$L_\infty$-Norm Computation for Continuous-Time Descriptor Systems Using Structured Matrix Pencils

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Abstract

In this paper we discuss an algorithm for the computation of the $L_\infty$-norm of transfer functions related to descriptor systems. We show how one can achieve this goal by computing the eigenvalues of certain skew-Hamiltonian/Hamiltonian matrix pencils and analyse arising problems. We also formulate and prove a theoretical result which serves as a basis for testing a transfer function matrix for properness. Finally we verify our results using a descriptor system related to mechanical engineering.

Index Terms

$H_\infty$ control, transfer function matrices, singular systems.

I. INTRODUCTION

In many applications from industry and technology computer simulations are performed using models which can be formulated by systems of differential equations. Often the equations underlie additional algebraic constraints which prevent the system from attaining every possible state. In this context we speak of descriptor systems (or singular systems). These systems...
naturally arise in a large variety of applications such as electrical circuit simulation, multibody dynamics with constraints or the semidiscretization of certain partial differential equations (see [1] and references therein). Very important characteristic values of such systems are the $L_\infty$-norms of the corresponding transfer functions. These norms have found important applications in robust control or model order reduction [1], [2], [3].

Consider a continuous-time linear time-invariant descriptor system

\[ \begin{align*}
E \dot{x}(t) &= Ax(t) + Bu(t), \\
y(t) &= Cx(t) + Du(t),
\end{align*} \]

(1)

with $E, A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{p \times n}$, $D \in \mathbb{R}^{p \times m}$, descriptor vector $x(t) \in \mathbb{R}^n$, control vector $u(t) \in \mathbb{R}^m$, and output vector $y(t) \in \mathbb{R}^p$. Here, $E$ usually is a singular matrix. By taking the Laplace transform [4] of both equations in (1) and inserting the first equation into the other one we obtain the matrix-valued transfer function of the system

\[ G(s) := C (sE - A)^{-1} B + D, \quad s \in \mathbb{C}, \]

(2)

which directly maps inputs to outputs in the frequency domain. For transfer functions with no poles on the imaginary axis, i.e., for those where the matrix pencil $\lambda E - A$ has no purely imaginary eigenvalues, we define the $L_\infty$-norm [2] by

\[ \|G\|_{L_\infty} := \sup_{\omega \in \mathbb{R}} \sigma_{\text{max}}(G(i\omega)), \]

where $\sigma_{\text{max}}$ denotes the maximum singular value. Note that even if (2) has no poles on the imaginary axis, the $L_\infty$-norm may be unbounded at infinity and hence does not exist. This motivates the following definition. We call a transfer function $G$ proper if \[ \lim_{\omega \to \infty} \|G(i\omega)\| < \infty \]

and strictly proper if \[ \lim_{\omega \to \infty} \|G(i\omega)\| = 0 \]

for any induced matrix norm $\| \cdot \|$. Otherwise we call it improper [5].

The remainder of this article is structured as follows. In Section II we present a new method for computing the $L_\infty$-norm for proper transfer functions related to descriptor systems. Algorithms for the computation of the $L_\infty$-norm of standard state-space systems rely on the relationship of the singular values of a transfer function and the imaginary eigenvalues of a specific Hamiltonian matrix, First, Byers developed a bisection method [6] which converges linearly to the norm. A few years later, several other authors improved this algorithm [7], [8] to obtain a quadratic rate of convergence. In this article we generalize this algorithm to descriptor systems which
leads to skew-Hamiltonian/Hamiltonian eigenvalue problems. We give a particular emphasis on computing the eigenvalues of the involved matrix pencils in a structure-preserving manner to increase reliability and accuracy of the method. In Section III we state and prove a theorem which serves as a basis for an algorithm which may check if a transfer function is proper. We verify our theoretical results using an example from mechanical engineering in Section IV. Finally, in Section V we give a short conclusion and state different possible directions of future research.

