Image bilevel thresholding based on stable transition region set

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A B S T R A C T
We utilize the linear system theory to establish a theory model of transition region. With the model, we reveal an important property of transition region, namely the gray level distribution symmetry. Utilizing the property, we propose a new thresholding framework based on stable transition region set. The elements of the stable transition region set are equal or close to each other in the average gray level. As an example of the proposed framework, we have shown that the feature transformation based on the multiscale gradient multiplication technology is an effective means of estimating the threshold. We have performed subjective and objective comparisons on both synthetic and real images. The experimental results show the segmentation quality of the proposed approach is superior to three conventional transition region-based thresholding methods.

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1. Introduction

Thresholding is one of the most important and effective methods for image segmentation. It is best suited for images consisting of object and background with different gray level values. Thresholding can serve a variety of applications, such as document processing [1–4], biomedical image analysis [5,6], automatic detection of moving objects [7,8], and vision-based nondestructive testing [9, 10], etc.

Transition region-based thresholding is a kind of novel approach for image segmentation in recent years. Transition region is geometrically located between objects and background, and is composed of pixels having intermediate gray levels between those of object and of background [11]. Once the transition region has been extracted, the final segmentation threshold will be determined by the mean or peak of the gray level histogram of the transition region [11–16]. Thus, the core issue to transition region-based thresholding method is how to extract the reasonable transition region.

The existing transition region-based thresholding methods can be roughly classified as the bidirectional and unidirectional restriction approaches. The bidirectional restriction methods automatically calculate the lower bound $f_l$ and the upper bound $f_h$ of the gray level distribution of the transition region. The pixels with gray level in the interval $[f_l, f_h]$ are treated as the transition region. The effective average gradient (EAG) approach [11] is a representative among the bidirectional restriction methods. The EAG method uses the clip transformation followed by computing the effective average gradient to obtain the lower bound $f_l$ and the upper bound $f_h$. The EAG method is fully automatic, and no limitations on shape and size of objects are imposed. The EAG method, however, faces two primary limitations: (1) for the relatively complex images, the gray level interval $[f_l, f_h]$ is often too loose, which will misclassify many object or background pixels into the transition region; (2) Groenewald et al. proved the existence of $f_l > f_h$ [17]. In this case, the transition region cannot be extracted. This drawback is also verified by our experiments in Section 7.3, where the case $f_l > f_h$ occurs 3 times among all 50 real images. Liang and Le modified the gradient operator with Gaussian weight to suppress the affects of noise [18]. But it, essentially, is still based on EAG method, and the above two drawbacks of EAG method still exist.

Generally, the unidirectional restriction methods firstly apply a predefined feature transformation to the gray level image to generate a new feature image. The pixels in the feature image with feature value that is greater than a given feature threshold $F$ are treated as the transition region. The feature threshold $F$ is a predefined monotone function of the independent variable $\lambda$. Given a $\lambda$ value, the corresponding $F$ value can be obtained, and the transition region can also be extracted. The typical unidirectional restriction approaches include higher gradient-based (HD) method [12,13], gray level difference-based (GLD) method [16], and local entropy-based (LE) method [14]. For the noisy images, the HD and GLD methods often misclassify most of object or background regions into transition region and cause serious deviation...
of the transition region. As a consequence, the two methods often yield bad segmentation results for the noisy images. The LE method describes pixel with the local entropy concept. The local entropy is larger for heterogeneous region but smaller for a homogeneous neighborhood. The pixels with larger local entropy value belong to the transition region with higher probability. However, when the heterogeneity of the object or background region is close to or even exceeds that of the transition region, it is difficult for the LE method to extract the transition region from the object or background. In addition to the above limitations of their own, the unidirectional restriction methods have a common drawback: the feature threshold \( F \) is computed with a fixed empirical \( \lambda \). But the segmentation thresholds obtained by these methods are sensitive to the parameters \( \lambda \), as illustrated by our experiments in Section 7.1. Therefore, it is doubtful the reliability, accuracy, and generality of these approaches.

The bidirectional and unidirectional restriction approaches have an implicit viewpoint, namely a reasonable threshold only corresponds to a reasonable transition region. Unlike this, our basic ideas lie in that a reasonable threshold can correspond to many different reasonable transition regions. In other words, these different reasonable transition regions can form a transition region set. The elements of the set are equal or close to each other in the average gray level, namely, the elements of the set have high stability in the average gray level. This paper utilizes this kind of stability to distinguish the reasonable transition regions from the unreasonable ones.

The rest of this paper is organized as follows. In Section 2, we establish a theory model of transition region with the linear system theory, and we also propose some new concepts relating to transition region. In Section 3, we reveal an important property of transition region, namely the gray level distribution symmetry. The property is used to establish a new thresholding framework in Section 4. In Section 5, as an example of the proposed framework, we show that the feature transformation based on the multiscale gradient multiplication technology is an effective means to estimate the final threshold. The proposed thresholding algorithm is presented in Section 6. The experimental results and discussion are presented in Section 7. Finally, the main conclusions are drawn in Section 8.

2. Theory model and related concepts of transition region

Blurring and noise lead to the quality degradation of the image. In particular, the former is the basic reason for the existence of transition region. Generally, two important assumptions are made about the degradation process. One assumption is that the blurring degradation process is caused by a linear shift invariance point spread function (PSF) that acts on the true image. Another assumption is about the noise which is assumed to be purely additive in nature. Under the two assumptions, the linear degradation caused by blurring and additive noise can be given by the following equations [19–21]:

\[
f(x, y) = f_{psf}(x, y) + n(x, y)
\]  
\[
f_{psf}(x, y) = h(x, y) * g(x, y)
\]

where

\[
f(x, y)\] represents the degraded image which is acquired by the imaging system, \( n(x, y) \) represents an additive noise introduced by the system, and is usually taken to be a zero mean Gaussian distributed white noise term. In this paper, we deal only with additive Gaussian noise, as it effectively models the noise in many different imaging scenarios. Furthermore, we let the probability density function of \( n(x, y) \) be

\[
r_n(x) = \frac{1}{\sqrt{2\pi \sigma_n^2}} e^{-\frac{x^2}{2\sigma_n^2}}
\]

In Eq. (2), \( g(x, y) \) is the original undegraded image, which consists of the object with gray level \( g_o \) and the background with gray level \( g_b \). \( h(x, y) \) represents the two-dimensional PSF of the imaging system, which usually can be well approximated by a Gaussian function [22–25].

