ON ENTROPIES FOR RANDOM PARTITIONS OF THE UNIT SEGMENT

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We prove the complete convergence of Shannon’s, paired, genetic and $\alpha$-entropy for random partitions of the unit segment. We also derive exact expressions for expectations and variances of the above entropies using special functions.

Keywords: paired entropy, genetic entropy, $\alpha$-entropy, random partitions, complete convergence

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1. INTRODUCTION

Entropy is a measure of uncertainty or information. Shannon’s entropy

$$H(p_0, \ldots, p_k) = -\sum_{i=0}^{k} p_i \log p_i$$

(cf. [21]) for probabilities $p_0, \ldots, p_k$, $\sum_{i=0}^{k} p_i = 1$, is the most common used measure of randomness. There are known many generalizations of this entropy (cf. [18]). Burbea and Rao [3] introduced $\phi$-entropy defined as

$$D^\phi_k = \sum_{i=0}^{k} \phi (p_i)$$

where $(p_0, \ldots, p_k)$ is a probability distribution and $\phi$ is a twice differentiable real function on $(0,1)$. Special cases of $\phi$-entropy are

- Shannon’s entropy if $\phi(x) = -x \log x$,
- paired entropy if $\phi(x) = -x \log x - (1 - x) \log(1 - x)$,
- genetic entropy if $\phi(x) = x - x^2 - x^2(1 - x)^2$ (cf. [17]).

Menendez et al. [18] generalized $\phi$-entropy and defined a family of $(h, \phi)$-entropies,

$$D^{(h,\phi)}_k = h \left( \sum_{i=0}^{k} \phi (p_i) \right),$$
where \( h : \mathbb{R} \to \mathbb{R} \) and \( \phi : (0, 1) \to \mathbb{R} \) are twice differentiable real functions. Examples of \((h, \phi)\)-entropies are as follows

- \( \alpha \) degree entropy if \( \phi(x) = x^\alpha \) and \( h(x) = \frac{x^{1-\alpha} - 1}{1-\alpha} \), \( \alpha > 0 \), \( \alpha \neq 1 \) (cf. [14]),
- \( \alpha \) order entropy if \( \phi(x) = x^\alpha \) and \( h(x) = \log x \), \( \alpha > 0 \), \( \alpha \neq 1 \) (cf. [20]).

The Shannon entropy of spacings is the quantity

\[
D_k^S = - \sum_{j=0}^{k} Y_{j,k} \log Y_{j,k},
\]

where \( Y_{j,k} = X_{j+1,k} - X_{j:k}, \quad 0 \leq j \leq k, \) and \( 0 \leq X_{1:k} \leq \ldots \leq X_{k:k} \leq 1, \) \( X_{0:k} = 0 \) and \( X_{k+1:k} = 1, \) are order statistics of a sample \((X_1, \ldots, X_n)\) from a distribution \( F. \) The asymptotic behaviour of Shannon’s entropy \( D_k^S \), when \( F \) is the uniform distribution, was studied in [7], [22] and [23]. Goldstein [7] proved that the sequence \((D_k^S - \log(k + 1))\) converges to \( \gamma - 1 \) in probability as \( k \to \infty \), where \( \gamma = 0.577215 \ldots \) is the Euler–Mascheroni constant. Slud [23] showed that the convergence holds almost surely. Shao and Jimenez [22] proved that if the spacings come from a continuous distribution \( F \) then the almost sure convergence of \( D_k^S \) to \( \gamma - 1 \) characterizes uniform distribution among continuous distributions. Some related problems were investigated in Ekstörn [10], Hall [11], [12] and Misra [19]. In that papers limit theorems and tests of uniformity for sums of \( m \)th spacings were considered. We are interested in the complete convergence of Shannon’s entropy of spacings \( D_k^S \).