II. COMPUTATION OF THE $\mathcal{L}_\infty$-NORM

In this section we derive an algorithm for the computation of the $\mathcal{L}_\infty$-norm for proper transfer functions related to descriptor systems of the form (1). In the sequel we assume that the matrix pencil $\lambda E - A$ is regular, i.e., $\det(\lambda E - A)$ is not identical to the zero polynomial.

A. Preliminaries

The computation of the $\mathcal{L}_\infty$-norm is connected to the computation of the eigenvalues of specific skew-Hamiltonian/Hamiltonian matrix pencils \cite{9}, \cite{10} $\lambda N - M_\gamma$ with

$$ N = \begin{bmatrix} E & 0 \\ 0 & E^T \end{bmatrix}, \quad M_\gamma = \begin{bmatrix} A & 0 \\ 0 & -A^T \end{bmatrix} + \begin{bmatrix} B & 0 \\ 0 & -C^T \end{bmatrix} \begin{bmatrix} -D & \gamma I \\ \gamma I & -D^T \end{bmatrix}^{-1} \begin{bmatrix} C & 0 \\ 0 & B^T \end{bmatrix}. $$

(3)

The matrix $M_\gamma$ can also be expressed as

$$ M_\gamma = \begin{bmatrix} A - BR^{-1}D^T C & -\gamma BR^{-1}B^T \\ \gamma C^T S^{-1}C & -A^T + C^T DR^{-1}B^T \end{bmatrix} $$

with the matrices $R = D^T D - \gamma^2 I$, and $S = DD^T - \gamma^2 I$. Using this we can formulate a theorem which connects the singular values of a transfer function matrix with the eigenvalues of the associated matrix pencil (3) as it has been done for standard systems in \cite{11}.

Theorem 1: Assume the matrix pencil $\lambda E - A$ is regular and has no finite eigenvalues on the imaginary axis, $\gamma > 0$ is not a singular value of $D$ and $\omega_0 \in \mathbb{R}$. Then, $\gamma$ is a singular value of $G(i\omega_0)$ if and only if $i\omega_0 N - M_\gamma$ is singular.

Proof: The argumentation follows the one of the proof of Theorem 1 in \cite{11}. Let $\gamma$ be a singular value of $G(i\omega_0)$. Then there exist nonzero vectors $u \in \mathbb{C}^m$, $v \in \mathbb{C}^p$ such that

$$ G(i\omega_0)u = \gamma v, \quad G(i\omega_0)^H v = \gamma u. $$
Thus,

\[(C(i\omega_0 E - A)^{-1} B + D)u = \gamma v, \quad (B^T (-i\omega_0 E^T - A^T)^{-1} C^T + D^T)v = \gamma u.\]  \hfill (4)

Define

\[r = (i\omega_0 E - A)^{-1} Bu, \quad s = (-i\omega_0 E^T - A^T)^{-1} C^T v.\]  \hfill (5)

Now solving for \(u\) and \(v\) in terms of \(r\) and \(s\) yields

\[
\begin{bmatrix}
u \\ v
\end{bmatrix} = \begin{bmatrix}
-D & \gamma I \\
\gamma I & -D^T
\end{bmatrix}^{-1} \begin{bmatrix}
C & 0 \\ 0 & B^T
\end{bmatrix} \begin{bmatrix}
r \\ s
\end{bmatrix}.
\]  \hfill (6)

Note, that (6) guarantees that

\[
\begin{bmatrix}
r \\ s
\end{bmatrix} \neq \begin{bmatrix}
0 \\ 0
\end{bmatrix}.
\]

From (5) we get

\[(i\omega_0 E - A)r = Bu, \quad (-i\omega_0 E^T - A^T)s = C^Tv.\]  \hfill (7)