In order to analyze some properties of transition region, one further assumption is made about the boundary of the object. There are two kinds of assumptions about the boundary. The first kind of assumption is that the boundary is composed of a finite sequence of straight line segments [22,26]. Therefore, for the vertical boundary segment, its local image can be represented as follows:

\[
g(x, y) = \begin{cases} 
  g_b & -W \leq x < 0 , \ -W \leq y \leq W \\
  g_o & 0 \leq x \leq W , \ -W \leq y \leq W 
\end{cases}
\]  

Correspondingly, we assume the Gaussian PSF \( h(x, y) \) with the following expression:

\[
h(x, y) = C \cdot \frac{1}{2\pi \sigma_{psf}^2} e^{-\frac{x^2+y^2}{2\sigma_{psf}^2}}
\]

where the coefficient \( C \) makes the Gaussian PSF \( h(x, y) \) satisfy the following constraint:

\[
\int_{-\omega}^{\omega} \int_{-\omega}^{\omega} h(x, y) dx dy = 1
\]

where \( \omega \) denotes the half width of the convolution operation range of the Gaussian PSF. The second kind of assumption is

\[
g(x, y) = \begin{cases} 
  g_b & x < 0 \\
  g_o & x \geq 0
\end{cases} \quad \text{and} \quad h(x, y) = \frac{1}{2\pi \sigma_{psf}^2} e^{-\frac{x^2+y^2}{2\sigma_{psf}^2}}
\]

Whether under the first kind of assumption or the second, we can obtain the same or similar conclusions for the subsequent propositions. However, the proof procedures are more concise with the second kind of assumption. Thus, we will adopt the second kind of assumption about the boundary to analyze some properties of transition region in the subsequent sections.

To facilitate the subsequent discussion, here we define some new concepts relating to the transition region.

Precise transition region (PTR): the points in the image \( f_{psf}(x, y) \) with gray level in the interval \( (g_b, g_o) \) are defined as the PTR of \( f_{psf}(x, y) \). According to Eqs. (1) and (2), we define that the images \( f(x, y) \) and \( f_{psf}(x, y) \) have identical PTR. An example of the PTR is shown in Fig. 1.

Loose transition region (LTR): an LTR is an approximation to the PTR, and the approximation degree is characterized by the difference between the LTR and the PTR in the average gray level. If the difference is small enough, the LTR is called the general transition region (GTR).

Transition region set: given a transition region, a set composed of some subsets of the transition region is defined as a transition region set. 

L-stable transition region set (L-STRS): given a transition region set, if the absolute value of the difference between the average gray level of its any two elements is not greater than \( L \), the transition region set is defined as an L-STRS. The value \( L \) and the element number \( M \) of the L-STRS are called the explicit and implicit parameters of the set stability, respectively. The set stability
is characterized by the parameters L and M. Given a value M, we define that the smaller the value L, the more stable a transition region set. Given a value L, we define that the larger the value M, the more stable a transition region set.

L-stable threshold set (L-STS): given an L-STRS, a new set, which is composed of the average gray level of each element in the L-STRS, is defined as an L-STS.

3. Gray level distribution symmetry of PTR

In this section, we will propose three propositions to show an important property of PTR, namely the gray level distribution symmetry.

**Proposition 1.** In the PTR of the image \( f_{psf}(x, y) \), if there is a point \( p \) with gray level \( f_p \), there will exist a point \( q \) with gray level \( f_q \), and the two points \( p \) and \( q \) satisfy the following two relationships: (1) \( p \) and \( q \) are symmetric about the boundary of object; (2) the gray levels of \( p \) and \( q \) satisfy \( f_p + f_q = 2g_{\text{steady}} \), where \( g_{\text{steady}} = (g_b + g_o)/2 \) (see Appendix A for the proof).

Proposition 1 shows that the boundary of object is the central axis of the PTR. Based on Proposition 1, we have the following two propositions, which reveal the gray level distribution symmetry of the PTR.

**Proposition 2.** In the PTR of the image \( f_{psf}(x, y) \), the gray level distribution of the points, which locate symmetrically at the both sides of the central axis, is symmetric with respect to \( x = g_{\text{steady}} \) (see Appendix B for the proof).

**Proposition 3.** In the PTR of the image \( f(x, y) \), the gray level distribution of the points, which locate symmetrically at the both sides of the central axis, is symmetric with respect to \( x = g_{\text{steady}} \) (see Appendix C for the proof).

According to Propositions 2 and 3, it is not difficult to obtain the following two corollaries.

**Corollary 1.** In the image \( f_{psf}(x, y) \), the gray level distribution of the PTR is symmetric about \( x = g_{\text{steady}} \).

**Corollary 2.** In the image \( f(x, y) \), the gray level distribution of the PTR is symmetric about \( x = g_{\text{steady}} \).

Fig. 2 illustrates the gray level distribution symmetry of the PTR in a relatively complex image. An original undegraded im-
the following two requirements: (1) a GTR strategy that removes gradually the points in the LTR can meet steady STRS (a visual example for the PTR. When obtaining such an LTR central axis of the PTR of the image PTRs are symmetric about respectively. As shown in Figs. 2(e)–(f), the gray level histograms of the PTRs in the images Figs. 2(b) and 2(c), respectively. As shown in Figs. 2(e)–(f), the gray level histograms of the PTRs are symmetric about x = 140.

Proposition 1 shows that the gray levels of the points on the central axis of the PTR of the image \( f_{psf}(x, y) \) are equal to \( g_{steady} \). If we segment the PTR of the image \( f_{psf}(x, y) \) with the gray level \( g_{steady} \), the misclassification error is zero in theory (the misclassification error reflects the percentage of background region wrongly classified into foreground, and conversely, foreground wrongly assigned to background. More details about misclassification error are given in Section 7). For the image \( f(x, y) \), if we segment its PTR with the gray level \( g_{steady} \), it is not hard to understand that the misclassification error is the minimum according to Proposition 3. Thus, the gray level \( g_{steady} \) is an ideal segmentation threshold for the image \( f_{psf}(x, y) \) and \( f(x, y) \). Further, the average gray levels of the PTRs of both \( f_{psf}(x, y) \) and \( f(x, y) \) are equal to \( g_{steady} \). This shows the average gray level of the PTR is an ideal segmentation threshold.

4. Proposed thresholding framework

Generally, it is difficult to obtain directly the PTR in practice. However, it is relatively easy to obtain an LTR \( R_{LTR} \) that includes the PTR. When obtaining such an LTR \( R_{LTR} \), let us assume that a strategy that removes gradually the points in the LTR can meet the following two requirements: (1) a GTR \( R_{GTR}^0 \) can be produced when the strategy is applied to the LTR \( R_{LTR} \); (2) when the strategy is applied to the GTR \( R_{GTR}^0 \), a series of new GTRs can be obtained. For each new GTR, the region in the intersection of the PTR and the new GTR is symmetric about the central axis. Then when the strategy is applied to the LTR \( R_{LTR} \) and the GTR \( R_{GTR}^0 \), a transition region set \( S_{STR} = S_{LTR} \cup S_{GTR} \) can be obtained, where \( S_{LTR} = \{ R_{LTR}^i \mid R_{LTR}^i \subseteq R_{LTR} \}, i = 1, 2, \ldots \} \) is an L1-STRS, and \( S_{GTR} = \{ R_{GTR}^j \mid R_{GTR}^j \subseteq R_{GTR} \} \) and \( R_{GTR}^0 \in S_{LTR} \), \( j = 1, 2, \ldots \} \) is an L2-STRS (a visual example for \( S_{STR} \) and \( S_{GTR} \) can be found in Fig. 3(b)). Further, two stable threshold sets \( L1-STRS \) and \( L2-STRS \) respectively corresponding to \( S_{LTR} \) and \( S_{GTR} \) can be obtained (see Fig. 3(c)). According to Propositions 1–3, it is easy to understand that each element value of the \( L2-STRS \) is close to \( g_{steady} \). Therefore, the \( L2-STRS \) can be used for determining the final segmentation threshold. Hence, if we can extract \( S_{GTR} \) from \( S_{STR} \), we will can obtain the final segmentation threshold. Now, the core issue is how to extract \( S_{GTR} \) from \( S_{STR} \).

