Basic Dynamics From a Pulse-Coupled Network of Autonomous Integrate-and-Fire Chaotic Circuits

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Abstract—This paper studies basic dynamics from a novel pulse-coupled network (PCN). The unit element of the PCN is an integrate-and-fire circuit (IFC) that exhibits chaos. We can give an iff condition for the chaos generation. Using two IFCs, we construct a master–slave PCN. It exhibits interesting chaos synchronous phenomena and their breakdown phenomena. We give basic classification of the phenomena and their existence regions can be elucidated in the parameter space. We then construct a ring-type PCN and elucidate that the PCN exhibits interesting grouping phenomena based on the chaos synchronization patterns. Using a simple test circuit, some of typical phenomena can be verified in the laboratory.

Index Terms—Chaos, grouping, integrate-and-fire, pulse-coupled neural networks, synchronization.

I. INTRODUCTION

INTEGRATE-AND-FIRE models (IFMs) are known as simple single neuron models and have been studied intensively. Applying a periodic stimulus, the IFMs exhibit interesting periodic and aperiodic phenomena [1]–[4]. Using the IFMs, pulse-coupled networks (PCNs) can be constructed. The PCNs exhibit a variety of synchronous and asynchronous phenomena [5]–[9]. The analysis of these phenomena is important not only as basic nonlinear problems but also as an approach to information processing functions in the brain. On the other hand, consideration of electric circuit versions of the PCNs is important for analysis of various synchronous phenomena; and for engineering applications, e.g., chaos-based communications systems [10], [11] and artificial neural networks [7]. That is, the circuit versions can contribute for studies of nonlinear electric circuits and artificial neural networks. Note that the nonlinear electric circuit does not imply just a realization method of given neural-network models. The circuit itself is an extremely important real physical system represented by Van der Pol circuit for limit cycle, Duffing circuit for chaos, and cellular neural networks (see [12]–[14] and references therein). This paper studies basic phenomena from a novel PCN based on our original integrate-and-fire chaotic circuit (IFC).

First, we introduce the configuration of the IFC: it consists of one capacitor, one inductor, one linear negative resistor, and one firing switch. The IFC is autonomous, is piecewise linear and has two state variables. The state variables can vibrate flexibly below the firing threshold and the IFC can generate chaos. We can give an iff condition for chaos generation in a theoretical sense in [15]. Note that usual single state IFMs without stimuli cannot generate chaos [1]–[4]. We also present a simple equivalent circuit of the IFC: it consists of two capacitors, one 2-port voltage-controlled current source and one firing switch; and is well suited for realization on a chip.

Second, using two IFCs, we construct a basic master–slave PCN. The slave IFC is pulse-coupled with the master IFC when the state of the master reaches the threshold. It exhibits interesting chaos synchronous phenomena and their breakdown phenomena. The stability of the synchronous phenomena can be confirmed by precise numerical simulation. We then give basic classification of the phenomena and their existence regions can be elucidated in the parameter space. The master–slave PCN can be implemented easily and typical phenomena can be verified in the laboratory. Chaos synchronization has been studied intensively [16]. For example, [11] studies chaos synchronization of a class of dynamical systems driven by chaotic pulse-train which is produced through equidistant sampling of a chaotic trajectory. In our system, the driving pulse-train is produced depending on the sub-state of the master and is nonequidistant.

Third, we develop the master–slave PCN into a ring-type PCN. We elucidate that the ring-type PCN can exhibit interesting grouping phenomena based on the various chaos synchronization patterns. The ring-type PCN has plural grouping patterns and exhibits one of them depending on the initial states. Number and kinds of the grouping phenomena can be elucidated easily. Such interesting synchronous phenomena are hot topics in the literature [17], [18], however, the analysis is hard because of complex nonlinearity. The ring-type PCN might be developed into novel artificial neural networks having efficient classification functions.

These results are based on our previous works. [19] and [20] have studied a dependent switched capacitor chaos generator (DSCG) having three parameters. The IFC can be regarded as a novel simple example of the DSCG. Note that analysis and implementation the IFC is much simpler than the DSCG and that it is extremely hard to provide an iff condition for chaos generation from the DSCG. [21] has studied mutually coupled two IFCs with fixed parameters and has given a preliminary numerical result.