From (7) we obtain

\[
\begin{pmatrix}
i\omega_0 & 0 \\ 0 & i\omega_0 E^T
\end{pmatrix} - \begin{pmatrix}
A & 0 \\ 0 & -A^T
\end{pmatrix} \begin{bmatrix}
r \\ s
\end{bmatrix} = \begin{bmatrix}
B & 0 \\ 0 & -C^T
\end{bmatrix} \begin{bmatrix}
u \\ v
\end{bmatrix},
\]

which, with (6) is equivalent to

\[
\begin{pmatrix}
A & 0 \\ 0 & -A^T
\end{pmatrix} + \begin{bmatrix}
B & 0 \\ 0 & -C^T
\end{bmatrix} \begin{bmatrix}
-D & \gamma I \\ \gamma I & -D^T
\end{bmatrix}^{-1} \begin{bmatrix}
C & 0 \\ 0 & B^T
\end{bmatrix} \begin{bmatrix}
r \\ s
\end{bmatrix} = i\omega_0 \begin{bmatrix}
E & 0 \\ 0 & E^T
\end{bmatrix} \begin{bmatrix}
r \\ s
\end{bmatrix}.
\]  \hfill (8)

Thus

\[
M_{\gamma} \begin{bmatrix}
r \\ s
\end{bmatrix} = i\omega_0 N \begin{bmatrix}
r \\ s
\end{bmatrix}.
\]

This proves one direction of Theorem 1.

Now we prove the converse direction. Suppose that the matrix pencil \(\lambda N - M_{\gamma}\) has the eigenvalue \(i\omega_0\), that is, (8) holds for some \(\begin{bmatrix}
r \\ s
\end{bmatrix} \neq \begin{bmatrix}
0 \\ 0
\end{bmatrix}\). Defining \(u\) and \(v\) by equation (6), clearly yields \(\begin{bmatrix}
u \\ v
\end{bmatrix} \neq 0\). (Otherwise \(\begin{bmatrix}
r \\ s
\end{bmatrix}\) would be zero, following from (5).) Then from (6) and (8), we conclude (4), which establishes that \(\gamma\) is a singular value of \(G(i\omega_0)\).
Next we state and prove a modified version of Theorem 2 from [11].

**Theorem 2:** Let $\gamma > \min_{\omega \in \mathbb{R}} \sigma_{\max}(G(i\omega))$ be not a singular value of $D$. Then $\|G\|_{L_\infty} \geq \gamma$ if and only if $\lambda N - M_\gamma$ has imaginary eigenvalues (i.e., at least one).

**Proof:** Assume first $\|G\|_{L_\infty} \geq \gamma$. From the definition of the $L_\infty$-norm and the continuity of $\sigma_{\max}(G(i\omega))$ it follows that there exists $\omega_0 \in \mathbb{R}$ such that $\sigma_{\max}(G(i\omega_0)) = \gamma$. Together with Theorem 1 we obtain that $i\omega_0 N - M_\gamma$ is singular, so the matrix pencil $\lambda N - M_\gamma$ has at least one purely imaginary eigenvalue.

If we assume on the other hand that $\lambda N - M_\gamma$ has purely imaginary eigenvalues, e.g., $i\omega_0$, Theorem 1 yields that $\gamma$ is a singular value of $(G(i\omega_0))$, hence $\|G\|_{L_\infty} \geq \gamma$.

**B. The Algorithm**

Using the two theorems from above we are able to state an iterative method which iterates over $\gamma$ and checks in each step if the matrix pencil $\lambda N - M_\gamma$ has purely imaginary eigenvalues. In Algorithm 1 we illustrate the generalization of the algorithm proposed in [7] to descriptor systems. A graphical interpretation of the method can also be found in [7]. As the algorithm for standard state-space systems, our generalization is still monotonically and quadratically converging and the relative error of the computed $L_\infty$-norm is at most $\varepsilon$, see [7], [8] for details.