Since each element of the \( L2-STRS \) is close to \( g_{steady} \), the value \( L2 \) should be relatively small according to the STRS and STS definitions (see Fig. 3(c)). In other words, \( S_{GTR} \) has relatively high stability. On the other hand, the points in the region that is in LTR but not in PTR often cannot satisfy the gray level distribution symmetry, which results in that there are relatively large differences among the elements of \( L1-STRS \). Thus, the value \( L1 \) should be relatively large (see Fig. 3(c)), that is, \( S_{LTR} \) has poor stability. Therefore, the distinction between \( S_{GTR} \) and \( S_{STR} \) in stability may be used for distinguishing \( S_{GTR} \) from \( S_{STR} \). A simply heuristic strategy is dividing the transition region set \( S_{STR} \) into a certain number of 1-STRSs, and then choosing a 1-STRS \( S^* \) with the maximum element number as the approximation to \( S_{GTR} \). Here we call the heuristic strategy as the maximizing 1-STRS strategy.

The above analyses indicate that the successful implementation of the maximizing 1-STRS strategy should satisfy the following two requirements as much as possible. First, the PTR should be highlighted to facilitate the extraction of the GTR \( R_{GTR}^0 \), which require us to design a good feature transformation function that highlights the PTR. Second, if we adopt the strategy that treats the points in the feature image with feature value greater than the feature threshold \( F \) as the transition region, then when the feature threshold \( F \) changes from the minimum feature value to the maximum one, the gray level distributions of the extracted regions should be symmetric about the \( x = g_{steady} \) as much as possible. In subsequent Section 5, we will show that the feature transformation based on multiscale gradient multiplication can meet the above two requirements.

5. Feature transformation based on multiscale gradient multiplication

The theory model of transition region described in Section 2 shows that the PTR is essentially a Gaussian smoothed step edge. The points in the PTR should generally have more significant difference than the ones in the object or background in gray level. There-
fore, to highlight the PTR, an alternative scheme is to apply the gradient-based feature transformation to the gray level image. Note that the gray level images are often corrupted by noise, especially Gaussian or Gaussian-like distribution noise, during the acquisition process. To calculate the gradient of noisy image, a widely adopted strategy is to apply the first derivative to the smoothed image, namely the so-called smoothed gradient estimation. A smoothed gradient operator that has received a lot of attention is the First Derivative of Gaussian (FDoG) filter, which estimates the gradient after smoothing with a Gaussian function. The FDoG filter can be derived under the criteria of detection and localization [27].

An important issue is the space scale of the FDoG filter. Small space scales are sensitive to edge signals but also prone to noise, whereas large scale filters are robust to noise but could filter out fine details. The existing some methods usually utilize the global or local image information to estimate the appropriate scale of the FDoG filter. However, it is difficult for these methods to meet the following requirement: noise is well suppressed and the accuracy of the edge location is similar to the response of the minimal scale filter. To overcome this dilemma, an alternative scheme is the so-called multiscale multiplication, namely, multiplying the responses of the filters at different scales. It has been shown that the multiscale multiplication approach can improve the edge localization accuracy while suppressing noise, which has been validated in space domain [28,29], wavelet domain [30] and odd Gabor transform domain [31].

The multiscale multiplication technology adopted in this paper is similar to the method proposed by Bao et al. [29]. Both methods multiply the responses of the FDoG filters at different scales. However, there are two main differences between the two methods. First, the method proposed by Bao et al. adopts the nonmaxima suppression technology to retain the edge points that have a local maximum. However, the nonmaxima suppression technology is not beneficial to construct the STRS, so it is not used by our method. Second, the filter scale and the filter time in the Bao et al.’s method are fixed, whereas, in our method, they are adaptively computed by analyzing the noise and the edge information. In Section 5.1, we will introduce the basic idea of the feature transformation based on multiscale gradient multiplication, and analyze its advantage on highlighting the transition region. Section 5.2 will analyze the distribution property of the feature value in the transition region. In Section 5.3, we will present how to automatically compute the filter scale and the filter time.

5.1. Multiscale gradient multiplication (MGM)

For a gray level image \( f(x, y) \), the gradient magnitude corresponding to a given filter scale \( \sigma \) can be obtained according to the following expression:

\[
A_{\sigma}(x, y) = \| \nabla G(x, y; \sigma) \ast f(x, y) \|
\]  

(6)

where

\[
G(x, y; \sigma) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2+y^2}{2\sigma^2}}
\]

Thus, given \( k \) space scales \( \sigma_1, \sigma_2, \ldots, \sigma_k \), the MGM image can be obtained with the following steps: firstly calculate the corresponding gradient magnitude image \( A_{\sigma_i}(x, y) \) for each space scale \( \sigma_i \) according to the expression (6), and then output the product \( A(x, y) = \prod_{i=1}^{k} A_{\sigma_i}(x, y) \) as the MGM image. The \( k \) space scales keep dyadic relationship (that is \( \sigma_0 = 2^{-k-1}\sigma_1 \), \( k \) is a positive integer), as it will incorporate sufficiently edge information of different scales while reduce the degree of correlation of the noise across scales. The similar strategy is adopted in [28–31].

The signals and noise have different singularity. The edge structures present observable gradient magnitudes along the scales, while noise decreases rapidly [29]. Based on this observation, it is easy to understand that multiplying the response at adjacent scales will amplify the edge structure and suppress the noise well. Fig. 4 illustrates the performance difference between the MGM method and the single scale gradient method in the edge location and the noise suppression. We can see that with single scale \( \sigma = 0.25 \), the edge is more accurately localized but the negative effect of noise is also obvious (see Fig. 4(c)). Along with the increase of scale, noise is better suppressed but traded off with a decreased accuracy in edge location (see Figs. 4(d)–(g)). By comparison, the MGM image combines the advantages of different scale gradient magnitude images. Figs. 4(k)–(l) show that in the corresponding MGM images, noise is better suppressed compared with \( A_{\sigma=0.25}(x, y) \), while the accuracy of the edge location is similar to the response \( A_{\sigma=0.25}(x, y) \) of the minimal filter scale. The above analyses show, even for the noisy images, using the MGM technology as the feature transformation function is beneficial to highlight the PTR, and facilitate the extraction of the GTR.