Recall that a sequence \((X_n)\) converges completely to \( X \) if for all \( \varepsilon > 0 \)

\[
\sum_{n=1}^{\infty} \Pr (|X_n - X| > \varepsilon) < \infty \quad (\text{cf. [16]}).
\]

We consider also the complete convergence of paired, genetic and \( \alpha \)-entropy of spacings. Namely, we investigate the asymptotic behaviour of \( D_k^S \),

\[
D_k^P = - \sum_{j=0}^{k} (Y_{j,k} \log Y_{j,k} + (1 - Y_{j,k}) \log (1 - Y_{j,k}))
\]

\[
D_k^G = \sum_{j=0}^{k} \left( Y_{j,k} (1 - Y_{j,k}) - Y_{j,k}^2 (1 - Y_{j,k})^2 \right)
\]

and

\[
D_k^\alpha = \frac{1}{2^{1-\alpha} - 1} \left( \sum_{j=0}^{k} Y_{j,k}^\alpha - 1 \right), \quad \alpha > 0, \quad \alpha \neq 1,
\]

where \( Y_{j,k} \) are uniform spacings, i.e. \( F \) is uniform distribution on \((0, 1)\).

The paper is organized as follows. In Section 2 we present definitions and auxiliary results containing some formulae for sums and integrals. The explicit expressions for expectations and variances of the above entropies are given in Section 3. Finally, Section 4 is devoted to the complete convergence of \( D_k^S, D_k^P, D_k^G \) and \( D_k^\alpha \).
2. PRELIMINARIES

Denote by \( H_k^{(r)} \), \( r \in \mathbb{N} \), \( k \in \mathbb{N} \), the harmonic number of order \( r \), i.e.

\[
H_k^{(r)} = \sum_{i=1}^{k} \frac{1}{i^r}, \quad r \geq 1
\]

(cf. [9]). For simplicity we write \( H_k := H_k^{(1)} \). For \( r > 1 \) we use Riemann \( \zeta \)-function and Hurwitz generalized \( \zeta \)-function defined, respectively, by

\[
\zeta(r) = H_\infty^{(r)} = \sum_{i=1}^{\infty} \frac{1}{i^r}, \quad \zeta(r; q) = \zeta(r) - H_q^{(r)} = \sum_{i=0}^{\infty} \frac{1}{(i + q)^r}, \quad q \geq 1.
\]

We also use the relation between the harmonic numbers, Psi (or Digamma) function

\[
\psi(x) = \frac{d}{dx} \log \Gamma(x)
\]

and the derivatives of Psi function (or Polygamma functions).

Here \( \Gamma(x) = \int_{0}^{\infty} t^{x-1} e^{-t} dt, \; x > 0 \), is the Gamma function. It is known that

\[
H_k := H_k^{(1)} = \gamma + \psi(k + 1) \tag{1}
\]

(cf. [8]) and for \( r \geq 2 \)

\[
H_k^{(r)} = \frac{(-1)^r}{(r-1)!} \left( \psi^{(r-1)}(1) - \psi^{(r-1)}(k + 1) \right)
\]

as

\[
\zeta(r; q) = \frac{(-1)^r}{(r-1)!} \psi^{(r-1)}(q).
\]

By \( B(x,y) \) we denote Beta function, i.e.

\[
B(x, y) = \int_{0}^{1} t^{x-1}(1 - t)^{y-1} dt, \quad x > 0, \quad y > 0. \tag{2}
\]

\((\lambda)_r\) denotes the Pochhammer symbol (or the shifted factorial) defined by

\[
(\lambda)_r := \frac{\Gamma(\lambda + r)}{\Gamma(r)} = \begin{cases} 1 & r = 0, \\ \lambda(\lambda + 1) \ldots (\lambda + r - 1) & r \in \mathbb{N}. \end{cases}
\]

Also let \( \beta(x) \) be the function defined by

\[
\beta(x) := \frac{1}{2} \left[ \psi \left( \frac{x + \frac{1}{2}}{2} \right) - \psi \left( \frac{x}{2} \right) \right] = \sum_{j=0}^{\infty} \frac{(-1)^j}{j + x}, \quad x > 0,
\]

and

\[
\beta'(x) = -\sum_{j=0}^{\infty} \frac{(-1)^j}{(j + x)^2} \quad (\text{cf. [8]}).
\]

