \tau \right) e^{-\mu \tau}.$$  \hspace{1cm} (18)

The first derivative of $Z(\tau)$ with respect to $\tau$ is given as

$$\frac{dZ}{d\tau} = \frac{Pe^{-\mu \tau}}{D[h(\tau)]^2} \left[ (S + \delta L) (-1 + \theta \lambda \tau) + M \left( \frac{AD}{P} \mu + \frac{\theta \lambda}{\mu} e^{-\mu \tau} - 1 + \theta \lambda \tau \right) \right.$$

$$\left. + \left( \frac{hP}{D} + \pi \theta \right) \lambda \left( AD \tau - \frac{P}{\mu^2} + \frac{P}{\mu^2} e^{-\mu \tau} + \frac{P}{\mu^2} \theta \lambda \tau \right) \right].$$

We can obtain, by setting $dZ/d\tau = 0$ and simplifying the resulting equation, that the optimal production run time $\tau^*$ as the unique nonnegative solution of the following equation:

$$(S + \delta L) \theta \lambda \tau + M \theta \lambda \left( \frac{1}{\mu} e^{-\mu \tau} + \tau \right) + \left( \frac{hP}{D} + \pi \theta \right) \lambda \left( AD \tau + \frac{P}{\mu^2} e^{-\mu \tau} + \frac{P \theta \lambda}{\mu^2} \tau \right)$$

$$= (S + \delta L) + M \left( 1 - \frac{AD}{P} \mu \right) + \left( \frac{hP}{D} + \pi \theta \right) \lambda \left( \frac{P}{\mu^2} \right).$$  \hspace{1cm} (18)

We define two functions

$$f_1(\tau) = \frac{1}{\mu} e^{-\mu \tau} + \tau, \quad \text{and} \quad f_2(\tau) = AD \tau + \frac{P}{\mu^2} e^{-\mu \tau} + \frac{P \theta \lambda}{\mu^2} \tau.$$
It is easy to obtain the following derivatives
\[
\frac{df_1(\tau)}{d\tau} = 1 - e^{-\mu \tau}, \quad \text{and} \\
\frac{df_2(\tau)}{d\tau} = AD - \frac{P}{\mu} e^{-\mu \tau} + \frac{P}{\mu^2} \theta \lambda = D \left( R + \frac{1}{\mu} \right) (1 - e^{-\mu \tau}) + P \lambda \mu e^{-\mu \tau} + \frac{P}{\mu} \left( 1 - e^{-\mu (\tau - u)} \right) e^{-\mu \tau} \\
+ P \theta \lambda e^{-\mu \tau} \left[ \frac{1}{\mu^2} (e^{\mu \tau} - 1) - \frac{1}{2} u^2 - \frac{1}{\mu} u \right].
\]

It is obvious that \(d^2 f_1(\tau)/d\tau > 0\) for \(\tau > 0\) since \(e^{\mu \tau} \geq 1 + \mu \tau + (\mu \tau)^2/2\) it is not difficult to see that \(d^2 f_2(\tau)/d\tau > 0\) for \(\tau > 0\). Hence, if we view the left-hand side of (18) as a function of \(\tau\), it is not only an increasing function of \(\tau > 0\) but also assumes all values of \(\tau > 0\). Since \(Z(\tau)\) is a continuously differentiable function of \(\tau\) with a derivative which changes sign only once at \(\tau^*\) from negative to positive, it follows that \(Z(\tau)\) assumes its global minimum at the point \(\tau^*\).

A closed-form solution is not readily available from (18), but numerical search approaches including Newton’s method or the bisection method can be developed to find a root of Eq. (18). Since the development of these methods are fairly straightforward, we omit the detailed discussion here. In fact, a root of Eq. (18) can be found by using the Goal Seek option in Microsoft Excel.

Using a Taylor series approximation to the exponential terms in (18), we obtain
\[
\frac{1}{2} \left( M \mu \theta + \frac{h P^2}{D} + \pi \theta \right) \lambda \tau^2 + \left[ (S + \delta L) \theta + \left( \frac{h P}{D} + \pi \theta \right) \left( AD - \frac{P}{\mu} + \frac{P}{\mu^2} \theta \lambda \right) \right] \lambda \tau - \left[ S + \delta L + M \frac{\mu P}{P} \left( AD - \frac{P}{\mu} + \frac{P}{\mu^2} \theta \lambda \right) \right] = 0.
\]

Note that for \(\tau > 0\), we obtain
\[
AD - \frac{P}{\mu} + \frac{P}{\mu^2} \theta \lambda = D \left( R + \frac{1}{\mu} \right) (1 - e^{-\mu \tau}) + P \lambda \mu e^{-\mu \tau} + P \theta \lambda e^{-\mu \tau} \left[ \frac{1}{\mu^2} (e^{\mu \tau} - 1) - \frac{1}{2} u^2 - \frac{1}{\mu} u \right] \geq 0.
\]