II. INTEGRATE-AND-FIRE CHAOTIC CIRCUIT

In this section, we introduce an IFC and give an iff condition for chaos generation. Fig. 1(a) shows the IFC that will be a unit element in the PCN. In the figure, −R, COMP, and MM denote a linear negative resistor, a comparator and a monostable multivibrator, respectively. If the capacitor voltage υ is below the
firing threshold $V_T$, the firing switch $S$ is opened and the circuit dynamics is described by

$$\begin{align*}
C &\frac{dv}{dt} = i, \\
L &\frac{di}{dt} = -v + R_i, \
\text{for } v(t) < V_T.
\end{align*}$$

(1)

If the capacitor voltage $v$ increases and reaches the threshold $V_T$, the COMP triggers the MM to output pulse signal $v_0$. The pulse signal closes the switch $S$ and $v$ is reset to the base voltage $E$ instantaneously, holding $i = \text{constant}^1$

$$(v(t^+), i(t^+)) = (E, i(t)), \quad \text{if } v(t) = V_T.$$  

(2)

Repeating this manner, the IFC generates a firing pulse-train

$$v_0(t) = \begin{cases} 
V_H, & \text{if } v(t) = V_T \\
V_L, & \text{for } v(t) < V_T
\end{cases}$$

(3)

where $V_H$ and $V_L$ are the high and low voltage levels of the MM, respectively. Fig. 1(b) shows the symbolized model of Fig. 1(a), where the gray circle represents the unit circuit and the arrow represents the firing rule. We assume that (1) has unstable complex characteristic root $\omega_\pm = \pm j\omega$

$$\omega^2 = \frac{1}{LC} - \left(\frac{R}{2L}\right)^2 > 0, \quad \delta = \frac{R}{2\omega L} > 0,$$

(4)

Hence the state vector $(v, i)$ can vibrate divergently below the threshold $V_T$. As shown in Figs. 2 and 3, the IFC exhibits chaotic attractor. Note that single state IFMs without stimulus cannot generate chaos. In the experiment, the negative resistor is implemented using op-amp [22]. The divergent vibration and the firing switch correspond to stretching and folding mechanisms, respectively, which are fundamental for chaos generation [22].

Using the following dimensionless variables and parameters:

$$\tau = \omega t, \quad q = \frac{v}{V_T}, \quad x = \frac{v}{V_T} \left(\frac{\dot{x}}{x} \frac{dx}{dt}\right)$$

$$\begin{align*}
y & = -\frac{\delta}{V_T} v + \frac{1}{\omega CV_T} i \\
z & = \frac{v_0 - V_L}{V_H - V_L}.
\end{align*}$$

(5)

\footnotesize{$^1$This approximation is a routine in the electrical engineering.}

Equations (1)–(3) are transformed into (6)–(8), respectively

$$\begin{align*}
\dot{x} &= \delta x \quad \text{for } x(\tau) < 1 \quad \text{(6)} \\
(y(\tau^+), x(\tau^+)) &= (q, y(\tau) + \delta(1 - q)), \quad \text{if } x(\tau) = 1 \quad \text{(7)} \\
z(\tau) &= \begin{cases} 
1, & \text{if } x(\tau) = 1 \\
0, & \text{for } x(\tau) > 1.
\end{cases} \quad \text{(8)}
\end{align*}$$

Note that this normalized equation has two parameters: the damping $\delta$ and the base level $q$ and that $\delta$ and $q$ can be controlled easily by $-R$ and $E$, respectively. For simplicity, the two parameters are assumed to be

$$0 < \delta, q < 0.$$  

(9)

This normalized equation is derived from an equivalent circuit in Appendix: that consists of one two-port voltage-controlled current source, two capacitors, COMP and MM; and is well suited for realization on a chip.

Below the threshold $x = 1$, (6) has the piecewise exact solution

$$\begin{bmatrix} x(\tau) \\ y(\tau) \end{bmatrix} = e^{\delta \tau} \begin{bmatrix} \cos \tau & \sin \tau \\ -\sin \tau & \cos \tau \end{bmatrix} \begin{bmatrix} x(0) \\ y(0) \end{bmatrix}, \quad \text{for } x(\tau) < 1$$

(10)

where $(x(0), y(0))$ denotes an initial state vector at $\tau = 0$. Using this solution, chaotic attractor can be calculated as shown in Fig. 3.