5.2. Distribution of MGM value in PTR

According to the convolution theory, we get:

\[
G(x, y; \sigma) \ast f(x, y) = \left( G(x, y; \sigma) \ast h(x, y) \right) \ast g(x, y)
\]

\[
+ G(x, y; \sigma) \ast n(x, y)
\]

(7)

Further, using the cascading theorem of the Gaussian function, the first item of the right side of Eq. (7) can be simplified to the following expression:

\[
\left( G(x, y; \sigma) \ast h(x, y) \right) \ast g(x, y) = G(x, y; \sigma_{\text{new}}) \ast g(x, y)
\]

(8)

where

\[
G(x, y; \sigma_{\text{new}}) = \frac{\sigma}{\sqrt{2\pi}\sigma_{\text{new}}} e^{-\frac{x^2+y^2}{2\sigma_{\text{new}}^2}}
\]

\[\sigma_{\text{new}} = \sqrt{\sigma^2 + \sigma_{\text{psf}}^2}\]

We propose the following three propositions to reveal the distribution property of gradient magnitude of the pixels in the PTR.

Proposition 4. For the image \( \| \nabla G(x, y; \sigma_{\text{new}}) \ast g(x, y) \| \), if there is a point \( p \) in the PTR, there will exist a point \( q \) in the PTR, and the two points \( p \) and \( q \) are symmetric about the central axis; (2) the gradient magnitudes of \( p \) and \( q \) are equal to each other (see Appendix D for the proof).

Proposition 5. For the image \( \| \nabla G(x, y; \sigma_{\text{new}}) \ast g(x, y) \| \), the points, which locate symmetrically at the both sides of the central axis as well as in the PTR, have the same gradient magnitude distribution (see Appendix E for the proof).

Proposition 6. For the image \( \| \nabla G(x, y; \sigma_{\text{new}}) \ast g(x, y) + G(x, y; \sigma) \ast n(x, y) \| \), the points, which locate symmetrically at the both sides of the central axis as well as in the PTR, have the same gradient magnitude distribution (see Appendix F for the proof).

According to Proposition 5, in the MGM image \( A(x, y) \) of the image \( f_{\text{psf}}(x, y) \), the points, which locate symmetrically at the both sides of the central axis as well as in the PTR, have the same MGM value distribution. According to Proposition 6 and the probability law for a function of random variables [32,33], in the MGM image \( A(x, y) \) of the image \( f(x, y) \), the points, which locate symmetrically at the both sides of the central axis as well as in the PTR, have the same MGM value distribution. Thus, once we obtain the MGM image of the image \( f_{\text{psf}}(x, y) \) or the image \( f(x, y) \), the number of the points, which locate at the both sides of the central
axis as well as in the PTR, and have MGM value greater than feature threshold $F$, is equal to each other in statistically. According to Propositions 1–3, the gray level distributions of the extracted points should be symmetric about $x = g_{\text{steady}}$. Obviously, this will be beneficial to generate an STRS with relatively high stability, and to implement successfully the maximizing 1-STRS strategy.

5.3. Automatic computation of filter scale

To estimate automatically the final segmentation threshold, the problem how to compute the space scales $\sigma_1, \sigma_2, \ldots, \sigma_k$ still need to be addressed. We choose the minimal scale $\sigma_1 = 0.25$, as the half-width of the minimal discretized filters is 1 when we truncate the Gaussian function at the position of the 4 standard deviation [34]. On the other hand, since the adjacent scales keep dyadic relationship, so if we can obtain the maximal filter scale $\sigma_k$, we also can obtain other $k - 2$ filter scales $\sigma_2, \ldots, \sigma_{k-1}$. Further, we will propose the following two propositions to estimate the maximal filter scale $\sigma_k$.

**Proposition 7.** Given a two-dimensional Gaussian function

$$G(x, y; \sigma) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x^2 + y^2)}{2\sigma^2}}$$

and a zero mean Gaussian distributed white noise image $n(x, y)$, the mean $E_{\text{noise}}$ of gradient magnitude of the noisy image $n(x, y)$ and the filter scale $\sigma$ have the relationship $E_{\text{noise}} = \frac{\rho}{\sigma}$, where $\rho = \sqrt{\frac{\pi}{2} \sigma_n}$ and $\sigma_n$ is the standard deviation of the noise image $n(x, y)$ (see Appendix G for the proof).

**Proposition 8.** Given a two-dimensional Gaussian function

$$G(x, y; \sigma_{\text{new}}) = \frac{1}{\sqrt{2\pi}\sigma_{\text{new}}} e^{-\frac{x^2 + y^2}{2\sigma_{\text{new}}^2}}$$

and a step edge image

$$g(x, y) = \begin{cases} g_0 & x < 0 \\ g_0 & x \geq 0 \end{cases}$$
the integral of gradient magnitude of all points on the line that is perpendicular to the step edge and pass through a point (x, y) is equal to $\sqrt{2\pi} |g_0 - g_b| \sigma$. Further, if the length of the step edge is $\tau$, the mean $E_{signal}$ of gradient magnitude of the step image and the filter scale $\sigma$ has the relationship $E_{signal} = \kappa \sigma$, where $\kappa = \frac{1}{\tau} \sqrt{2\pi} |g_0 - g_b|$ (see Appendix H for the proof).

According to Propositions 7 and 8, we construct a function about the filter scale $\sigma^* \gamma(\sigma^*) = E_{signal} + E_{noise}$. The function $\gamma(\sigma^*)$ has a unique global minimum at $\sigma^* = \frac{1}{2} \sqrt{\frac{\sigma_n}{|g_0 - g_b|}} \tau / \iota$.

When $\sigma \in (0, \sigma^*)$, the function $E_{signal}$ is monotonically increasing, while the function $\gamma(\sigma)$ is monotonically decreasing. This shows: (1) the response of the FDoG filter to noise decays rapidly; (2) the average gradient magnitude of the noise is still greater than that of the edge. When $\sigma \in (\sigma^*, +\infty)$, the function $E_{noise}$ is monotonically decreasing, while the function $\gamma(\sigma)$ is monotonically increasing. This suggests: (1) the noise is suppressed to a great extent; (2) the average gradient magnitude of the edge is greater than that of the noise. Based on the above analysis, we hold that assigning $\sigma^*$ to the maximal filter scale $\sigma_0$ is appropriate. However, it is difficult to obtain $\sigma_n$, $g_0$, $g_b$ and $\iota$ for the real image. Therefore, it is difficult to obtain the explicit expression of $\sigma^*$ for the real image. In practice, we will utilize the characteristic that $\gamma(\sigma)$ has a unique global minimum to estimate the maximal filter scale $\sigma_0$. We compute the gradient magnitude images at different filter scale, and we choose the filter scale that the corresponding average gradient magnitude is minimal as the maximal filter scale $\sigma_k$. Since the adjacent scales keep dyadic relationship, so if the obtained scale $\sigma_k$ is not equal to the multiple of $2^{k-1}$ of $\sigma_1$ ($k$ is a positive integer), we will replace it with $2^{\lceil \log_2(\frac{\sigma_0}{\sigma_1}) \rceil} \sigma_1$.

6. Proposed thresholding method

According to the preceding two sections, here we propose an STRS-based thresholding method. The method includes the following 4 steps:

Step 1: For an input gray level image $f(x, y)$, first compute automatically all filter scales $\sigma_1, \ldots, \sigma_k$ according to the scheme described in Section 5.3, and then compute the MGM image $A(x, y)$ according to Section 5.1.