As we have to decide if the matrix pencils $\lambda N - M_\gamma$ have purely imaginary eigenvalues, a special emphasis should to given to the accuracy of the eigenvalue computation. This is necessary to ensure reliable results as inaccuracy in the eigenvalues could force the algorithm to produce wrong results. Therefore we apply a new structure-exploiting and -preserving approach to compute the eigenvalues of the arising skew-Hamiltonian/Hamiltonian matrix pencils as described in [9], [10]. By applying this method, simple imaginary eigenvalues do not experience any error in their real parts. This follows from the fact that the spectrum of every skew-Hamiltonian/Hamiltonian matrix pencil is symmetric with respect to the imaginary axis. That is, eigenvalues occur in pairs $(\lambda, -\lambda)$ if they are real, or in quadruples $(\lambda, -\lambda, \bar{\lambda}, -\bar{\lambda})$ if they are complex. Furthermore, the new eigenvalue solver only allows structured perturbations, i.e., skew-Hamiltonian/Hamiltonian ones. So if we perturb a simple imaginary eigenvalue $\lambda$ in its real part there does not exist its symmetric counterpart $-\bar{\lambda}$, in other words, only an error in the imaginary part is possible. In Fig. 1 we show the purely imaginary eigenvalues computed by the new structure-preserving method compared with the output of the QZ algorithm [12] for an
example matrix pencil which confirms the theoretical results. Note that in this example, using the QZ algorithm for computing the imaginary eigenvalues would force the $L_\infty$-norm algorithm to fail, since an error of about $2 \cdot 10^{-12}$ in the real parts is too large to be considered as zero.

C. Choice of the Initial Lower Bound

As already mentioned in Algorithm 1 it is necessary to determine an appropriate initial value $\gamma_{lb} < \|G\|_{L_\infty}$. This is important as a good choice of this value might decrease the computational costs of the algorithm drastically. First we observe that many systems attain their $L_\infty$-norm at the frequencies $\omega = 0$ or $\omega = \infty$. Furthermore there exist some heuristics to determine additional test frequencies at which we evaluate the transfer function. We propose to apply the method given in [14]. The computation of $\sigma_{\max}(G(i\omega))$ ($\omega \neq \infty$) is rather simple but in contrast to the standard system case, evaluating $\sigma_{\max}(G(\infty))$ is a more difficult task. To achieve this goal we separate finite and infinite eigenvalues in $\lambda E - A$ using the QZ algorithm with eigenvalue reordering and subsequently solve a particular generalized Sylvester equation [15], [16], [17], [18]. This leads to an additive decomposition of the transfer function (2) into a strictly proper part $G_{sp}$ and a polynomial part $P(s)$. Therefore we can drop $G_{sp}$ if we consider $\lim_{\omega \rightarrow \infty} \sigma_{\max}(G(i\omega))$. Since we assume that $G$ is proper, $P$ has to be a constant polynomial and hence

$$\sigma_{\max}(G(\infty)) = \sigma_{\max}(P(0)).$$  \hspace{1cm} (9)

We refer to [19] for more details.
D. Improving the Accuracy and Reliability of the Eigenvalue Computation

Naively computing the matrix $M_\gamma$ in (3) could be very ill-advised because it contains a lot of matrix products and inverses. The matrices $R$ and $S$ could be ill-conditioned and even if they are not, forming ”matrix-times-its-transpose” products like $BR^{-1}B^T$ suffers from the same kind of numerical instability as forming the normal equations to solve linear least square problems (see Example 5.3.2 in [12]). When explicitly computing the blocks of $M_\gamma$ this could easily corrupt the entries of the matrix by rounding errors and hence highly perturb the eigenvalues of the matrix pencil $\lambda N - M_\gamma$. In particular purely imaginary eigenvalues can be easily moved away from the imaginary axis by this kind of errors which forces our algorithm for computing the $L_\infty$-norm to produce wrong results. Therefore it is desirable to work directly on the original data without explicitly forming matrix products and inverses.