In order to derive a one-dimensional (1-D) return map, we define some objects as shown in Fig. 4: the domain of the return map $L = \{(x, y) | x = 0, y \geq 0\}$, the threshold line $L_T = \{(x, y) | x = 1\}$ and the base line $L_q = \{(x, y) | x = q\}$. Let a point on these objects be represented by their $y$-coordinate.
We also define a key point $D$ on $L$ such that a trajectory started from $D$ touches $L_T$ at time $\tau_d$

$$D = e^{\delta \tau_d} (\sin \tau_d - \delta \cos \tau_d), \quad \tau_d = \frac{\pi}{2} + \tan^{-1} \delta.$$  

Let us consider the trajectory started from a point $y_0$ on $L$ at $\tau = 0$. If $0 < y_0 < D$, the trajectory returns to $L$ at $\tau = 2\pi$ without reaching $L_T$ and the return point $y_1$ is given by

$$y_1 = f_1(y_0) \equiv e^{2\delta \tau_d} y_0.$$  

If $D \leq y_0$, the trajectory hits the threshold $L_T$ and is reset to the base $L_q$. The hitting point $y_a$ and the reset point $y_b$ are given by

$$y_a = e^{\delta \tau_a} y_0 \cos \tau_a, \quad y_b = y_a + \delta (1 - q)$$  

where the hitting time $\tau_a$ is the root of the implicit equation

$$e^{\delta \tau_a} y_0 \sin \tau_a - 1 = 0, \quad \text{for } 0 < \tau_a \leq \tau_b.$$  

Then the trajectory restarts from $y_b$ at time $\tau_a$ and returns to $L$. The return point $y_1$ is given by

$$y_1 = f_2(y_b) \equiv e^{\delta \tau_a} (-q \sin \tau_b + y_b \cos \tau_b),$$

$$\tau_a = \frac{\pi}{2} + \tan^{-1} \frac{y_b}{q},$$

where we note $f_2(D) > 0$. Since the trajectory must return to $L$, we can define the 1-D return map

$$f: L \to L, \quad y_0 \mapsto y_1.$$  

Fig. 5 shows shapes of the return map. Using (11)-(14), the return map can be described analytically

$$y_1 = f(y_0) = \begin{cases} f_1(y_0), & \text{for } 0 < y_0 < D \\ f_2(y_0), & \text{for } D \leq y_0. \end{cases}$$

**Definition 1:** If there exists some subset $I$ in $L$ such that $f(I) \subseteq I$, then $I$ is said to be an invariant interval. If an invariant interval $I$ exists and if $|Df(y)| > 1$ is satisfied for almost all $y \in I$, $f$ on $I$ is ergodic and has a positive Lyapunov exponent [15], where $Df$ denotes the slope of $f$. In this case, $f$ is said to generate chaos.

**Theorem 1:** Let $y_r \equiv f_2(D), y_r \equiv f_1(D)$ and $I \equiv [y_r, y_r]$. $I$ is an invariant interval and $f$ generates chaos if and only if $f_2(y_r) \leq y_r$.

**Proof:** (see Fig. 5): First, if $f_2(y_r) > y_r$, the orbit of the return map diverges and an invariant interval cannot exist hence $f$ cannot generate chaos: $f_2(y_r) \leq y_r$ is a necessary condition for chaos generation.

Since $Df_1 > 1$ and $f_2(D) > 0$, $f_2(y_r) \leq y_r$ guarantees $f(I) \subseteq I$. We then introduce explicit expressions of $y_0$ and $y_r$

$$y_0 = e^{\delta \tau_0} \sqrt{1 + y_r^2}$$

$$\tau_0 = \frac{\pi}{2} - \tan^{-1} y_r$$

$$y_r = y_0 + \delta (1 - q)$$

$$y_r = e^{\delta \tau_0} \sqrt{q^2 + y_r^2}$$

$$\tau_0 = \frac{\pi}{2} + \tan^{-1} \frac{y_r}{q}$$

Using these, we can calculate the slope of $f$ for $D \leq y_0$

$$Df_2(y_0) = \frac{dy_1}{dy_0} \frac{dy_2}{dy_0} \cdot \left( \frac{dy_0}{dy_0} \right)^{-1}$$