We need the following lemmas.
Lemma 1. The following summation formulae hold true

\[
\sum_{j=1}^{k} \binom{k}{j} \frac{(-1)^{j+1}}{j} = \psi(k + 1) + \gamma \quad (\text{cf. [8], 0.155.4}),
\]

\[
\sum_{j=0}^{k} \binom{k}{j} (-1)^{j} H_{j+r} = -B(k, r + 1), \quad r \in \mathbb{N} \quad (\text{cf. [25]}),
\]

\[
\sum_{j=1}^{k+1} \frac{(-1)^{j-1}}{j^2} = \frac{\pi^2}{12} + (-1)^{k} \beta'(k + 3) \quad (\text{cf. [13], 5.12.50}),
\]

\[
\sum_{j=1}^{k+1} \frac{H_{j}}{j} = \frac{1}{2} (\psi(k + 2) + \gamma)^2 + \frac{1}{2} \left( \frac{\pi^2}{6} - \psi'(k + 2) \right) \quad (\text{cf. [9], 6.71}),
\]

\[
\sum_{j=1}^{\infty} \frac{1}{j(k + 1 + j)^2} = \frac{\psi(k + 2) + \gamma - \psi'(k + 2)}{(k + 1)^2} - \frac{1}{k + 1}, \quad k \in \mathbb{N} \quad (\text{cf. [13], 6.1.82}).
\]

Lemma 2.

\[
\sum_{j=0}^{k} \binom{k+1}{j} \frac{(-1)^{j}}{(k + 1 - j)(j + 1)} = \frac{(-1)^{k} (\psi(k + 3) + \gamma)}{k + 2} + \frac{1}{(k + 2)^2},
\]

\[
\sum_{j=0}^{k+1} \binom{k+2}{j} \frac{(-1)^{j} H_{j}}{k + 2 - j} = (-1)^{k+1} \left( (\gamma + \psi(k + 3))^2 - \psi'(k + 3) \right) + 2\beta'(k + 3),
\]

\[
\sum_{j=0}^{k-2} \binom{k-2}{j} \frac{(-1)^{j}}{(k - j)^2(j + 1)(j + 2)} = \frac{(-1)^{k} (\gamma + \psi(k + 3) - 1)}{k(k^2 - 1)(k + 2)}
\]

\[
+ \frac{1}{(k^2 - 1)(k + 1)} - \frac{1}{k(k - 1)(k + 2)^2},
\]

\[
\sum_{j=0}^{k-2} \binom{k-2}{j} \frac{(-1)^{j} H_{j+2}}{(k - j)(j + 1)(j + 2)} = -\frac{2(k + 2)\beta'(k + 3) + 1}{k(k^2 - 1)(k + 2)^2} + \frac{1}{(k - 1)(k + 1)^2}
\]

\[
+ \frac{(-1)^{k} \left( (\gamma + \psi(k + 3))^2 - (\gamma + \psi(k + 3) - \psi'(k + 3)) \right)}{k(k^2 - 1)(k + 2)}.
\]

Proof. To prove (8) we use

\[
\frac{1}{(k + 1 - j)(j + 1)} = \frac{1}{k + 2} \left( \frac{1}{k + 1 - j} + \frac{1}{j + 1} \right)
\]
and next by (3) and (1)

\[\sum_{j=0}^{k} \binom{k+1}{j} \frac{(-1)^j}{(k+1-j)(j+1)} = \frac{1}{k+2} \sum_{j=0}^{k} \binom{k+1}{j} \frac{(-1)^j}{k+1-j} + \frac{1}{k+2} \sum_{j=0}^{k+1} \binom{k+1}{j} \frac{(-1)^j}{j+1} + \frac{(-1)^k}{(k+2)^2}\]

\[= \frac{(-1)^k}{k+2} \sum_{j=1}^{k+1} \binom{k+1}{j} \frac{(-1)^{j+1}}{j} + 1 + \frac{(-1)^k}{(k+2)^2} = \frac{(-1)^k \psi(k+3) + \gamma}{k+2} + \frac{1}{(k+2)^2}.\]

Now we prove (9). Let

\[S_k := \sum_{j=0}^{k} \binom{k+1}{j} \frac{(-1)^j H_j}{k+1-j}.\]