Hence, a positive root of (19) can be obtained as follows:
\[
\tau^*_{\text{app}} = \frac{-\left[ (S + \delta L) \theta + \Omega \Pi \right] M \mu \theta + \Pi \Omega}{M \mu \theta + P \Omega} + \left\{ \left[ (S + \delta L) \theta + \Omega \Pi \right]^2 + 2 \frac{M \mu \theta + P \Pi}{\lambda} \left[ S + \delta L + M \frac{\mu}{P} \Pi \right] \right\}^{1/2},
\]

where \(\Omega = \frac{h P}{D} + \pi \theta\), and \(\Pi = AD - \frac{P}{\mu} + \frac{P}{\mu^2} \theta \lambda\).

Also note that when both \(\theta \to 0\) and \(\mu \to 0\), (20) reduces to the production run time given in the conventional EPQ model.

5. Numerical examples

In this section we provide some numerical results to illustrate the model developed in this paper by considering various parameter sets for a problem. The input data sets are given in Table 1.

Each cost component as a function of the production uptime is given in Fig. 2. It shows that the restoration (corrective maintenance) cost is independent of the production uptime. The inventory holding cost and the defective cost are increasing functions of the production uptime. However, the defective cost becomes flat with increasing production uptime. This is due to the fact that the actual number of defective units is dictated by the occurrence of the process failure. Both the setup cost and the cost of lost sales are decreasing functions of the production uptime. They approach a constant when the production uptime increases.

The graphs of the expected total cost as a function of the production uptime for \(\mu = 0.01, 1, 10, \theta = 0.5, 5\), and \(R = 0.05\) are given in Figs. 3 and 4. The curves become flatter as \(\mu\) increases because, in the case of a large \(\mu\) value, the cost of machine failure becomes a dominant factor. As the value of \(\mu\) increases, the curves move up. This is because the total cost is raised mainly due to a large increase in the maintenance cost that is a direct result of the rapid increase of the rate of deterioration. For a large \(\theta\) value, the optimal \(\tau^*\) shifts slightly to the left. This implies
Table 1
Input data

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Fig. 2</th>
<th>Figs. 3 and 4</th>
<th>Fig. 5</th>
<th>Table 2</th>
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Fig. 2. Each cost component as a function of production uptime.

that in order to reduce deterioration cost, the production uptimes should be reduced. Also, with a large θ value, the deterioration cost becomes significant with a longer production uptime and a smaller µ value. The implication is that with a long production uptime the machine will produce more deteriorating items, and a small value implies that it is less likely for a breakdown to occur before the end of a production run.

In Fig. 5, we show the expected total cost as a function of the production uptime for µ = 0.01, 1, 10, θ = 0.5, and R = 0.2. This is a case with a long period of time for a corrective maintenance activity. The gap between the expected total cost functions of a large µ value and a small µ value is widened. This is due to the fact that with a long repair time, the factor of lost sales is becoming a significant factor.

Let τ_{app} and τ_{opt} denote the approximate solutions obtained by applying (20) and solutions obtained using the Goal Seek option in Excel to (18), respectively, for a pair of values of µ and θ. A relative error for a given pair of µ and θ
Fig. 4. The expected total cost with $\theta = 5$ and $R = 0.05$.

Fig. 5. The expected total cost with $\theta = 0.5$ and $R = 0.2$.

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is defined as follows:

$$
\varepsilon_1 = \frac{|\tau_{\text{app}} - \tau_{\text{opt}}|}{\tau_{\text{opt}}}.
$$

The relative errors for various sets of $\mu$ and $\theta$ are provided in Table 2. It is not difficult to see from Table 2 that when $\mu$ is small, approximation solutions obtained using Eq. (20) work well regardless of the magnitude of $\theta$, and vice versa. On the other hand, a large value of $\mu$ implies that the machine becomes less reliable (i.e., it is more likely
for a machine breakdown to occur during a production run). In this case, solutions obtained using (20) perform poorly. However, when both $\mu$ and $\theta$ become larger, (20) performs relatively well.

6. Conclusions

An EPQ model has been developed in this paper for a deteriorating product to be manufactured on a machine that is subject to random breakdowns and a fixed corrective maintenance period. The objective is to determine an optimal production uptime so as to minimize the expected total costs per unit time consisting of setup, corrective maintenance, inventory carrying, deterioration, and lost sales costs. The time to deterioration is assumed to be exponentially distributed. This implies that the rate of deterioration is constant. Hence, it is possible to obtain a relatively simple expression for a near optimal solution. Numerical examples have been solved to show the impact of random machine breakdowns and a constant deterioration rate. The new EPQ model effectively balances all cost components mentioned above. Future research directions include models with effects such as price-dependent demand and other distributions for deterioration and time-to-breakdown.

References


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