$$= \frac{y_0 + \delta q}{y_0 e^{2\delta \tau_0}} \cdot \frac{y_0}{y_0}$$

$$= \frac{y_0 e^{2\delta \tau_0}}{y_0 e^{2\delta \tau_0} + \delta q}$$

Since $e^{2\delta (\tau_0 + \tau_a)} > 1$, $0 < y_1 < y_0$ guarantees $Df_2 > 1$. Also, it can be proven that $f_2(y_r) \leq y_r$ guarantees $f_2(y_0) \leq y_0$ (i.e., $y_1 \leq y_0$) for all $y_0 \in [D, y_r]$: If $f_2(y_0) > y_0$ for some $y_0 \in [D, y_r]$, the graph $y_0 = f_2(y_0)$ cannot intersect the line $y_1 = y_0$ for $y_0 \in [y_r, y_r]$ because the slope $Df_2$ at the intersection cannot exceed one. It guarantees $f_2(y_r) > y_r$ that contradicts $f_2(y_r) \leq y_r$. That is, $f_2(y_r) \leq y_r$ guarantees $f_2(y_0) \leq y_0$ and $Df_2(y_0) > 1$ for all $y_0 \in [D, y_r]$: $f_2(y_r) \leq y_r$ is a sufficient condition for chaos generation.

The condition $f_2(y_r) \leq y_r$ is an inequality of the parameters and is satisfied in the gray region in Fig. 6. Hereafter, we focus on the parameters in this region. Note that the length of $I$ relates
to the size of the chaotic attractor and that the corresponding chaotic attractor cannot exceed the band \( \{(x, y) \mid -e^{\pi} < x \leq 1\} \) in the phase space (see Fig. 4).

III. MASTER–SLAVE PCN

Fig. 7(a) shows a master–slave PCN consisting of two IFCs. It is a basic system of large scale PCNs. The master is an independent IFC. If the coupling switch \( S_{12} \) is opened all the time, the slave is also an independent IFC. They are described by

\[
\begin{align*}
\frac{dv_1(t)}{dt} &= i_1, \\
\frac{L_i}{dt} &= -v_1 + Ri_1, & \text{for } v_1(t) < V_T, \\
(v_1(t^+), i_1(t^+)) &= (E_1, i_1(t)), & \text{if } v_1(t) = V_T \\
v_{v1}(t) &= \begin{cases} 
V_H, & \text{if } v_1(t) = V_T \\
V_L, & \text{for } v_1(t) > V_T
\end{cases}
\end{align*}
\]

Then the master–slave pulse-coupling is realized by the compulsory firing switch \( S_{12} \): if the master voltage \( v_1 \) reaches the threshold voltage \( V_T \), \( S_{12} \) is closed and \( v_2 \) is reset to the secondary base voltage \( E_2 \) instantaneously, holding \( i_2 = \text{constant} \). That is, the following compulsory firing rule is added to the slave’s firing rule:

\[
\begin{align*}
\frac{dv_2(t)}{dt} &= i_2, \\
\frac{L_i}{dt} &= -v_1 + Ri_2, & \text{for } v_2(t) < V_T \\
(v_2(t^+), i_2(t^+)) &= (E_1, i_2(t)), & \text{if } v_2(t) = V_T \\
v_{v2}(t) &= \begin{cases} 
V_H, & \text{if } v_2(t) = V_T \\
V_L, & \text{for } v_2(t) < V_T
\end{cases}
\end{align*}
\]  