Step 2: Construct a set $U$ that includes all possible MGM values of the MGM image $A(x, y)$ but without repetition. Suppose that the set $U$ has been sorted in ascending order. Traverse the set $U$. During the traversing procedure, first, assign the current accessing element value to the feature threshold $F_t$; next, treat the pixels in the $A(x, y)$ with pixel value greater than $F_t$ as a transition region; then, compute the mean $\mu_g$ of gray level of the corresponding transition region in the image $f(x, y)$; finally, put the value $\mu_g$ in a new set $V_{0g}$.

Step 3: Obtain the final segmentation threshold using the maximizing 1-STS strategy implemented as follows: Traverse the set $V_{0g}$ obtained from Step 2. During the traversing procedure, treat the current accessing element $\mu_x$ as the temporary center $\mu^{(s)}_x$, and classify any element $\mu_j$ that satisfies the relationship $|\mu_j - \mu^{(s)}_x| \leq 0.5$ into the corresponding 1-STS $T_j$. After traversing, select a 1-STS $T_k$ with maximal cardinality, and output the corresponding $\mu^{(s)}_k$ as the final segmentation threshold $t$ (the transition region corresponding to $\mu^{(s)}_k$ is treated as the final transition region extracted by the STRS method).

Step 4: Segment the image $f(x, y)$ using the threshold $t$.

7. Experimental results and discussion

To assess its effectiveness, the proposed STRS method is compared with 3 conventional transition region-based thresholding methods used widely in the literature, i.e., the EAG method [11], the LE method [14], and the GLD method [16]. The test images include not only a large number of synthetic images, but also 50 real images from different application areas. The 50 real images contain small or big objects, and are noisy or smooth. The objects in those images can be exactly distinguished from the background by a single global threshold.

In order to verify the performance of all compared thresholding methods, the optimal thresholded image is manually created using visual inspection and used as a gold standard (ground-truth image). For the accuracy quantitative evaluation, the misclassification error (ME) measure [35,36] is adopted in this paper. The ME reflects the percentage of background pixels wrongly classified into foreground, and conversely, foreground pixels wrongly assigned to background. For a two class segmentation problem, the ME can be simply expressed as:

$$ME = 1 - \frac{|B_{gt} \cap B_t| + |O_{gt} \cap O_t|}{|B_{gt}| + |O_{gt}|}$$

where $B_{gt}$ and $O_{gt}$ represent the background and foreground (object) of the ground-truth image, respectively; $B_t$ and $O_t$ denote the background and foreground pixels in the segmented image by threshold $t$, and $| \cdot |$ denotes the cardinality of a set. The ME varies from 0 for a perfectly classified image to 1 for a totally wrongly classified image.

7.1. Experiments on synthetic images

In the experiments on synthetic images, the sensibility test for the LE, GLD, and STRS methods to the feature threshold is firstly analyzed. The LE and GLD methods belong to the unidirectional restriction approaches. The pixels in the feature image with feature value greater than the feature threshold $F$ are treated as the transition region. The average gray level of the transition region in the gray level image $f(x, y)$ is treated as the segmentation threshold. For the LE method, $F = \lambda F_{max}$, where the $F_{max}$ is the maximal feature value in the corresponding feature image, and the recommended interval of the parameter $\lambda$ is [0.8, 0.9] [14]; for the GLD method, $F = F_{max}; \lambda = |\mu_{GLD}|$ and $F < F_{max}$, where $\mu_{GLD}$ and $F_{max}$ are the mean, standard deviation, and maximal feature value of the corresponding feature image, and the recommended interval of the parameter $\lambda$ is [1.5] [16]. Since the feature threshold $F$ of both LE and GLD methods is the linear function about the feature parameter $\lambda$, the relationship of the segmentation threshold to the feature parameter $\lambda$ can also reflect the relationship of the segmentation threshold to the feature threshold $F$. Thus, for the LE and GLD methods, we will directly analyze the relationship of the segmentation threshold to the feature parameter $\lambda$.

The optimal threshold for the image Fig. 2(b) is equal to 140 according to the standard of minimizing the ME value. Figs. 5(a)–(b) show the relationships of the segmentation threshold to the feature parameter $\lambda$ for the LE and GLD methods about the image Fig. 2(b), respectively. Fig. 5(c) represents the relationship of the segmentation threshold to the feature threshold $F$ for the STRS method about the image Fig. 2(b). We can find from Figs. 5(a)–(c) that the segmentation thresholds obtained by the LE or GLD method increase almost linearly along with the increase of the parameter $\lambda$, while, for the STRS method, 96.4% segmentation thresholds are close to 140.

The optimal threshold for the image Fig. 2(c) is equal to 143 according to the standard of minimizing the ME value. Figs. 6(a)–(b)
show the relationships of the segmentation threshold to the feature parameter $\lambda$ for the LE and GLD methods about the image Fig. 2(c), respectively. Fig. 6(c) represents the relationship of the segmentation threshold to the feature threshold $F$ for the STRS method about the image Fig. 2(c). The LE method treats the pixels with higher local entropy as the transition region. However, for the image corrupted by Gaussian noise, the local entropy values of most pixels in the background and the objects are almost equal to those of the transition region. Thus, it is difficult for the LE method to highlight the transition region (see Fig. 6(d)). The
GLD method implements the feature transformation by computing the absolute difference between the gray level of each input pixel and the average gray level of the pixels in its corresponding local neighborhood. This kind of feature transformation is very sensitive to the noise. As a result, the GLD method fails to highlight the transition region (see Fig. 6(e)). For the noisy image, both LE and GLD methods cannot effectively highlight the transition region, which makes them have difficulty to extract the reasonable transition region. In fact, when a relatively small value is assigned to the feature parameter \( \lambda \), the two methods will wrongly classify a large number of pixels in the background and the objects into the transition region. Contrarily, if a relatively large value is assigned to the feature parameter \( \lambda \), the two methods will wrongly classify many pixels in the transition region into the background or the objects. Further, we perform such an experiment that applying the maximizing 1-STRS strategy to the LE and GLD methods for determining the final segmentation threshold. We find that when the feature parameter \( \lambda \) is relatively small, the two methods may generate their own 1-STRS with maximum element number, but since the obtained transition region includes many false transition region pixels, there will be a relatively great difference between the final segmentation threshold and the ideal threshold (see Figs. 6(a)–(b)). On the other hand, when the feature parameter \( \lambda \) is relatively large, the segmentation thresholds corresponding to the different \( \lambda \) values are obviously different (see Figs. 6(a)–(b)). By comparison, although the segmentation thresholds obtained by the proposed STRS method also vary with the feature threshold \( F \), there are 93.65% segmentation thresholds in the relatively reasonable interval [137, 144]. In particular, there are 43.31% segmentation thresholds in the interval [139.6056, 140.6056], and these thresholds form a 1-STRS with maximum element number.

