JOURNAL OF FORMALIZED MATHEMATICS
Volume 11, Released 1999, Published 2003
Inst. of Computer Science, Univ. of Białystok

A Characterization of Concept Lattices. Dual Concept Lattices

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Summary. In this article we continue the formalization of concept lattices following [6]. We give necessary and sufficient conditions for a complete lattice to be isomorphic to a given formal context. As a by-product we get that a lattice is complete if and only if it is isomorphic to a concept lattice. In addition we introduce dual formal concepts and dual concept lattices and prove that the dual of a concept lattice over a formal context is isomorphic to the concept lattice over the dual formal context.

MML Identifier: CONLAT_2.

WWW: http://mizar.org/JFM/Vol11/conlat_2.html

The articles [13], [5], [17], [8], [14], [2], [12], [18], [9], [16], [15], [1], [11], [4], [3], [19], [7], and [10] provide the notation and terminology for this paper.

1. Preliminaries

Let C be a FormalContext and let C_1 be a strict FormalConcept of C. The functor $^@C_1$ yielding an element of ConceptLattice$C$ is defined by:

(Def. 1) $^@C_1 = C_1$.

Let C be a FormalContext. Observe that ConceptLattice$C$ is bounded.

We now state four propositions:

(1) For every FormalContext C holds $\bot_{\mathrm{ConceptLattice}C} = \mathrm{Concept}-\mathrm{with}-\mathrm{all}-\mathrm{Attributes}C$ and $\top_{\mathrm{ConceptLattice}C} = \mathrm{Concept}-\mathrm{with}-\mathrm{all}-\mathrm{Objects}C$.

(2) Let C be a FormalContext and D be a non empty subset of $2^{\text{the objects of } C}$. Then $(\mathrm{ObjectDerivation}C)(\bigcup D) = \bigcap\{(\mathrm{ObjectDerivation}C)(O); O$ ranges over subsets of the objects of C: $O \in D\}$.

(3) Let C be a FormalContext and D be a non empty subset of $2^{\text{the attributes of } C}$. Then $(\mathrm{AttributeDerivation}C)(\bigcup D) = \bigcap\{(\mathrm{AttributeDerivation}C)(A); A$ ranges over subsets of the attributes of C: $A \in D\}$.

(4) Let C be a FormalContext and D be a subset of ConceptLattice$C$. Then $\bigsqcap_{\mathrm{ConceptLattice}C} D$ is a FormalConcept of C and $\bigsqcup_{\mathrm{ConceptLattice}C} D$ is a FormalConcept of C.

Let C be a FormalContext and let D be a subset of ConceptLattice$C$. The functor $\bigsqcap_C D$ yielding a FormalConcept of C is defined as follows:

(Def. 2) $\bigsqcap_C D = \bigsqcap_{\mathrm{ConceptLattice}C} D$.

The functor $\bigsqcup_C D$ yielding a FormalConcept of C is defined as follows:

(Def. 3) $\bigsqcup_C D = \bigsqcup_{\mathrm{ConceptLattice}\,C} D$.

One can prove the following propositions:

(5) For every FormalContext C holds $\bigsqcup_C(\emptyset_{\mathrm{ConceptLattice}\,C}) = \mathrm{Concept}-\mathrm{with}-\mathrm{all}-\mathrm{Attributes}\,C$ and $\bigsqcap_C(\emptyset_{\mathrm{ConceptLattice}\,C}) = \mathrm{Concept}-\mathrm{with}-\mathrm{all}-\mathrm{Objects}\,C$.

(6) For every FormalContext C holds $\bigsqcup_C(\Omega_{\text{the carrier of ConceptLattice}\,C}) = \mathrm{Concept}-\mathrm{with}-\mathrm{all}-\mathrm{Objects}\,C$ and $\bigsqcap_C(\Omega_{\text{the carrier of ConceptLattice}\,C}) = \mathrm{Concept}-\mathrm{with}-\mathrm{all}-\mathrm{Attributes}\,C$.

(7) Let C be a FormalContext and D be a non empty subset of ConceptLattice$C$. Then

(i) the extent of $\bigsqcup_C D = (\mathrm{AttributeDerivation}\,C)((\mathrm{ObjectDerivation}\,C)(\bigcup\{$the extent of $\langle E,I\rangle$; E ranges over subsets of the objects of C, I ranges over subsets of the attributes of C: $\langle E,I\rangle \in D\}))$, and

(ii) the intent of $\bigsqcup_C D = \bigcap\{$the intent of $\langle E,I\rangle$; E ranges over subsets of the objects of C, I ranges over subsets of the attributes of C: $\langle E,I\rangle \in D\}$.

(8) Let C be a FormalContext and D be a non empty subset of ConceptLattice$C$. Then

(i) the extent of $\bigsqcap_C D = \bigcap\{$the extent of $\langle E,I\rangle$; E ranges over subsets of the objects of C, I ranges over subsets of the attributes of C: $\langle E,I\rangle \in D\}$, and

(ii) the intent of $\bigsqcap_C D = (\mathrm{ObjectDerivation}\,C)((\mathrm{AttributeDerivation}\,C)(\bigcup\{$the intent of $\langle E,I\rangle$; E ranges over subsets of the objects of C, I ranges over subsets of the attributes of C: $\langle E,I\rangle \in D\}))$.

(9) Let C be a FormalContext and C_1 be a strict FormalConcept of C. Then $\bigsqcup_{\mathrm{ConceptLattice}\,C}\{\langle O,A\rangle; O$ ranges over subsets of the objects of C, A ranges over subsets of the attributes of C: $\bigvee_{o:\text{object of }C}$ $(o \in$ the extent of C_1 $\wedge$ $O = (\mathrm{AttributeDerivation}\,C)((\mathrm{ObjectDerivation}\,C)(\{o\}))$ $\wedge$ $A = (\mathrm{ObjectDerivation}\,C)(\{o\}))\} = C_1$.