Equations (17) and (18) are transformed into the normalized equations (19) and (20) shown at the bottom of the next page. Using the following dimensionless variables and parameters:

\[
\begin{align*}
x_k &= \frac{v_k}{V_T} \\
y_k &= -\frac{\delta}{V_T} y_k + \frac{1}{\omega C V_T} z_k \\
z_k &= \frac{v_{v1} - V_L}{V_H - V_L}, & k \in \{1, 2\} \\
q &= \frac{E_1}{V_T} \\
q_2 &= \frac{E_2}{V_T}
\end{align*}
\]
we introduce a dummy slave. That is, removing the self firing threshold of the slave and applying the following similarity to the slave state variable
\[
\begin{bmatrix}
  x'_2 \\
  y'_2
\end{bmatrix} = \frac{1}{\gamma} \begin{bmatrix}
  x_2 \\
  y_2
\end{bmatrix}, \quad \gamma \equiv \frac{q_2}{q}
\]
we obtain the dummy slave
\[
\begin{bmatrix}
  x'_2 \\
  y'_2
\end{bmatrix} = \begin{bmatrix}
  \delta & 1 \\
  -1 & \delta
\end{bmatrix} \begin{bmatrix}
  x'_2 \\
  y'_2
\end{bmatrix}
\]
for \( x_1(\tau) < 1 \)
\[
(x'_2(\tau^+), y'_2(\tau^+)) = (q, y'_2(\tau) + \delta(x'_2(\tau) - q))
\]
if \( x_1(\tau) = 1. \) (21)

We consider the synchronous phenomena between the master and the dummy slave. The dummy slave has the compulsory firing only. It is a useful tool to analyze basic dynamics of the “real” master–slave PCN described by (19) and (20). Using this, we can calculate the stability of the synchronous state and clarify that the slave attractor becomes similar to the master one. Since the master’s firing causes the dummy slave’s compulsory firing, the initial states can be \((q, y_2(0))\) and \((q, y'_2(0))\) without loss of generality. Let \( \tau_n \) be the \( n \)th firing time of the master and let \( T_n \equiv \tau_n - \tau_{n-1} \). Just after the \( n \)th firing, the states of the master and the dummy slave are given by
\[
\begin{bmatrix}
  x_1(\tau_n) \\
  y_1(\tau_n)
\end{bmatrix} = e^{\beta T_n} \begin{bmatrix}
  x_1(\tau_{n-1}) \\
  y_1(\tau_{n-1})
\end{bmatrix}
\]
\[
\begin{bmatrix}
  x_2(\tau_n) \\
  y_2(\tau_n)
\end{bmatrix} = e^{\beta T_n} \begin{bmatrix}
  x_2(\tau_{n-1}) \\
  y_2(\tau_{n-1})
\end{bmatrix}
\]
where \( n \) is a positive integer and \( \tau_0 = 0 \). Using these, we define the average error expansion ratio
\[
\alpha \equiv \frac{1}{N} \sum_{n=1}^{N} \ln \alpha_n
\]
\[
\alpha_n = \frac{|y_h(\tau_n^+) - y_h(\tau_n^-)|}{|y_h(\tau_{n-1}^+) - y_h(\tau_{n-1}^-)|} = |e^{\beta T_n} (\cos T_n + \delta \sin T_n)|
\]
where \( N \) is a sufficiently large integer. Note that this quantity depends on the firing time interval of the master. Calculating (23), we have confirmed \( \alpha < 0 \) in the black region in Fig. 9 where the synchronous state is stable. If \( \alpha > 0 \) is satisfied, the error between the master and the dummy slave expands. Therefore, the dummy slave diverges and the attractor of the dummy slave cannot exist. \( \alpha < 0 \) is satisfied in the gray region in Fig. 9.

If the “dummy” slave can synchronize with the master, the “real” slave can synchronize with the master having similar ratio
\[
\begin{bmatrix}
  x'_2 \\
  y'_2
\end{bmatrix} \rightarrow \begin{bmatrix}
  x_1 \\
  y_1
\end{bmatrix}
\]
implies
\[
\begin{bmatrix}
  x_2 \\
  y_2
\end{bmatrix} \rightarrow \gamma \begin{bmatrix}
  x_1 \\
  y_1
\end{bmatrix}
\]
where the slave trajectory does not reach the threshold. Note that the master attractor cannot exceed the band \( \{(x_1, y_1) | e^{\beta \pi} < x \leq 1\} \) in the phase space as commented after Theorem 1. Therefore, if
\[
q_2 \in Q_2 \equiv \{ q_2 | -|q| < q_2 < |q|e^{3\pi} \}
\]
is satisfied, we can observe the following synchronous phenomena.