Using

\[\binom{k+2}{j} = \binom{k+1}{j} + \binom{k+1}{j-1}, \quad j = 0, \ldots, k+1, \quad \binom{k}{-1} = 0,\]

we get

\[S_{k+1} = \frac{k+1}{k+2} \sum_{j=0}^{k+1} \binom{k+2}{j} \frac{(-1)^j H_j}{k+1-j} + \sum_{j=0}^{k+1} \binom{k+1}{j-1} \frac{(-1)^j H_j}{k+2-j}\]

\[= \frac{k+2}{k+2} \sum_{j=0}^{k+2} \binom{k+2}{j} \frac{(-1)^j H_j}{k+1-j} - \frac{(-1)^k H_{k+2}}{k+2} - \sum_{j=0}^{k} \frac{\binom{k+1}{j} (-1)^j}{(j+1)(k+1-j)} - S_k.\]

Then by (4) and (8) we get the recurrence relation

\[S_{k+1} = -\frac{2}{(k+2)^2} + \frac{2(-1)^k H_{k+2}}{k+2} - S_k, \quad k = 0, 1, \ldots\]

where \(S_0 = 0.\) Hence

\[S_{k+1} = 2(-1)^k \sum_{j=1}^{k+2} \frac{(-1)^{j-1}}{j^2} - 2(-1)^k \sum_{j=1}^{k+2} \frac{H_j}{j},\]

which by (6) and (5) gives (9).

Now consider (10). We see that

\[\sum_{j=0}^{k-2} \binom{k-2}{j} \frac{(-1)^j}{(j+1)(j+2)(k-j)^2} = \frac{1}{k(k^2-1)(k+2)} \sum_{j=2}^{k+1} \binom{k+2}{j} \frac{(-1)^j k+1-j}{k+2-j},\]
and changing the order of summation we have
\[
\sum_{j=0}^{k-2} \binom{k-2}{j} \frac{(-1)^j}{(k-j)^2(j+1)(j+2)} = \frac{(-1)^k}{k(k^2-1)(k+2)} \sum_{j=1}^{k} \binom{k+2}{j} \frac{(-1)^j j - 1}{j}
\]
\[
= \frac{(-1)^k}{k(k^2-1)(k+2)} \left( \sum_{j=1}^{k+2} \binom{k+2}{j} (-1)^j - 1 + \sum_{j=1}^{k+2} \binom{k+2}{j} \frac{(-1)^j+1}{j} \right)
\]
\[
+ \frac{1}{(k-1)(k+1)^2} - \frac{1}{k(k-1)(k+2)^2}.
\]

Finally using (3) we get (10).

Now we prove (11). Applying the same evaluations as above we see that
\[
\sum_{j=0}^{k-2} \binom{k-2}{j} \frac{(-1)^j H_{j+2}}{(j+1)(j+2)(k-j)^2} = \frac{1}{k(k^2-1)(k+2)} \sum_{j=0}^{k-2} \binom{k+2}{j} (-1)^j H_{j+2} \frac{k-1-j}{k-j}
\]
\[
= \frac{1}{k(k^2-1)(k+2)} \left( \sum_{j=1}^{k+2} \binom{k+2}{j} (-1)^j H_j - (-1)^k H_{k+2} - \sum_{j=1}^{k+1} \binom{k+2}{j} \frac{(-1)^j H_j}{k+2-j} \right)
\]
\[
+ \frac{1}{(k-1)(k+1)^2},
\]
which after using (4) and (9) gives (11).

**Corollary 1.**

\[
\sum_{j=0}^{k-2} \binom{k-2}{j} \frac{(-1)^j (H_{j+2} - 1)}{(j+1)(j+2)(k-j)^2} = \frac{1}{k(k^2-1)(k+2)} \sum_{j=1}^{k} \binom{k+2}{j} \frac{(\gamma + \psi(k+3) - 1)^2 - \psi'(k+3)}{k(k^2-1)(k+2)}
\]
\[
- \frac{2\beta'(k+3)}{k(k^2-1)(k+2)} + \frac{1}{(k^2-1)(k+2)^2},
\]

(12)