According to the above analyses, two conclusions about the LE and GLD methods can be drawn: (1) the segmentation threshold (that is the average gray level of the extracted transition region) obtained by the two methods is sensitive to the feature parameter \( \lambda \), and for different gray level images, the corresponding \( \lambda \) values are different for obtaining the reasonable segmentation thresholds. This shows it is improper for the two methods to extract the transition region with a fixed \( \lambda \) value for different gray level images. (2) The maximizing 1-STRS strategy is inapplicable to the two methods.

The second experiment on synthetic images is to compare the segmentation results of the LE, GLD, EAG, and STRS methods for noise-free and noisy images. Considering the maximizing 1-STRS strategy is inapplicable to the LE and GLD methods, the two methods adopt the recommended \( \lambda \) values of their own, that is, \( \lambda = 0.85 \) for the LE method [14] and \( \lambda = 1.5 \) for the GLD method [16]. Figs. 7(a)–(d) show the transition regions extracted by the LE, GLD, EAG, and STRS methods from the image in Fig. 2(b), and Figs. 7(e)–(h) illustrate the corresponding segmentation results of the 4 methods. For the noise-free image in Fig. 2(b), the optimal threshold is equal to 140 according to the standard of minimizing the ME value, and the corresponding ME value is 0.0033569. The segmentation thresholds obtained respectively by the GLD, EAG, and STRS methods approximate 140, and all of their ME values are 0.0033722. By comparison, the segmentation result of the LE method is worst, and its threshold and ME value are 142.8569 and 0.0035553, respectively. Transition region is geometrically located between objects and background, and is composed of pixels having intermediate gray levels between those of object and of background [11]. In this view, the transition regions extracted by the EAG and STRS methods are more accurate than the ones extracted by the LE and GLD methods (see Figs. 7(a)–(d)).

For the noisy image in Fig. 2(c), the optimal threshold is equal to 143 according to the standard of minimizing the ME value, and the corresponding ME value is 0.032547. The segmentation thresholds obtained by the LE, GLD, and EAG methods are 129.1601, 128.1117, and 127.6144, respectively. There are relatively large diff-
ference between the three thresholds and the optimal threshold. The reason for this kind of relatively large difference is that the transition regions extracted by the three methods are very unreasonable, and include too many false transition region pixels (see Figs. 8(a)–(c)). By comparison, the transition region extracted by the STRS method is more accurate (see Fig. 8(d)). As a result, its segmentation result is best among 4 compared methods, which can also be verified by quantitative comparison where its segmentation threshold and ME value are 140.1056 and 0.033401, respectively.

7.2. Experiments on real images

To verify the effectiveness in the practical applications, the proposed STRS method is further compared with the LE, GLD, and EAG methods on 50 real images from different application areas. Due to the limited space, only the segmentation results of 3 representative images are showed in detail, and the comparison of average segmentation performance on all 50 real images will be illustrated in Section 7.3. In all experiments in this subsection, the LE and GLD methods adopt the recommended $\lambda$ values of their own, that is, $\lambda = 0.85$ for the LE method [14] and $\lambda = 1.5$ for the GLD method [16].

An infrared image and the corresponding ground-truth image are displayed in Figs. 9(a)–(b), respectively. The infrared image mainly consists of a target object and two backgrounds with different gray level distribution range. As a result, the gray level histogram of the image is a multimodal distribution (see Fig. 9(c)). The LE method treats the pixels with higher local entropy as the transition region. However, for this infrared image, the local entropy values of partial pixels in the background are almost equal to those of the transition region. Thus, the LE method misclassifies partial background region into transition region (see Fig. 9(e)). As a result, the segmentation result of the LE method contains more background region than other 3 methods (see Fig. 9(i)). The EAG method finds a gray level interval [104, 236] for extracting transition region. Because the interval is too loose, almost entire object region is wrongly assigned to transition region (see Fig. 9(g)). As a result, the relatively dark part of the object region is wrongly classified into the background (see Fig. 9(k)). The GLD and STRS methods more accurately extract transition region (see Figs. 9(f) and (h)), and then obtain better segmentation results than other 2 methods (see Figs. 9(j) and (l)).

Figs. 10(a)–(b) show a thermal image of glass-fiber reinforced plastics and the corresponding ground-truth image, respectively. Both object and background have a unimodal gray level distribution, but the object is much less than the background in size, which leads to a relatively great difference in height between two peaks of the histogram. The thresholds obtained by the LE, GLD, EAG, and STRS methods are 168.0408, 157.0203, 162.9974, and 212.9028, respectively. The corresponding ME values of 4 methods are 0.35956, 0.52161, 0.45129, and 0.0062866, respectively. Among these methods, the proposed STRS method more accurately extract transition region of the image and obtains better segmentation result than other 3 methods (see Figs. 10(e)–(l)). This can also be verified by quantitative comparison, where the ME value of the STRS method is the smallest of 4 compared methods.

Fig. 11(a) shows an image containing a small object on the dark background, and Fig. 11(b) is the corresponding ground-truth image. The size of the object is not commensurate with the background, thus the image histogram only contains one obvious peak formed by the texture noise of the background (see Fig. 11(c)). The lower bound $f_l$ and the upper bound $f_u$ of the gray level distribution of the transition region computed by the EAG method are 132 and 78, respectively. Because the lower bound $f_l$ is greater than the upper bound $f_u$, the transition region extracted by the EAG method is an empty set, which indicates the failure of the EAG method on segmenting the image Fig. 11(a). The transition regions extracted by the LE and GLD methods contain too many false transition region pixels. As a result, both of them obtain very
bad segmentation results, and their ME values are greater than 0.8. By comparison, the transition region extracted by the STRS method is well located between the object and background (see Fig. 11(g)), and the segmentation result is very close to the ground-truth image (see Fig. 11(j)).

7.3. Comparison of average segmentation performance

Fig. 12 shows the quantification comparison of the segmentation performance for 4 thresholding methods on 50 real test images. The abscissa represents the image number, and the ordinate denotes the ME value. The 4 vertical bars above each image number denote the ME values of the LE, GLD, EAG, and STRS methods in order from left to right. Note that, for the EAG method, if the extracted transition region is an empty set, we let the corresponding ME value be 1. Considering most ME values of the STRS method are less than 0.01, whereas some ME values of other 3 methods are greater than 0.8, only the ME values between 0 and 0.1 are shown in Figs. 12(a)–(d) for examining clearly the performance of 4 thresholding methods, while an overall perspective is shown in Figs. 12(e)–(h).

Two statistical results can be achieved from Fig. 12. First, the numbers of the ME value greater than 0.01 are 36, 37, and 36 for the LE, GLD, and EAG methods, respectively. In particular, the case that the transition region extracted by the EAG method is the empty set occurs 3 times among all 50 real images. By comparison, the number of the ME value greater than 0.01 is only 6 for the proposed STRS method. Second, the 4 thresholding algorithms can be ranked from 1st to 4th according to the ME values for each test image, the smaller the ME value, the higher the rank. Among all 50 real test images, the proposed STRS method ranks first with 48 times, second with 2 times.