- In-phase master–slave chaos synchronization with similar ratio \( |\gamma| \equiv |q_2/q| \) for \(-|q| < q_2 < 0\).
- Antiphase master–slave chaos synchronization with similar ratio \( |\gamma| \) for \( 0 < q_2 < |q|e^{3\pi} \).

As \( q_2 \) approaches to zero, the slave attractor size also approaches to zero. If \( q_2 \notin Q_2 \), the slave trajectory reaches the threshold and we can observe the following phenomena:

- In-phase like synchronization breakdown for \( q_2 \leq -|q| \).
- Antiphase like synchronization breakdown for \( |q|e^{3\pi} \leq q_2 < 1 \).

\[
\begin{align*}
\text{Unit 1:} & \quad \begin{bmatrix}
  \dot{x}_1 \\
  \dot{y}_1
\end{bmatrix} = \begin{bmatrix}
  \delta & 1 \\
  -1 & \delta
\end{bmatrix} \begin{bmatrix}
  x_1 \\
  y_1
\end{bmatrix}, \quad \text{for} \ x_1(\tau) < 1 \\
(x_1(\tau^+), y_1(\tau^+)) &= (q, y_1(\tau) + \delta(1 - q)), \quad \text{if} \ x_1(\tau) = 1 \\
z_1(\tau) &= \begin{cases} 
1, & \text{if} \ x_1(\tau) = 1 \\
0, & \text{for} \ x_1(\tau) < 1
\end{cases} \\
\text{Unit 2:} & \quad \begin{bmatrix}
  \dot{x}_2 \\
  \dot{y}_2
\end{bmatrix} = \begin{bmatrix}
  \delta & 1 \\
  -1 & \delta
\end{bmatrix} \begin{bmatrix}
  x_2 \\
  y_2
\end{bmatrix}, \quad \text{for} \ x_2(\tau) < 1, \ \text{and} \ x_1(\tau) < 1 \\
(x_2(\tau^+), y_2(\tau^+)) &= \begin{cases} 
(q, y_2(\tau) + \delta(1 - q)), & \text{if} \ x_2(\tau) = 1 \\
(q_2, y_2(\tau) + \delta(x_2(\tau) - q_2)), & \text{if} \ x_1(\tau) = 1 \\
z_2(\tau) &= \begin{cases} 
1, & \text{if} \ x_2(\tau) = 1 \\
0, & \text{for} \ x_2(\tau) < 1
\end{cases}
\end{cases}
\end{align*}
\]
Fig. 8. Typical phenomena from the master–slave PCN in the laboratory. \(-R=1[MΩ], L=100[mH], C=2[mF], V_T=1[V], E=-0.3[V]\) (normalized parameters: \(\delta=0.07, q_0=0.3, q_2=E_0/V_T\)). (a) In-phase-like synchronization breakdown, \(E_2=-0.35[V]\). (b) In-phase synchronization of chaos, \(E_2=0.15[V]\). (c) Inhibition of the slave’s oscillation, \(E_2=0[V]\). (d) Antiphase synchronization of chaos, \(E_2=-0.15[V]\). (e) Antiphase like synchronization breakdown, \(E_2=0.35[V]\).

Fig. 9. Master–slave chaos synchronization existence region (black region) for \(N=10^4\). In the white region, master and slave attractors do not exist. In the gray region, slave attractor does not exist.

IV. RING-TYPE PCN

In this section, we consider a ring type-PCN consisting of six IFCs as illustrated in Fig. 10, where the same symbols as Fig. 7(b) are used: the unit IFCs are represented by the gray circles, the self firing rules are represented by the arrows with \(q\) and the compulsory firing rules are represented by the arrows with \(q_c\). The dynamics is described by (26), shown at the bottom of the next page. That is, each IFC is pulse-coupled with its closest neighbor units in bidirection. As the state of Unit \(i\) reaches the threshold, it is reset to the primary base level \(q\). At the same time, the states of its closest units, Unit \(j\) and Unit \(k\), are reset to the secondary base levels \(q_j\) and \(q_k\), respectively. At the resetting moment, the states of the other units are held to be constant. On the other hand, as the state of either closest unit, Unit \(j\) or Unit \(k\), reaches the threshold, the state of Unit \(i\) is reset to the secondary base level \(q_c\). Note again that each unit has two firing bases: the primary base \(q\) for the self firing and the secondary base \(q_c\) \((i=1 \sim 6)\) for the compulsory firing by the closest neighbor units. For simplicity, we focus \((\delta, q)\) on the black region in Fig. 9 and select \(q_i\) \((i=1 \sim 6)\) as the control parameters.
Fig. 11. Grouping patterns. The gray and white circles realize the masters and slaves, respectively. The arrows realize “active” self and compulsory firings, respectively.

