On the Domination Number of Cartesian Products of Two Directed Paths

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Abstract. Let \( D = (V, A) \) be a directed graph of order \( p \). A subset \( S \) of the vertex set \( V(D) \) is a dominating set of \( D \) if for each vertex \( v \in V(D) \) there exists a vertex \( u \in S \) such that \( (u, v) \) is an arc of \( D \). The domination number of \( D \), \( \gamma(D) \), is the order of a smallest dominating set of \( D \). In this paper we calculate the domination number of the cartesian product of two directed paths \( P_m \) and \( P_n \) for general \( m \) and \( n \).

Keywords: Graphs, Directed graphs, directed paths, Cartesian product, domination number

1. Introduction

All directed graphs in this paper are finite without loops and multiple arcs (shortly digraph). Let \( D \) be a digraph, \( V(D) \) and \( A(D) \) refer to the vertex and arc sets respectively. For all \( u \in V(D) \), the sets \( O(u) = \{ v : (u, v) \in A(D) \} \) and \( I(u) = \{ v : (v, u) \in A(D) \} \) are called the outset and inset of \( u \), respectively. We refer to outdegree \( |O(u)| \) by \( d^+(u) \) and indegree \( |I(u)| \) by \( d^-(u) \). The maximum outdegree and indegree of all vertices in \( D \) are denoted by \( \Delta^+(D) \) and \( \Delta^-(D) \), respectively. The minimum outdegree and indegree of all vertices in \( D \) are denoted by \( \delta^+(D) \) and \( \delta^-(D) \), respectively. For all \( H \subseteq V(D) \), the outset of \( H \) is defined as \( O(H) = \bigcup_{u \in H} O(u) \). Similarly, the inset of \( H \) is defined as \( I(H) = \bigcup_{u \in H} I(u) \). The closed outset of \( H \) is \( O[H] = O(H) \cup H \). The closed inset of \( H \) is \( I[H] = I(H) \cup H \). Related to definition an outset \( H \), we can say \( S \subseteq V(D) \) is dominating set of \( D \), if and only if \( O[S] = V(D) \). The size of the smallest dominating set of \( D \) is domination number \( \gamma(D) \).

The cartesian product \( D_1 \times D_2 \) of two digraphs \( D_1 \) and \( D_2 \) is the digraph with vertex set \( V(D_1 \times D_2) = V(D_1) \times V(D_2) \) and \( ((u_1,v_1),(u_2,v_2)) \in A(D_1 \times D_2) \) if and only if \( u_1 = v_2 \) and \((u_2, v_1) \in A(D_2) \) or \( u_2 = v_2 \) and \((u_1, v_1) \in A(D_1) \).

For finding domination number of grid graphs, Jacobson and Kinch in [7], were calculated the domination number of cartesian product of undirected paths \( P_m \) and \( P_n \).
for \( m = 1, 2, 3, 4 \). The cases \( m = 5, 6 \) were calculated by Chang and Clark [14]. Also, Chang et al., [13], established the upper bounds of cartesian product of undirected paths \( P_m \) and \( P_n \) for \( 5 \leq m \leq 10 \) and arbitrary \( n \). In [10], Gravier and Mollard gave un upper and lower bounds of general cartesian product of two undirected paths.

Determination of the domination number of digraphs seems to be a more difficult problem than the domination number of undirected graphs. Whereas the model is more general, dominating sets for digraphs have more applications than undirected graphs. Dominating sets of small cardinality are the most interesting. For a general digraph, finding a minimum dominating set is NP-hard, see [11] (which follow from a simple reduction from the undirected case). Some approximation results can be found in [6]. For a summary of results known about the domination number of a digraph, see Ghosal et al., [3], Merz, Stewart [12], Langley et al., [5] and Haines et al., [15]. There are a few results about domination number in digraphs can be found in [16], [2] and [1]. Of particular interest are the following on the lower and the upper bounds of a dominating set of an arbitrary digraph \( D \) on \( p \) vertices. Here, we mention of some previous results. In [8], Shaheen established the domination number of the cartesian product of directed cycles \( C_m \times C_n \) for \( m = 3, 4, 5, 6 \) and arbitrary \( n \). Liu et al., [4], they studied the domination number of \( P_m \times P_n \) for \( m = 2, 3, 4, 5, 6 \) and arbitrary \( n \). Also, in [9] we studied the domination number of \( P_m \times P_n \) when \( m = 7, 8 \) and arbitrary \( n \). In [4] and [9], the following results are proved.