**Lemma 3.**

\[
\int_0^1 x^{p-1} \left( \log \frac{1}{x} \right)^q dx = \frac{1}{p^{q+1}} \Gamma(q+1), \quad p > 0, \quad q > -1 \quad \text{(cf. [8], 3.653.2), (13)}
\]

and for \( p > 1, \ q > 1 \)

\[
\int_0^1 x^{p-1} (1-x)^{q-1} \log \frac{1}{x} dx = B(p, q)(\psi(p+q) - \psi(p)) \quad \text{(cf. [8], 3.628.1), (14)}
\]

\[
\int_0^1 x^p \log \frac{1}{x} \log \frac{1}{1-x} dx = \frac{\psi(p+2) + \gamma}{(p+1)^2} - \frac{\psi'(p+2)}{p+1},
\]

(15)
for \( r \in \mathbb{N} \) and \( q \in \mathbb{N} \)
\[
\int_{0}^{1} x^{r-1} \left( \log \frac{1}{1-x} \right)^{q} \, dx
= \frac{q!}{r^{1+2a_{2}+\ldots+qa_{q}}} \prod_{i=1}^{q} \frac{(-1)^{ia_{i}}(\psi(i-r-1)-(i-r+1)(1)-(1)_{i})}{a_{i}!}(1)_{i}
\]
(cf. [2]),
where the summation is over all \( a_{i} \in \mathbb{N} \), \( i = 1, \ldots, q \), and \( c(j,q) \) are the unsigned Stirling numbers of the first kind. In particular
\[
\int_{0}^{1} x^{r-1} \left( \log \frac{1}{1-x} \right)^{2} \, dx = \left( \psi(r+1) + \gamma \right)^{2} + \frac{\pi^{2}}{6} - \frac{\psi'(r+1)}{r}.
\]

**Proof.** We prove (15). Using the expansion
\[
\log \frac{1}{1-x} = \sum_{j=1}^{\infty} \frac{x^{j}}{j}, \quad |x| < 1,
\]
and (13) we get
\[
\int_{0}^{1} x^{r} \log \frac{1}{x} \log \frac{1}{1-x} \, dx = \sum_{j=1}^{\infty} \frac{1}{j(j+r+1)^{2}},
\]
which by (7) proves (15). \( \square \)

### 3. MEAN AND VARIANCE OF ENTROPY FOR RANDOM PARTITIONS

In this section we present formulae for the expectation and variance of Shannon’s, paired, genetic and \( \alpha \)-entropy of spacings. In what follows we use Darling’s (cf. [4]) moment formulae for the statistic \( \sum_{j=0}^{k} f(Y_{j,k}) \), which are given by
\[
E \left( \sum_{j=0}^{k} f(Y_{j,k}) \right) = k(k+1) \int_{0}^{1} (1-x)^{k-1} f(x) \, dx,
\]
(17)
\[
E \left( \sum_{j=0}^{k} f(Y_{j,k}) \right)^{2} = k(k+1) \int_{0}^{1} (1-x)^{k-1} f^{2}(x) \, dx
+ k^{2}(k^{2}-1) \int_{0}^{1} \int_{0}^{1-x} (1-x-y)^{k-2} f(x)f(y) \, dy \, dx,
\]
(18)
and
\[
E \left( \sum_{j=0}^{k} f(Y_{j,k}) \sum_{j=0}^{k} g(Y_{j,k}) \right) = k(k+1) \int_{0}^{1} (1-x)^{k-1} f(x)g(x) \, dx
+ k^{2}(k^{2}-1) \int_{0}^{1} \int_{0}^{1-x} (1-x-y)^{k-2} f(x)g(y) \, dy \, dx,
\]
(19)
for any real functions $f$ and $g$ such that the above moments exist.

First we consider $D_k^S$. The results regarding the moments of Shanon’s entropy of spacings are partially known (cf. [7, 23]) but we establish explicit formulae in terms of Polygamma functions.