This PCN can exhibit the grouping phenomena based on the in-phase or antiphase master–slave synchronization of chaos. Fig. 11 shows grouping patterns consisting of two groups: one master and two slaves exist in each group, the masters are located in the diagonal position of the network and the slaves are located next to their masters. In a likewise manner to the result by condition (25), we can say that the following condition is necessary for Unit \( i \) to become the slave:

\[
q_i \in Q_i \equiv \{q_i \mid -|q| < q_i < |q|e^{\delta \pi} \}.
\]  

If the Unit \( i \) is the slave, the similar ratio between the master and the Unit \( i \) is given by

\[
|\gamma| = \left| \frac{q_i}{q} \right|.
\]  

Since each slave state cannot reach the threshold, the slave cannot output the pulse signal and the pulse-coupling to the neighbor slave disappears: the slave can synchronize to the master in the same group, but cannot synchronize to the units in the other group. As a result, two synchronous groups emerge. Basically, the behavior of the PCN can be classified as the following:

- **Pattern A**: Unit 1 and 4 become the masters and the other units become the slaves.
- **Pattern B**: Unit 2 and 5 become the masters and the other units become the slaves.
- **Pattern C**: Unit 3 and 6 become the masters and the other units become the slaves.

Note that a slave unit \( i \) exhibits in-phase synchronization to the master if \(-|q| < q_i < 0\) with similar ratio \(|\gamma|\) and exhibits antiphase synchronization to the master if \(0 < q_i < |q|e^{\delta \pi}\) with similar ratio \(|\gamma|\). We then summarize the phenomena.

1) The PCN has Patterns A, B and C; and exhibits one of them depending on the initial states if

\[
q_i \in Q_i \text{ for all } i,
\]  

2) The PCN has only Pattern A if

\[
(q_1, q_4) \in \overline{Q_1} \cap Q_4 \text{ and } q_i \in Q_i \text{ for } i \in \{2, 3, 5, 6\}.
\]  

3) The PCN has only Pattern B if

\[
(q_2, q_5) \in \overline{Q_2} \cap Q_5 \text{ and } q_i \in Q_i \text{ for } i \in \{1, 3, 4, 6\}.
\]  

4) The PCN has only Pattern C if

\[
(q_3, q_6) \in \overline{Q_3} \cap Q_6 \text{ and } q_i \in Q_i \text{ for } i \in \{1, 2, 4, 5\}.
\]  

5) The PCN cannot have any grouping pattern and exhibits synchronization breakdown if conditions (29)–(32) are not satisfied.

Fig. 12 shows the typical phenomena in the numerical simulations: Unit 2 and 5 become the master and the other units synchronize to Unit 2 or 5. That is, the PCN exhibits Pattern B and the parameters satisfy condition (31). These results can be applied easily to a large scale ring-type PCN consisting of \( 3N \) IFCs. Each unit forms the synchronous groups with the closest neighbor units and the PCN can have \( N \) synchronous groups.

\[
\begin{align*}
\text{Unit } i : & \begin{bmatrix} \dot{x}_i \\ \dot{y}_i \end{bmatrix} = \begin{bmatrix} \delta & 1 \\ -1 & \delta \end{bmatrix} \begin{bmatrix} x_i \\ y_i \end{bmatrix}, & \text{for } x_i(\tau) < 1, x_j(\tau) < 1, \text{ and } x_k(\tau) < 1 \\
& \begin{cases} (q_i, y_i(\tau) + \delta(1 - q_i)), & \text{if } x_i(\tau) = 1 \\
(q_i, y_i(\tau) + \delta(x_i(\tau) - q_i)), & \text{if } x_i(\tau) = 1 \text{ or } x_k(\tau) = 1 
\end{cases} \\
z_i(\tau) = & \begin{cases} 1, & \text{if } x_i(\tau) = 1 \\
0, & \text{for } x_i(\tau) < 1 
\end{cases} \\
i \in \{1, 2, 3, 4, 5, 6\} \\
j = & (i \bmod 6) + 1 \\
k = & ((i + 4) \bmod 6) + 1.
\end{align*}
\]  