**Theorem 1.1** [4]:
\[
\gamma(P_1 \times P_n) = \gamma(P_n) = \left\lceil \frac{n}{2} \right\rceil, \quad \gamma(P_2 \times P_n) = n, \quad \gamma(P_3 \times P_n) = n + \left\lceil \frac{n}{4} \right\rceil, \quad \gamma(P_4 \times P_n) = n + \left\lceil \frac{2n}{3} \right\rceil, \quad \gamma(P_5 \times P_n) = 2n + 1, \quad \gamma(P_6 \times P_n) = 2n + \left\lceil \frac{n+2}{3} \right\rceil.
\]

**Theorem 1.2** [9]:
\[
\gamma(P_7 \times P_n) = 2n + \left\lceil \frac{2n}{3} \right\rceil + 1, \quad \gamma(P_8 \times P_n) = 3n + 2.
\]

2. Main results

We study dominating sets in grid digraphs using one technique: by given a minimum of upper dominating set \( S \) of \( P_m \times P_n \) and then we establish that \( S \) is a minimum dominating set of \( P_m \times P_n \) for arbitrary \( m \) and \( n \). Definitely, we have \( \gamma(P_m \times P_n) = |S| \).

The \( j \)'th column of \( V(P_m \times P_n) \) is \( K_j = \{(i, j) : i = 1, \ldots, m\} \). If \( S \) is a dominating set for \( P_m \times P_n \), then we denote \( W_j = K_j \cap S \). Let \( s_j = |W_j| \), where the sequence \((s_1, \ldots, s_m)\) is called a dominating sequence corresponding to \( S \). Also, the \( i \)'th row of \( V(P_m \times P_n) \) is \( R_i = \{(i, j) : j = 1, \ldots, n\} \). For \( 2 \leq i \leq m \), the vertices of \( i \)-row are dominated by vertices of \( i \)-row or \((i-1)\)-row.

**Remark 2.1:** For any minimum dominating set \( S \) for \( P_m \times P_n \), we have \((1, 1) \in S \) and \( s_1 \geq \left\lceil \frac{m}{2} \right\rceil \). Furthermore \( s_1 + s_2 \geq m \).

**Proof.** Since \( d'(1, 1) = 0 \), always we need \((1, 1) \in S \). The vertices of the first column of \( P_m \times P_n \) can only dominated by themselves and each vertex can dominate two vertices. So, we necessarily need \( \left\lceil \frac{m}{2} \right\rceil \). Also, we have \((1, 1) \in K_1 \cap S \) and for each
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(i, 1) ∈ K₁ ∩ S we need (i + 1, 1) ∈ K₁ ∩ S or (i + 1, 2) ∈ K₂ ∩ S. Hence, |K₁ ∩ S| + |K₂ ∩ S| = s₁ + s₂ ≥ m.

Lemma 2.2. For m, n ≥ 2 and m > k + 1, we have
\[ \gamma(P_m \times P_n) \geq \gamma(P_k \times P_n) + \gamma(P_{m-k-1} \times P_n), \]
where k ≥ 1.

Proof. Let S be a minimum dominating set of P_m × P_n. By moving the vertices of R_{k+1} ∩ S to R_{k+2} ∩ S we get S is a dominating set of P_k × P_n and P_{m-k-1} × P_n with complete dominated. □

Lemma 2.3. For m, n ≥ 4, we have
\[ \gamma(P_m \times P_n) \geq \gamma(P_2 \times P_n) + \gamma(P_{m-3} \times P_n) + 1. \]

Proof. From Lemma 2.2 we have \( \gamma(P_m \times P_n) \geq \gamma(P_2 \times P_n) + \gamma(P_{m-3} \times P_n) \). Let S be a minimum dominating set of P_m × P_n, where that
\[ \sum_{i=1}^{m} R_i \cap S. \]
It is clear that
\[ \gamma(P_2 \times P_n) \leq \sum_{i=1}^{m} R_i \cap S \]
and \( \gamma(P_{m-3} \times P_n) \leq \sum_{i=1}^{m} R_i \cap S \). We will prove that \( \gamma(P_2 \times P_n) \leq |\sum_{i=1}^{m} R_i \cap S| - 1 \) or \( \gamma(P_{m-3} \times P_n) \leq |\sum_{i=1}^{m} R_i \cap S| - 1 \). Suppose that \( \gamma(P_2 \times P_n) = |\sum_{i=1}^{m} R_i \cap S| \) holds.