This can be achieved by extending the matrix pencil (3) to

$$
\lambda N - M_\gamma = \begin{bmatrix}
\lambda E - A & 0 & -B & 0 \\
0 & \lambda E^T + A^T & 0 & C^T \\
-C & 0 & -D & \gamma I_p \\
0 & -B^T & \gamma I_m & -D^T
\end{bmatrix},
$$

which can be shown to have the same finite eigenvalues as the original matrix pencil (3) [19]. However, we lose the skew-Hamiltonian/Hamiltonian structure by this operation which we can recover by performing the following steps. First we observe that the dimension of the matrix pencil (10) is $2n + m + p$ which is odd if and only if $m + p$ is odd. As every skew-Hamiltonian/Hamiltonian matrix pencil has an even dimension we append a zero column to both $B$ and $D$ and define $\tilde{B} := \begin{bmatrix} B & 0 \end{bmatrix}$, $\tilde{D} := \begin{bmatrix} D & 0 \end{bmatrix}$, $\tilde{m} = m + 1$ if $m + p$ is odd. This step is equivalent to the introduction of an artificial input with no influence on the system’s behavior and its $L_\infty$-norm. If $m + p$ is even we simply set $\tilde{B} := B$, $\tilde{D} := D$, $\tilde{m} := m$. By permuting and scaling the block rows and columns of (10) we obtain the even matrix pencil [20]

$$
\lambda \tilde{N} - \tilde{M}_\gamma = \begin{bmatrix}
0 & -\lambda E^T - A^T & -C^T & 0 \\
\lambda E - A & 0 & 0 & -\tilde{B} \\
-C & 0 & \gamma I_p & -\tilde{D} \\
0 & -\tilde{B}^T & -\tilde{D}^T & \gamma I_{\tilde{m}}
\end{bmatrix},
$$
Now we can exploit the symmetries of the matrix $\hat{M}_\gamma$ and repartition its blocks. We define $k = \frac{\tilde{m} + p}{2}$ and obtain

$$
n \begin{bmatrix} n & C^T \ 0 \end{bmatrix} =: n \begin{bmatrix} R_{11} & R_{12} \ R_{21} & R_{22} \end{bmatrix},
$$

and further for the lower right block

$$
p \begin{bmatrix} p & \tilde{m} \end{bmatrix} =: \begin{bmatrix} k & k \ end{bmatrix}
$$

$$
\begin{bmatrix} -\gamma I_p & \tilde{D} \ \tilde{D}^T & -\gamma I_{\tilde{m}} \end{bmatrix} =: \begin{bmatrix} S_{11} & S_{12} \ S_{12}^T & S_{22} \end{bmatrix}
$$

with $S_{11} = S_{11}^T$ and $S_{22} = S_{22}^T$. By again permuting and scaling the block rows and columns we obtain the extended skew-Hamiltonian/Hamiltonian matrix pencil

$$
\lambda \mathbf{N} - \mathbf{M} = 
\begin{bmatrix}
\lambda E - A & -R_{21} & 0 & -R_{22} \\
-R_{12}^T & -S_{12}^T & -R_{22}^T & -S_{22} \\
0 & R_{11} & \lambda E^T + A^T & R_{12} \\
R_{11}^T & S_{11} & R_{21}^T & S_{12}
\end{bmatrix}
$$

which can now be treated by the new structure-preserving approach for computing the eigenvalues.

**III. TESTING PROPERNESS OF A TRANSFER FUNCTION MATRIX**

When calculating the $L_\infty$-norm we have to ensure that the corresponding transfer function is proper. Often one knows from the modeling that the transfer function is proper. However, if one does not know about this property one can check this by the procedure following from the next theorem.

**Theorem 3:** Let $(E; A, B, C, D)$ be a descriptor system with C-controllable and C-observable fast subsystem [4] and transfer function $G$. Let furthermore

$$
UEV = \begin{bmatrix} T & 0 \\
0 & 0 \end{bmatrix}
$$

be a decomposition of the matrix $E$ by a generalized state-space transform with nonsingular matrices $U, V \in \mathbb{R}^{n \times n}$ and a full-rank matrix $T \in \mathbb{R}^{r \times r}$. If we apply the same transformations
to the matrix $A$ and partition the blocks as in (11), i.e.,

$$UAV = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

(12)

with $A_{11} \in \mathbb{R}^{r \times r}$, $A_{12} \in \mathbb{R}^{r \times (n-r)}$, $A_{21} \in \mathbb{R}^{(n-r) \times r}$, $A_{22} \in \mathbb{R}^{(n-r) \times (n-r)}$, then $G$ is proper if and only if $A_{22}$ is invertible.