(26)
Finally, we suggest a basic application example of this ring-type PCN to an orientation detector. In the PCN, we can regard the secondary bases \( q_1 \) as inputs and the self-firing pulse-trains \( z_2 \) as outputs. If the inputs satisfy (30)–(32); the PCN exhibits Pattern A, B, and C, respectively, as shown in Fig. 11. These three synchronous patterns correspond to three orientations. If the number of the IFCs and pulse-couplings increases, the PCN can exhibit more synchronous patterns: it can detect more orientation. Such applications examples will be discussed in more detail elsewhere.

**V. CONCLUSION**

We have studied basic dynamics in a PCN based on our original IFC. The IFC has two states and exhibits chaotic attractor. Using the 1-D return map, we have given an iff condition for chaos generation. Using two IFCs, we have constructed a master–slave PCN that exhibits interesting chaos synchronous phenomena and their breakdown patterns. The existence region of each phenomenon have been elucidated in the parameter space. Then, we have developed the master–slave PCN into a ring-type PCN that exhibits grouping phenomena based on the chaos synchronization from the master–slave PCN. We have clarified basic mechanism for the grouping phenomena. This is the first analysis result on the basic dynamics in the large scale pulse-coupled network consisting of our original integrate-and-fire type chaotic oscillators. Using a simple test circuit, some of typical phenomena have been verified in the laboratory. Future problems include

1) More detailed classification of synchronous phenomena and their breakdown patterns.
2) Analysis of bifurcation phenomena of various synchronous patterns.
3) Development into a large scale chaotic PCN and its application to artificial neural networks with flexible functions.

**APPENDIX**

Fig. 13 shows the unit IFC consisting of one two-port voltage-controlled current source (ab. 2PVCCS), two capacitors, one comparator (COMP) and one monostable multivibrator (MM). The operation of the firing switch S is the same as that in Fig. 1. The 2PVCCS is made of the linear OTA pair and is characterized by

\[
\begin{bmatrix}
  v_1 \\
  v_2
\end{bmatrix} =
\begin{bmatrix}
  0 & g \\
  -g & g
\end{bmatrix}
\begin{bmatrix}
  v_1 \\
  v_2
\end{bmatrix},
\]

Then, the circuit dynamics is described by

\[
\begin{align*}
\frac{d}{dt} \begin{bmatrix}
  C_1 & v_1 \\
  C_2 & v_2
\end{bmatrix} &= \begin{bmatrix}
  0 & g \\
  -g & g
\end{bmatrix} \begin{bmatrix}
  v_1 \\
  v_2
\end{bmatrix}, & \text{for } v_i(t) < V_T \\
(v_1(t^+), v_2(t^+)) &= (V_L, v_2(t)), & \text{if } v_i(t) = V_T \\
v_0(t) &= \begin{cases}
  V_H, & \text{if } v(t) = V_T \\
  V_L, & \text{if } v(t) > V_T.
\end{cases}
\end{align*}
\]
We assume that (34) has unstable complex characteristic root
\[ \delta \omega \pm j\omega \]
\[ \omega^2 = \frac{g^2}{C_1 C_2} - \left( \frac{g}{2\omega C_2} \right)^2 > 0, \quad \delta = \frac{g}{2\omega C_2} > 0. \] (35)
The state vector \((v_1, v_2)\) can vibrate divergently below the threshold \(V_T\). Using the dimensionless variables and parameters
\[ x = \frac{v_1}{V_T}, \quad y = \frac{1}{V_T} \left( -\delta v_1 + \frac{g}{\omega C_4} v_2 \right), \quad z = \frac{v_0 - V_L}{V_H - V_L}, \quad q = \frac{E}{V_T}. \] (36)
Equation (34) is transformed into (6)–(8). That is, the circuit in Fig. 13 is equivalent to that in Fig. 1. This equivalent circuit is well suited for realization on a chip.

REFERENCES

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