Then we must prove the second inequality. By addition the vertices R_{i} ∩ S to R_{i+1} ∩ S, the resulting set is dominating set of P_{m-3} × P_n. If m = 4 or n = 4, then by (1), (2) and (4) gets the required. Let m, n ≥ 5, so we distinguish the following cases:

Case 1. If \{(3, j), (4, j)\} ⊂ S, \{(3, j), (3, j +1)\} ⊂ S, or \{(3, j), (4, j -1)\} ⊂ S, then we can delete (3, j), (3, j +1), (3, j) respectively, where 1 ≤ j, j -1 ≤ n.

Assume that the Case 1 does not exist. Then we have another two cases:

Case 2. (3, 1) ∉ S, then \{(2, 1),(4, 1)\} ⊂ S. To dominate (3, 2) we must have at least one of \{(2, 2), (3, 2)\} belongs to S. If (2, 2) ∈ S, we get a contradiction with \( \gamma(P_2 \times P_n) = |\sum_{i=1}^{m} R_i \cap S| \), because \{(1, 1), (2, 1), (2, 2)\} belongs to S. And if (3, 2) ∈ S, this is analogous to Case 1.

Case 3. Let (3,1) ∈ S, then if one of \{(4,1), (3,2)\} belongs to S, then we are in Case 1. Assume that no one of \{(4,1), (3,2)\} belongs to S. This implies that (4, 2) and (5, 1) are in S. If (4, 3) ∈ S then we can delete (4, 2) after moving (3, 1) ∈ S to (4, 1) ∈ S. Also, if (3, 3) ∈ S, then we are in Case 1. Otherwise, (2, 3) ∈ S to dominate (3, 3). Therefore, (3, 4), (4, 4) are not dominated and so we have the following subcases:

Subcase 3.1. If (3, 4) ∈ S, then we can move (4, 2) to (4, 3) and delete (3,4), (because (4, 1) ∈ S instead (3,1) ∈ S).

Subcase 3.2. If \{(2, 4), (4, 4)\} ⊂ S, then (3, 5) is not dominated, so at least one of
\{(2, 5), (3, 5)\} belongs to \(S\). If \(2,5) \in \mathcal{S}\), this is a contradiction with \(\gamma(P_2 \times P_n) = \sum_{i=1}^{n} R_i \cap S\), (because (2, 4) is not necessary). Also, if \(3, 5) \in \mathcal{S}\), then we can delete it, (because \{(4, 4), (3, 5)\} are in \(S\).

Finally, deduced that in all the cases we can delete at least one vertex from \(\sum_{i=1}^{n} R_i \cap S\) and the resulting set is still a dominating set of \(P_{m-3} \times P_n\). Thus,

\[\gamma(P_{m-3} \times P_n) \leq \left| \sum_{i=3}^{n} R_i \cap S \right| - 1. \]

**Note 2.4.** \([\lfloor m/2 \rfloor + \lfloor m/2 \rfloor] = m, \]
\([\lfloor m/3 \rfloor + \lfloor m/6 \rfloor] = \lfloor m/2 \rfloor\) for \(m \equiv 0, 2 \pmod{3}\).
\([\lfloor m/3 \rfloor + \lfloor (m-1)/6 \rfloor] - 1 = \lfloor m/2 \rfloor\) for \(m \equiv 1 \pmod{3}\).
\([\lfloor (n - 2)/3 \rfloor + \lfloor (n - 3)/3 \rfloor + \lfloor (n - 4)/3 \rfloor = n - 2\) for \(n \equiv 0, 1, 2 \pmod{3}\).
\([\lfloor (n - 3)/3 \rfloor + \lfloor (n - 4)/3 \rfloor] = \lfloor 2n/3 \rfloor - 2\).