(10) Let C be a FormalContext and C_1 be a strict FormalConcept of C. Then $\bigsqcap_{\mathrm{ConceptLattice}\,C}\{\langle O,A\rangle; O$ ranges over subsets of the objects of C, A ranges over subsets of the attributes of C: $\bigvee_{a:\text{Attribute of }C}$ $(a \in$ the intent of C_1 $\wedge$ $O = (\mathrm{AttributeDerivation}\,C)(\{a\})$ $\wedge$ $A = (\mathrm{ObjectDerivation}\,C)((\mathrm{AttributeDerivation}\,C)(\{a\})))\} = C_1$.

Let C be a FormalContext. The functor $\gamma(C)$ yielding a function from the objects of C into the carrier of ConceptLattice$C$ is defined by the condition (Def. 4).

(Def. 4) Let o be an element of the objects of C. Then there exists a subset O of the objects of C and there exists a subset A of the attributes of C such that $(\gamma(C))(o) = \langle O,A\rangle$ and $O = (\mathrm{AttributeDerivation}\,C)((\mathrm{ObjectDerivation}\,C)(\{o\}))$ and $A = (\mathrm{ObjectDerivation}\,C)(\{o\})$.

Let C be a FormalContext. The functor δ_C yields a function from the attributes of C into the carrier of ConceptLattice$C$ and is defined by the condition (Def. 5).

(Def. 5) Let a be an element of the attributes of C. Then there exists a subset O of the objects of C and there exists a subset A of the attributes of C such that $\delta_C(a) = \langle O,A\rangle$ and $O = (\mathrm{AttributeDerivation}\,C)(\{a\})$ and $A = (\mathrm{ObjectDerivation}\,C)((\mathrm{AttributeDerivation}\,C)(\{a\}))$.

The following propositions are true:

(11) Let C be a FormalContext, o be an object of C, and a be an Attribute of C. Then $(\gamma(C))(o)$ is a FormalConcept of C and $\delta_C(a)$ is a FormalConcept of C.

(12) For every FormalContext C holds $\mathrm{rng}\,\gamma(C)$ is supremum-dense and $\mathrm{rng}(\delta_C)$ is infimum-dense.

(13) Let C be a FormalContext, o be an object of C, and a be an Attribute of C. Then o is connected with a if and only if $(\gamma(C))(o) \sqsubseteq \delta_C(a)$.

2. THE CHARACTERIZATION

Next we state the proposition

(14) Let L be a complete lattice and C be a FormalContext. Then ConceptLattice$C$ and L are isomorphic if and only if there exists a function g from the objects of C into the carrier of L and there exists a function d from the attributes of C into the carrier of L such that rng$g$ is supremum-dense and rng$d$ is infimum-dense and for every object o of C and for every Attribute a of C holds o is connected with a iff $g(o) \sqsubseteq d(a)$.

Let L be a lattice. The functor Context$L$ yielding a strict non quasi-empty ContextStr is defined by:

(Def. 6) Context$L = \langle$the carrier of L, the carrier of L, LattRel$(L)\rangle$.

The following two propositions are true:

(15) For every complete lattice L holds ConceptLattice Context$L$ and L are isomorphic.

(16) For every lattice L holds L is complete iff there exists a FormalContext C such that ConceptLattice$C$ and L are isomorphic.

3. DUAL CONCEPT LATTICES

Let L be a complete lattice. Observe that L° is complete.

Let C be a FormalContext. The functor C° yields a strict non quasi-empty ContextStr and is defined as follows:

(Def. 7) $C^{\circ} = \langle$the attributes of C, the objects of C, (the information of $C)^{\smile}\rangle$.

The following propositions are true:

(17) For every strict FormalContext C holds $(C^{\circ})^{\circ} = C$.

(18) For every FormalContext C and for every subset O of the objects of C holds $(\text{ObjectDerivation}C)(O) = (\text{AttributeDerivation}C^{\circ})(O)$.

(19) For every FormalContext C and for every subset A of the attributes of C holds $(\text{AttributeDerivation}C)(A) = (\text{ObjectDerivation}C^{\circ})(A)$.

Let C be a FormalContext and let C_1 be a ConceptStr over C. The functor ${C_1}^{\circ}$ yields a strict ConceptStr over C° and is defined by:

(Def. 8) The extent of ${C_1}^{\circ} =$ the intent of C_1 and the intent of ${C_1}^{\circ} =$ the extent of C_1.

Let C be a FormalContext and let C_1 be a FormalConcept of C. Then ${C_1}^{\circ}$ is a strict FormalConcept of C°.

We now state the proposition

(20) For every FormalContext C and for every strict FormalConcept C_1 of C holds $({C_1}^{\circ})^{\circ} = C_1$.

Let C be a FormalContext. The functor DualHomomorphism$C$ yielding a homomorphism from $(\text{ConceptLattice}C)^{\circ}$ to ConceptLatticeC° is defined by:

(Def. 9) For every strict FormalConcept C_1 of C holds $(\text{DualHomomorphism}C)(C_1) = {C_1}^{\circ}$.

We now state two propositions:

(21) For every FormalContext C holds DualHomomorphism$C$ is isomorphism.

(22) For every FormalContext C holds ConceptLatticeC° and $(\text{ConceptLattice}C)^{\circ}$ are isomorphic.

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Received August 17, 1999

Published January 2, 2004