**Proof:** Since $U$, $V$ realize a generalized state-space transform we first have to update $B$ and $C$. Define

$$UB = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad CV = \begin{bmatrix} C_1 & C_2 \end{bmatrix}$$

with $B_1 \in \mathbb{R}^{r \times m}$, $B_2 \in \mathbb{R}^{(n-r) \times m}$, $C_1 \in \mathbb{R}^{p \times r}$, $C_2 \in \mathbb{R}^{p \times (n-r)}$. The system $(E; A, B, C, D)$ is assumed to have a $C$-controllable fast subsystem, so it follows that

$$\text{rank} \begin{bmatrix} E & B \end{bmatrix} = \text{rank} \begin{bmatrix} T & 0 & B_1 \\ 0 & 0 & B_2 \end{bmatrix} = n$$

which means that the matrix $B_2$ must have full rank. By a similar argument

$$\text{rank} \begin{bmatrix} E \\ C \end{bmatrix} = \text{rank} \begin{bmatrix} T & 0 \\ 0 & 0 \end{bmatrix} = n$$

holds and hence $C_2$ has to be a full-rank matrix. Now we write the transfer function of our descriptor system in terms of the transformed matrices, that is

$$G(s) = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} sT - A_{11} & -A_{12} \\ -A_{21} & -A_{22} \end{bmatrix}^{-1} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} + D.$$  

Applying the Banachiewicz inversion formula for the inverse of a nonsingular partitioned matrix [21] yields the following result:

$$K(s) = \begin{bmatrix} Q(s) + Q(s)A_{12}S^{-1}(s)A_{21}Q(s) & Q(s)A_{12}S^{-1}(s) \\ S^{-1}(s)A_{21}Q(s) & S^{-1}(s) \end{bmatrix}$$

with $Q(s) = (sT - A_{11})^{-1}$ and the Schur complement $S(s) = -A_{22} - A_{21}Q(s)A_{12}$. Now we consider $\lim_{s \to \infty} K(s)$. First of all we observe that $\lim_{s \to \infty} Q(s) = 0$ because the matrix pencil $sT - A_{11}$ does not have any infinite eigenvalues.
Assume the matrix $A_{22}$ is invertible. Then \[ \lim_{s \to \infty} S^{-1}(s) = -A_{22}^{-1} \] holds and consequently
\[ \lim_{s \to \infty} K(s) = \begin{bmatrix} 0 & 0 \\ 0 & -A_{22}^{-1} \end{bmatrix} \] and thus
\[ \lim_{s \to \infty} \|G(s)\|_2 = \|D - C_2 A_{22}^{-1} B_2\|_2 < \infty \] which means that $G$ is proper.

Let now $A_{22}$ be a singular matrix. The expression $Q(s)$ can be expanded into a Laurent series at $s = \infty$ (see, e.g., [5]) which yields
\[ Q(s) = \sum_{i = -\infty}^{\infty} Q_i s^i \] for constant coefficients $Q_i$. Since \[ \lim_{s \to \infty} Q(s) = 0 \] the matrices $Q_i$ for $i \geq 0$ have to be zero. In this way also $S(s)$ can be expanded into a Laurent series at the expansion point $s = \infty$, i.e.,
\[ S(s) = \sum_{i = -\infty}^{0} S_i s^i. \]
Since $A_{22}$ is assumed to be singular, \[ \lim_{s \to \infty} \lambda_{\min}(S(s)) = 0, \] where $\lambda_{\min}$ denotes the smallest eigenvalue in magnitude. Hence \[ \lim_{s \to \infty} |\lambda_{\max}(S^{-1}(s))| = \infty \] with the largest eigenvalue in magnitude $\lambda_{\max}$. So $S^{-1}(s)$ has a Laurent series representation at the expansion point $s = \infty$,
\[ S^{-1}(s) = \sum_{i = -\infty}^{\infty} \tilde{S}_i s^i, \] with degree larger or equal than 1. Consequently, the entries of $K(s)$ at the block positions $(1, 1), (1, 2), \text{ and } (2, 1)$ have a lower degree than the entry at block position $(2, 2)$ because they contain $Q(s)$ as a factor. Since the matrices $B_2$ and $C_2$ have full rank, the product $C_2 S^{-1}(s) B_2$ has the same degree as $S^{-1}(s)$. So we can write the transfer function $G$ as
\[ G(s) = H(s) + C_2 S^{-1}(s) B_2 + D, \]
where $H(s)$ contains only terms that have lower degree than $S^{-1}$. Since for $s \to \infty$ the maximum eigenvalue of $S^{-1}$ tends to infinity in modulus, the maximum singular value of $S^{-1}$ tends to infinity, thus also \[ \lim_{s \to \infty} \sigma_{\max}(G(s)) = \infty \] which means that $G$ is improper. 