**Proposition 1.** The expectation and the variance of $D_k^S$ are given by

$$E D_k^S = \psi(k + 2) + \gamma - 1, \tag{20}$$

$$\text{var} D_k^S = \frac{\pi^2 - 6}{3(k + 2)} - \zeta(2, k + 2). \tag{21}$$

**Proof.** Using (17) and (14) with $f(x) = -x \log x$ we get

$$E D_k^S = k(k + 1) \int_0^1 x(1 - x)^{k - 1} \log \frac{1}{x} \, dx$$

$$= k(k + 1) B(2, k) (\psi(k + 2) - \psi(2)) \quad \text{(cf. [7, 23])}$$

which gives (20).

Now by (18) we have

$$E \left( D_k^S \right)^2 = k(k + 1) \int_0^1 (1 - x)^{k - 1} x^2 \log^2 x \, dx$$

$$+ k^2(k^2 - 1) \int_0^1 \int_0^{1-x} (1 - x - y)^{k - 2} x \log xy \log y \, dy \, dx,$$

and substituting $y = (1 - x)t$ in the second integral we get

$$E \left( D_k^S \right)^2 = k(k + 1) \int_0^1 \left( x^{k+1} \log \frac{1}{1 - x} - 2x^k \log \frac{1}{1 - x} + x^{k-1} \log \frac{1}{1 - x} \right) \, dx$$

$$+ k^2(k^2 - 1) \int_0^1 t(1 - t)^{k - 2} \, dt \left( \int_0^1 x^k \log \frac{1}{x} \log \frac{1}{1 - x} \, dx - \int_0^1 x^{k+1} \log \frac{1}{x} \log \frac{1}{1 - x} \, dx \right)$$

$$+ k^2(k^2 - 1) \int_0^1 x(1 - x)^k \log \frac{1}{x} \, dx \int_0^1 t(1 - t)^{k - 2} \log \frac{1}{t} \, dt,$$

Therefore by (2), (16) and (14) we obtain

$$E \left( D_k^S \right)^2 = (\gamma + \psi(k + 2) - 1)^2 - \zeta(2, k + 2) + \frac{\pi^2 - 6}{3(k + 2)},$$

which with (20) leads to (21).

Since

$$\zeta(2, k + 2) \geq \frac{1}{(k + 2)(k + 3)} + \frac{1}{(k + 3)(k + 4)} + \frac{1}{(k + 4)(k + 5)} + \ldots = \frac{1}{k + 2},$$

we get
Corollary 2. \[ \text{var } D_k^S \leq \frac{\pi^2 - 9}{3(k + 2)}. \]

For paired entropy $D_k^P$ we have

Proposition 2. The expectation and the variance of $D_k^P$ are given by

\begin{align*}
\mathbb{E}D_k^P &= \psi(k + 1) + \gamma, \\
\text{var } D_k^P &= \frac{\pi^2 - 6}{3(k + 2)} - \zeta(2, k + 2) - \frac{k}{k + 2} \zeta \left(2, \frac{k + 2}{2}\right) + \frac{k(2k + 3)}{(k + 1)^2(k + 2)}. \tag{22} \\
\end{align*}

Proof. Write \( D_k^S = -\sum_{j=0}^{k} (1 - Y_{j,k}) \log(1 - Y_{j,k}). \) Then using (17) with \( f(x) = -(1 - x) \log(1 - x), \) we have

\[ \mathbb{E}D_k^S = k(k + 1) \int_0^1 (1 - x)^k \log \frac{1}{1 - x} \, dx = \frac{k}{k + 1}. \tag{24} \]

Since \( \mathbb{E}D_k^P = \mathbb{E}D_k^S + \text{var } D_k^S \) then by (20) we get (22).