**Theorem 2.5.** For \(m, n \geq 4\), we have

\[\gamma(P_m \times P_n) = m + \left\lfloor (m - 1)/3 \right\rfloor \left\lfloor (n - 2)/3 \right\rfloor + \left\lfloor m/3 \right\rfloor \left\lfloor (n - 3)/3 \right\rfloor + \left\lfloor (m + 1)/3 \right\rfloor \left\lfloor (n - 4)/3 \right\rfloor.\]

**Proof.** We define the following sets:

\(S_1 = \{(2i - 1, 1) : 1 \leq i \leq \lfloor m/2 \rfloor - 1, (n - 1, 1)\} \cup \{(3i, 2) : 1 \leq i \leq \lfloor m/3 \rfloor, (6i - 4, 2) : 1 \leq i \leq \lfloor m/6 \rfloor\} \cup \{(3i - 2, 3j) : 1 \leq i \leq \lfloor m/3 \rfloor \text{ and } 1 \leq j \leq \lfloor (n - 2)/3 \rfloor\} \cup \{(3i - 1, 3j + 1) : 1 \leq i \leq \lfloor m/3 \rfloor \text{ and } 1 \leq j \leq \lfloor (n - 3)/3 \rfloor\} \cup \{(1, 3j + 2), (3i, 3j + 2) : 1 \leq i \leq \lfloor m/3 \rfloor \text{ and } 1 \leq j \leq \lfloor (n - 4)/3 \rfloor\}.\)

\(S_2 = \{(2i - 1, 1) : 1 \leq i \leq \lfloor m/2 \rfloor - 1, (n - 1, 1)\} \cup \{(3i - 2, 2) : 1 \leq i \leq \lfloor m/3 \rfloor, (6i - 2, 2) : 1 \leq i \leq \lfloor (m - 1)/6 \rfloor - 1\} \cup \{(3i - 1, 3j) : 1 \leq i \leq \lfloor m/3 \rfloor \text{ and } 1 \leq j \leq \lfloor (n - 2)/3 \rfloor\} \cup \{(1, 3j + 1), (3i, 3j + 1) : 1 \leq i \leq \lfloor m/3 \rfloor \text{ and } 1 \leq j \leq \lfloor (n - 3)/3 \rfloor\} \cup \{(3i - 2, 3j + 2) : 1 \leq i \leq \lfloor m/3 \rfloor \text{ and } 1 \leq j \leq \lfloor (n - 4)/3 \rfloor\}.\)

\(S_3 = \{(2i - 1, 1) : 1 \leq i \leq \lfloor m/2 \rfloor - 1, (n - 1, 1)\} \cup \{(3i - 2, 2) : 1 \leq i \leq \lfloor m/3 \rfloor, (6i - 2, 2) : 1 \leq i \leq \lfloor (m - 6)/2 \rfloor - 1\} \cup \{(3i - 1, 3j) : 1 \leq i \leq \lfloor m/3 \rfloor \text{ and } 1 \leq j \leq \lfloor (n - 2)/3 \rfloor\} \cup \{(1, 3j + 1), (3i, 3j + 1) : 1 \leq i \leq \lfloor m/3 \rfloor \text{ and } 1 \leq j \leq \lfloor (n - 3)/3 \rfloor\} \cup \{(3i - 2, 3j + 2) : 1 \leq i \leq \lfloor m/3 \rfloor \text{ and } 1 \leq j \leq \lfloor (n - 4)/3 \rfloor\}.\)

See Figure 1, for illustration the sets \(S_1, S_2, \text{ and } S_3\). We note that \(S_1, S_2, \text{ and } S_3\) are dominating sets of \(P_m \times P_n\) for \(m \equiv 0 \pmod{3}, m \equiv 1 \pmod{3}\) and \(m \equiv 2 \pmod{3}\), respectively. Where, each one of \(S_1\), for \(i = 1, 2, 3\) has the following form:

\(S_1 = \{(\lfloor m/2 \rfloor, \lfloor m/2 \rfloor, \lfloor m/3 \rfloor, \lfloor m/3 \rfloor, [m + 3]/3, \ldots, [m/3], [m/3], \lfloor m + 3)/3 \rfloor, \ldots\}.\)

\(S_2 = \{(\lfloor m/2 \rfloor, \lfloor m/2 \rfloor, \lfloor m/3 \rfloor, \lfloor m/3 \rfloor, [m/3], [m/3], \ldots, \lfloor m/3 \rfloor, [m/3], \ldots\}.\)

\(S_3 = \{(\lfloor m/2 \rfloor, \lfloor m/2 \rfloor, \lfloor m/3 \rfloor, \lfloor m/3 \rfloor, \ldots, \lfloor m/3 \rfloor, [m/3], \ldots\}.\)

We can write one form for the three sets \(S_1, S_2, \text{ and } S_3\) as follows:

\(S : \{(\lfloor m/2 \rfloor, \lfloor m/2 \rfloor, \lfloor m - 3)/3 \rfloor, \lfloor m/3 \rfloor, \lfloor m/3 \rfloor, \lfloor m/3 \rfloor, \lfloor m/3 \rfloor, \lfloor m/3 \rfloor, \ldots\}.\)

This implies that

\[\gamma(P_m \times P_n) \leq m + \lfloor (m - 1)/3 \rfloor \lfloor (n - 2)/3 \rfloor + \lfloor m/3 \rfloor \lfloor (n - 3)/3 \rfloor + \lfloor (m + 1)/3 \rfloor \lfloor (n - 4)/3 \rfloor. \]

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By applying (9) for \( m \equiv 0, 1, 2 \pmod{3} \), we deduce that:
\[
\gamma(P_m \times P_n) \leq m(n + 1)/3 + \lceil (n - 4)/3 \rceil \quad \text{for } m \equiv 0 \pmod{3}.
\]
\[
\gamma(P_m \times P_n) \leq (mn + m - n + 2)/3 + \lceil (n - 3)/3 \rceil + \lceil (n - 4)/3 \rceil
\quad = (mn + m - n - 4)/3 + \lceil 2n/3 \rceil \quad \text{for } m \equiv 1 \pmod{3}.
\]
\[
\gamma(P_m \times P_n) \leq (mn + m + n - 2)/3 \quad \text{for } m \equiv 2 \pmod{3}.
\] (10) (11) (12)

The upper bounds gave by (10), (11) and (12). So, we need the lower bound. By Lemma 2.3, we get
\[
\gamma(P_m \times P_n) \geq \gamma(P_2 \times P_n) + \gamma(P_{m-3} \times P_n) + 1.
\]

Then, by repeating this processing gets
\[
\gamma(P_m \times P_n) \geq \gamma(P_2 \times P_n) + \gamma(P_{m-3} \times P_n) + 1
\quad \geq 2 \gamma(P_2 \times P_n) + \gamma(P_{m-6} \times P_n) + 2 \geq \ldots \geq k \gamma(P_2 \times P_n) + \gamma(P_{m-3k} \times P_n) + k, \quad \text{where } m - 3k > 0.
\]

Hence, we distinguish three cases.

For \( m \equiv 0 \pmod{3} \).
If \( m = 6 \), then by Theorem 1.1(6) is the required. Assume that \( m > 6 \), then
\[
\gamma(P_m \times P_n) \geq ((m - 6)/3) \gamma(P_2 \times P_n) + \gamma(P_6 \times P_n) + (m - 6)/3 = ((m - 6)/3) n + 2 n + \lceil (n + 2)/3 \rceil + (m - 6)/3 = mn + 1/3 + (n - 4)/3.
\] (13)

For \( m \equiv 1 \pmod{3} \).
If \( m = 4 \), then by Theorem 1.1(4) is the required. Assume that \( m > 4 \), then
\[
\gamma(P_m \times P_n) \geq ((m - 4)/3) \gamma(P_2 \times P_n) + \gamma(P_4 \times P_n) + (m - 4)/3 = ((m - 4)/3) n + n + \lceil 2n/3 \rceil + (m - 4)/3 = (mn + m + n - 4)/3 + \lceil 2n/3 \rceil.
\] (14)

For \( m \equiv 2 \pmod{3} \).
\[
\gamma(P_m \times P_n) \geq ((m - 2)/3) \gamma(P_2 \times P_n) + \gamma(P_2 \times P_n) + (m - 2)/3 = ((m - 2)/3) n + n + (m - 2)/3
\quad = (mn + m + n - 2)/3.
\] (15)

By applying (10) together with (13), (11) together with (14) and (12) together with (15), we get
\[
\gamma(P_m \times P_n) = mn + 1/3 + (n - 4)/3 \quad \text{for } m \equiv 0 \pmod{3}.
\]
\[
\gamma(P_m \times P_n) = (mn + m - n - 4)/3 + \lceil 2n/3 \rceil \quad \text{for } m \equiv 1 \pmod{3}.
\]
\[
\gamma(P_m \times P_n) = (mn + m + n - 2)/3 \quad \text{for } m \equiv 2 \pmod{3}.
\] This completes the proof. □
Acknowledgement

The author would like to thank the Saudi Basic Industries Corporation (SABIC) for supporting this research through the Institute of Research and Consulting Studies University of Umm Al-Qura

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Received: April, 2012