A numerical algorithm for testing a transfer function for properness consists of the following basic steps. First we remove all uncontrollable or unobservable infinite poles of the system using the method from [22]. With this procedure it is always possible to fulfill the assumptions.
of Theorem 3. Second we perform a URV decomposition [12] to transform the matrix $E$ to the compressed form in (11), followed by updating the matrices $A$, $B$, and $C$. Finally we use an RRQR decomposition [23], [24] to determine the rank of the submatrix $A_{22}$ in (12).

IV. NUMERICAL EXAMPLE

Consider a damped mass-spring system with holonomic constraints illustrated in Fig. 2 [1]. The $i$th mass $m_i$ is connected to the $(i + 1)$st mass by a spring and a damper with constants $k_i$ and $d_i$, respectively, and also to the ground by a spring and a damper with constants $\kappa_i$ and $\delta_i$, respectively. Additionally, the first mass is connected to the last one by a rigid bar and it is influenced by the control $u(t)$. The vibration of this system can be described by a second order descriptor system. By standard linearization methods we obtain a first order descriptor system of the form

$$
\dot{p}(t) = v(t),
$$

$$
M \ddot{v}(t) = -Kp(t) - Dv(t) + F^T \lambda(t) + B_2u(t),
$$

$$
0 = Fp(t),
$$

$$
y(t) = C_1p(t),
$$

with algebraic index 3, where $p \in \mathbb{R}^g$ is the position vector, $v(t) \in \mathbb{R}^g$ is the velocity vector, $\lambda(t) \in \mathbb{R}$ is the Lagrange multiplier, $M = \text{diag}(m_1, \ldots, m_g)$ is the mass matrix, $D$ and $K$ are the tridiagonal damping and stiffness matrices, respectively. For our experiments we take $g = 5$,  

Fig. 2. Constrained damped mass-spring system with $g$ masses


TABLE I

Convergence History of the \( \mathcal{L}_\infty \)-Norm Algorithm for the Constrained Mass-Spring System

<table>
<thead>
<tr>
<th>iteration ( i )</th>
<th>iterate ( \gamma_{ib}(i) )</th>
<th>rel. error</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.15756820450368172</td>
<td>0.00893922316548</td>
</tr>
<tr>
<td>2</td>
<td>0.15899264646186639</td>
<td>2.497728868261070 \cdot 10^{-5}</td>
</tr>
<tr>
<td>3</td>
<td>0.15899661773662260</td>
<td>1.865781701059065 \cdot 10^{-10}</td>
</tr>
<tr>
<td>4</td>
<td>0.15899661776628779</td>
<td>6.982683281081451 \cdot 10^{-16}</td>
</tr>
</tbody>
</table>

\( m_1 = \ldots = m_5 = 100 \) and

\[
\begin{align*}
    k_1 = \ldots &= k_4 = \kappa_2 = \ldots = \kappa_4 = 2, & \kappa_1 = \kappa_5 = 4, \\
    d_1 = \ldots &= d_4 = \delta_2 = \ldots = \delta_4 = 5, & \delta_1 = \delta_5 = 10.
\end{align*}
\]

Furthermore we assume that we can accelerate the first mass of the system, i.e., \( B_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix}^T \) and that we observe the position of the first mass, i.e., \( C_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix} \).