To assess the average segmentation performance of each method on all 50 test images, two methods have been considered. The first method utilizes the mean and standard deviation of the ME values of each algorithm for all test images. In the second method, the 4 thresholding algorithms are ranked from 1st to 4th according to the ME value for each test image. Then the mean and standard deviation of the ranks of each algorithm are used for the assessment of its segmentation performance. For both methods, the smaller the mean, the better the segmentation quality, and the smaller the standard deviation, the more stable the thresholding algorithm. The first method is called the overall assessment based on the average ME, and the second one is called the overall assessment based on the average rank. The results based on the two overall assessments are shown in Tables 1 and 2, respectively. In Table 1, the average MEs of the LE, GLD, EAG, and STRS methods are 0.10313, 0.14648, 0.12559, and 0.00455, respectively. Obviously, the average ME of the proposed STRS method is far less than those of other 3 methods, which indicates the average segmentation quality of the STRS method is best among 4 compared methods for the 50 test images. Furthermore, the smallest stan-
Fig. 10. Comparison between the results of various methods for an image with asymmetric bimodal distribution. (a) Original image; (b) corresponding ground-truth image; (c) gray level histogram of the original image; (d) relationship of the segmentation threshold to the feature threshold $F$ for the STRS method about the image in Fig. 10(a), where the red points mark the corresponding 1-STS with maximal cardinality obtained by the STRS method, and the arrow points out the central element of the corresponding 1-STS with maximal cardinality; (e)–(h) show the transition regions extracted by the LE, GLD, EAG, and STRS methods, respectively; (i) segmentation result of the LE method ($t = 168.0408$, $ME = 0.35956$); (j) segmentation result of the GLD method ($t = 157.0203$, $ME = 0.52161$); (k) segmentation result of the EAG method ($t = 162.9974$, $ME = 0.45129$); (l) segmentation result of the STRS method ($t = 212.9028$, $ME = 0.0062866$). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Table 1

<table>
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<th>Methods</th>
<th>Mean</th>
<th>Standard deviation</th>
<th>Rank</th>
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Table 2

<table>
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<th>Rank</th>
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</thead>
<tbody>
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<td>1</td>
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<tr>
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<td>EAG</td>
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<td>0.90351</td>
<td>3</td>
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<tr>
<td>GLD</td>
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<td>0.80407</td>
<td>4</td>
</tr>
</tbody>
</table>

standard deviation also manifests that the STRS method has relatively stable performance. The similar conclusion can be found from Table 2, where the average rank of the STRS method is equal to 1.04, which indicates that the proposed method may get the first place for most test images.

8. Conclusions and future work

We propose a new flexible framework capable of solving the bilevel thresholding problem more effective than three conventional transition region-based thresholding methods. This framework is based on the transition region model established by the linear system theory and the new STRS concept. By analyzing the transition region model, we reveal its gray level distribution symmetry. Utilizing the symmetry property, we analyze the existence of the STRS, and propose a heuristic strategy called the maximizing 1-STRS strategy to extract the STRS. As an example of the proposed framework, we show that the feature transformation based on the multiscale gradient multiplication technology is an effective means to extract the STRS and to estimate the final threshold. The experiment results on a variety of synthetic and real images demonstrate the usefulness of the proposed approach and its superiority to three conventional transition region-based thresholding methods.

Although the proposed theory is based on the assumption of additive Gaussian noise, some additional experimental results have demonstrated that the proposed STRS method can deal with these scenarios contaminated by several kinds of noise with non-Gaussian distribution, such as uniform noise, Poisson noise, and salt & pepper noise. The experimental results can be found in the supplementary material to this article. Although there are still some theory obstacles for the detailed proof at present, we have observed that the extracted transition regions still possess the gray level distribution symmetry (see the supplementary material). In our future works, we will undertake the in-depth analysis for the gray level distribution symmetry by the relevant probability statistics theories.

Acknowledgments

We would like to thank Dr. E.E. Kuruoglu and two anonymous reviewers for their careful reading and providing insightful com-
Fig. 11. Comparison between the results of various methods for an image with unimodal distribution. (a) Original image; (b) corresponding ground-truth image; (c) gray level histogram of the original image; (d) relationship of the segmentation threshold to the feature threshold $F$ for the STRS method about the image in Fig. 11(a), where the red points mark the corresponding 1-STS with maximal cardinality obtained by the STRS method, and the arrow points out the central element of the corresponding 1-STS with maximal cardinality; (e)–(g) show the transition regions extracted by the LE, GLD, and STRS methods, respectively; (h) segmentation result of the LE method (the points mark the corresponding 1-STS with maximal cardinality obtained by the STRS method, and the arrow points out the central element of the corresponding 1-STS with maximal cardinality; (i) segmentation result of the GLD method (the points mark the corresponding 1-STS with maximal cardinality obtained by the STRS method, and the arrow points out the central element of the corresponding 1-STS with maximal cardinality; (j) segmentation result of the STRS method ($t = 96.1502$, $ME = 0.90289$); (i) segmentation result of the GLD method ($t = 98.6428$, $ME = 0.83435$); (j) segmentation result of the STRS method ($t = 143.4667$, $ME = 0.00020637$). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)
Fig. 12. Quantification comparison of the segmentation performance for 4 thresholding methods on 50 test images. The abscissa represents the image number, and the ordinate denotes the ME value. The 4 vertical bars above each image number denote the ME values of the LE, GLD, EAG, and STRS methods in order from left to right. Only the ME values between 0 and 0.1 are shown in (a)–(b) for examining clearly the performance of 4 thresholding methods. Overall perspectives are shown in (c)–(d).

\[ P\{f_p - \Delta h \leq X \leq f_p + \Delta h\}, \] and assume also the probability of the points in the exterior PTR with gray levels in the interval \([f_q - \Delta h, f_q + \Delta h]\) is \(P\{f_q - \Delta h \leq X \leq f_q + \Delta h\}\), where \(\Delta h\) is a sufficiently small positive number. According to Proposition 1, for any point \(p_i\) in the inner PTR with gray level \(f_i \in [f_p - \Delta h, f_p + \Delta h]\), there will exist a point \(p_e\) in the exterior PTR with gray level \(f_e \in [f_q - \Delta h, f_q + \Delta h]\), where \(f_i + f_e = 2g_{\text{steady}}\) and \(p_i\) and \(p_e\) are symmetric about the central axis. In other words, points \(p_i\) and \(p_e\) occur in pairs. Thus, we should have \(P\{f_p - \Delta h \leq X \leq f_p + \Delta h\} = P\{f_q - \Delta h \leq X \leq f_q + \Delta h\}\), which indicates that, in the PTR of the image \(f_{\text{psf}}(x, y)\), the gray level distribution of the points, which locate symmetrically at the both sides of the central axis, is symmetric with respect to \(x = g_{\text{steady}}\). □