We obtain \( \sigma_{\max}(G(0)) = 9.722222222221683 \cdot 10^{-2} \), \( \sigma_{\max}(G(\infty)) = 1.77630918782653439 \cdot 10^{-17} \), and \( \sigma_{\max}(G(i \omega_p)) = 0.15756820450368172 \), where \( \omega_p \) denotes the test frequency which gives the largest maximum singular value of the transfer function matrix. In this way we already obtain a very good initial estimation of the \( \mathcal{L}_\infty \)-norm which is \( \|G\|_{\mathcal{L}_\infty} \approx 0.15899661776628790 \).

In Table I we illustrate the convergence history and the relative error of the method for a relative accuracy of \( \tau = 1000\varepsilon \). After already four iterations we obtain convergence.

For the properness test we obtain the following results. When removing all uncontrollable or unobservable infinite poles using the method from [22], the system’s order reduces from \( n = 11 \) to \( n_r = 6 \) and we obtain a reduced system with a nonsingular descriptor matrix \( E \). Therefore the matrix \( A_{22} \) in (12) is empty and Theorem 3 holds vacously.

V. CONCLUSIONS AND OUTLOOK

We presented an extended method for the computation of the \( \mathcal{L}_\infty \)-norm for descriptor systems with special emphasis on exploiting the structure of the involved skew-Hamiltonian/Hamiltonian matrix pencils. We also showed how these matrix pencils can be extended to ensure reliability and improve the accuracy of the eigenvalue computation. In this way we could also improve the
results of the $\mathcal{L}_\infty$-norm computation. We also introduced a theoretical result which can be used to check whether a transfer function matrix is proper or improper. There are still open problems concerning the norm computation. First we remark that we did not have a look on discrete-time systems in this article. In principle our method can also be applied to these but we then would have to deal with pencils with symplectic structure. However, there exist possibilities to transform these to more convenient structures, see e.g., [19], [25]. Second one could still improve the convergence order of the algorithm using the method explained in [26]. Finally, as our algorithm has computational costs of $O(n^3)$ it is not reasonable to apply it to large sparse systems. There exist iterative schemes to estimate the $\mathcal{L}_\infty$-norm for large sparse standard state-space systems [27], [28]. Such a method is still unknown for descriptor systems.

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REFERENCES


Algorithm 1 Two-Step Algorithm for Computing the $\mathcal{L}_\infty$-Norm

**Input:** Continuous linear time-invariant descriptor system $(E; A, B, C, D)$ with transfer function $G$, tolerance $\varepsilon$.

**Output:** $\|G\|_{\mathcal{L}_\infty}$.

1. Compute an initial value $\gamma_{lb} < \|G\|_{\mathcal{L}_\infty}$.
2. repeat
   3. Set $\gamma := (1 + 2\varepsilon)\gamma_{lb}$.
   4. Form the matrix pencil $M_\gamma - \lambda N$ and compute its eigenvalues.
   5. if no imaginary eigenvalues then
      6. $\gamma_{ub} = \gamma$, break.
   7. else
      8. Set $\{i\omega_1, \ldots, i\omega_k\} = $ finite imaginary eigenvalues with $\omega_i \geq 0$ for $i = 1, \ldots, k$.
      9. Set $m_j = \frac{1}{2}(\omega_j + \omega_{j+1})$, $j = 1, \ldots, k - 1$.
     10. Compute the largest singular value of $G(im_j)$ for $j = 1, \ldots, k - 1$.
      11. Set $\gamma_{lb} = \max_{1 \leq j \leq k-1} \sigma_{\text{max}}(G(im_j))$.
   12. end if
3. until break
4. Set $\|G\|_{\mathcal{L}_\infty} = \frac{1}{2}(\gamma_{lb} + \gamma_{ub})$. 