Appendix C. Proof for Proposition 3

Proof. \(f(x, y)\) is the degraded version of the image \(f_{\text{psf}}(x, y)\) after adding Gaussian white noise with mean 0 and variance \(\sigma_n^2\) to it. According to the linear combination law of the Gaussian random variables [32], we have that, for the points with gray level \(f_p\) in the inner PTR of the image \(f_{\text{psf}}(x, y)\), after adding Gaussian white noise, their gray level distribution obeys the Gaussian distribution, and the corresponding probability density function is

\[
\phi_1(x; f_p, \sigma_n) = \frac{1}{\sqrt{2\pi \sigma_n}} e^{-\frac{(x-f_p)^2}{2\sigma_n^2}}
\]

Similarly, for the points with gray level \(f_q = 2g_{\text{steady}} - f_p\) in the exterior PTR, after adding Gaussian white noise, their gray level distribution obeys the Gaussian distribution with probability density function

\[
\phi_2(x; f_q, \sigma_n) = \frac{1}{\sqrt{2\pi \sigma_n}} e^{-\frac{(x-f_q)^2}{2\sigma_n^2}}
\]

Further, let us construct a new function \(\psi(x; f_p, f_q, \sigma_n) = \phi_1(x; f_p, \sigma_n) + \phi_2(x; f_q, \sigma_n)\). It is easy to prove that \(\psi(g_{\text{steady}} - x; f_p, f_q, \sigma_n) = \psi(g_{\text{steady}} + x; f_p, f_q, \sigma_n)\). Therefore, the function \(\psi(x; f_p, f_q, \sigma_n)\) is symmetric about \(x = g_{\text{steady}}\). Thus, the following
Appendix D. Proof for Proposition 4

Proof.
\[\left|\nabla G(x, y; \sigma_{new}) \ast g(x, y)\right| = \left|\frac{\partial G(x, y; \sigma_{new})}{\partial x} \ast g(x, y)\right| = \left|\frac{\partial G(x, y; \sigma_{new})}{\partial x} \ast g(x - \alpha, y)\right| \int d\alpha d\beta\]

\[\|g_0 - g_b\| \int e^{\frac{-\beta^2}{2\sigma_{new}^2}} d\beta \int e^{\frac{-\alpha^2}{2\sigma_{new}^2}} d\alpha\]

If the coordinates of points p and q are respectively (x, y) and (−x, y), then p and q are symmetric about x = 0, and obviously, p and q have the same gradient magnitude.

The above proof is based on the assumption that the direction of the boundary of the object is vertical. In fact, the same proof is applicable to the boundary of any orientation after transforming its direction to vertical one.

Appendix E. Proof for Proposition 5

Proof. For the image \(\|\nabla G(x, y; \sigma_{new}) \ast g(x, y)\|\), suppose the probability of the points in the inner PTR with gradient magnitudes in the interval \([\delta - \Delta h, \delta + \Delta h]\) is \(P_{inner}[\delta - \Delta h \leq X \leq \delta + \Delta h]\), and assume also the probability of the pixels points in the exterior PTR with gradient magnitudes in the interval \([\delta - \Delta h, \delta + \Delta h]\) is \(P_{exterior}[\delta - \Delta h \leq X \leq \delta + \Delta h]\), where \(\Delta h\) is a sufficiently small positive number. According to Proposition 4, for any point \(p_1\) in the inner PTR with gradient magnitude \(\delta_1 \in [\delta - \Delta h, \delta + \Delta h]\), there will exist a point \(p_2\) in the exterior PTR with gradient magnitude \(\delta_2 \in [\delta - \Delta h, \delta + \Delta h]\), where \(p_1\) and \(p_2\) are symmetric about the central axis. In other words, points \(p_1\) and \(p_2\) occur in pairs. Thus, we should have \(P_{inner}[\delta - \Delta h \leq X \leq \delta + \Delta h] = P_{exterior}[\delta - \Delta h \leq X \leq \delta + \Delta h]\). Hence, we can draw such a conclusion: for the image \(\|\nabla G(x, y; \sigma_{new}) \ast g(x, y)\|\), the points, which locate symmetrically at the both sides of the central axis as well as in the PTR, have the same gradient magnitude distribution.

Appendix F. Proof for Proposition 6

Proof. Let
\[\chi_1 = \frac{\partial G(x, y; \sigma_{new})}{\partial x} \ast g(x, y), \quad \chi_2 = \frac{\partial G(x, y; \sigma)}{\partial y} \ast n(x, y)\]

\[\eta_1 = \frac{\partial G(x, y; \sigma_{new})}{\partial y} \ast g(x, y) \quad \text{and} \quad \eta_2 = \frac{\partial G(x, y; \sigma)}{\partial y} \ast n(x, y)\]

where the probability density function of \(n(x, y)\) is given by
\[r_n(z) = \frac{1}{\sqrt{2\pi\sigma_n}} e^{-\frac{z^2}{2\sigma_n^2}}\]

According to the linear combination law of the Gaussian random variables [32], the random variable \(\chi_2\) will obey the Gaussian distribution with probability density function
\[r_{\chi_2}(z) = \frac{1}{\sqrt{2\pi\sigma_{\chi_2}}} e^{-\frac{z^2}{2\sigma_{\chi_2}^2}}\]

where
\[\sigma_{\chi_2}^2 = \int e^{\frac{\delta^2}{2\sigma_{\chi_2}^2}} d\delta\]

Similarly, the random variable \(\eta_2\) will also obey the Gaussian distribution with probability density function
\[r_{\eta_2}(z) = \frac{1}{\sqrt{2\pi\sigma_{\eta_2}}} e^{-\frac{z^2}{2\sigma_{\eta_2}^2}}\]

in the directions perpendicular and parallel to the boundary, respectively. Further, by the probability law for a function of random variables [32,33], in the image \(\|\nabla G(x, y; \sigma_{new}) \ast g(x, y) + G(x, y; \sigma) \ast n(x, y))\|\), the points, which locate symmetrically at the both sides of the central axis have identical probability density function
\[\frac{1}{\sqrt{2\pi\sigma_{\chi_2}}} e^{-\frac{(z-\mu)^2}{2\sigma_{\chi_2}^2}} \quad \text{and} \quad \frac{1}{\sqrt{2\pi\sigma_{\eta_2}}} e^{-\frac{z^2}{2\sigma_{\eta_2}^2}}\]

Appendix H. Proof for Proposition 8

Proof. According to Proposition 4, we have
\[\|\nabla G(x, y; \sigma_{new}) \ast g(x, y)\| = |g_0 - g_b| \left(\frac{\sigma}{\sigma_{new}}\right) e^{-\frac{z^2}{2\sigma_{new}^2}}\]
Then the integral of gradient magnitude of all points on the line that is perpendicular to the boundary and pass through a point \((x, y)\) is
\[
\int_{-\infty}^{+\infty} |g_o - g_b| \left( \frac{\sigma}{\sigma_{\text{new}}} \right) e^{-\frac{x^2}{2\sigma_{\text{new}}^2}} \, dx = \sqrt{2\pi} |g_o - g_b| \sigma
\]
Further, if the length of the boundary is \(\tau\), and the size of the image is \(\tau\), then the average gradient magnitude of the image is \(\frac{1}{\tau} \sqrt{2\pi} |g_o - g_b| \sigma\).

**Supplementary material**

The online version of this article contains additional supplementary material.

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**References**


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