Now using (18) we have

\[ \mathbb{E} \left( D_k^S \right)^2 = k(k + 1) \int_0^1 (1 - x)^{k+1} \log^2(1 - x) \, dx + k^2(k^2 - 1) \]

\[ \cdot \int_0^1 \int_0^{1-x} (1 - x - y)^{k-2}(1 - x) \log(1 - x)(1 - y) \log(1 - y) \, dy \, dx := A_k + B_k, \]

say. By (13)

\[ A_k := k(k + 1) \int_0^1 (1 - x)^{k+1} \log^2(1 - x) \, dx = \frac{2k(k + 1)}{(k + 2)^3}. \]

Next setting \( t = 1 - y \) in \( B_k \) we have

\[ B_k = k^2(k^2 - 1) \left( \int_0^1 \int_0^1 (t - x)^{k-2}(1 - x) \log(1 - x)t \log t \, dt \, dx \right. \]

\[ \left. - \int_0^1 \int_0^x (t - x)^{k-2}(1 - x) \log(1 - x)t \log t \, dt \, dx \right) . \]

Then using the binomial expansion for \((t-x)^{k-2}\) in the first integral and substituting \( t = zx \) in the second we get

\[ B_k = k^2(k^2 - 1) \sum_{j=0}^{k-2} \binom{k-2}{j} (-1)^j \int_0^1 x^j(1 - x) \log \frac{1}{1 - x} \, dx \int_0^1 t^{k-j-1} \log \frac{1}{t} \, dt \]

\[ - (-1)^k k^2(k^2 - 1) \left( \int_0^1 x^{k}(1 - x) \log(1 - x) \log x \, dx \int_0^1 z(1 - z)^{k-2} \, dz \right. \]

\[ + \int_0^1 x(1 - x)^k \log \frac{1}{x} \, dx \int_0^1 z(1 - z)^{k-2} \log \frac{1}{z} \, dz \left) . \right] \]
Applying (2), (14) and (15) we see that

\[ B_k = k^2(k^2 - 1) \sum_{j=0}^{k-2} \binom{k-2}{j} (-1)^j \frac{(H_j + 2 - 1)}{(j+1)(j+2)(k-j)^2} \]

\[ - \frac{(-1)^kk^2}{k+2} \left( (\gamma + \psi(k+3) - 1)^2 - \psi'(k+3) \right). \]

Hence by (12) we get

\[ B_k = \frac{k^2}{(k+2)^2} \frac{2k}{k+2} \beta'(k+3). \]

Therefore

\[ E \left( \frac{D_k^S}{k} \right)^2 = -\frac{2k}{k+2} \beta'(k+3) + \frac{k(k^2 + 4k + 2)}{(k+2)^3}, \]

which by (24) and the equality \( \beta'(k+3) = -\beta'(k+2) - \frac{1}{(k+2)^2} \) gives

\[ \text{var} D_k^S = \frac{2k}{k+2} \beta'(k+2) + \frac{k}{(k+1)^2(k+2)}. \]

Now using (19) with \( f(x) = -(1-x) \log(1-x) \) and \( g(x) = -x \log x \) we get

\[ E D_k^S D_k^S = k(k+1) \int_0^1 (1-x)^k x \log x \log(1-x) dx \]

\[ + k^2(k^2 - 1) \int_0^1 \int_0^{1-x} (1-x-y)^{k-2} (1-x) \log(1-x)y \log y dy dx := C_k + E_k, \]

say. For \( C_k \) we have

\[ C_k := k(k+1) \int_0^1 (1-x)^k x \log x \log(1-x) dx \]

\[ = k(k+1) \int_0^1 (x^k \log x \log(1-x) - x^{k+1} \log x \log(1-x)) dx, \]

which by (15) gives

\[ C_k = \frac{k(2k+3)}{(k+1)(k+2)^2} (\gamma + \psi(k+2)) - \frac{2k(k+1)}{(k+2)^3} - \frac{k}{k+2} \zeta(2, k+2). \]

Substituting \( y = (1-x)t \) in \( E_k \) we get

\[ E_k = k^2(k^2 - 1) \left( \int_0^1 (1-x)^{k+1} \log \frac{1}{1-x} dx \int_0^1 t(1-t)^{k-2} \log \frac{1}{t} dt \right) \]

\[ + \int_0^1 (1-x)^{k+1} \log \frac{1}{1-x} \, dz \int_0^1 t(1-t)^{k-2} dt \right), \]

which by (13) and (14) gives

\[ E_k = \frac{k(k+1)}{(k+2)^2} (\gamma + \psi(k+2)) - \frac{k(k^2 + 2k + 2)}{(k+2)^3}. \]
Therefore
\[ \mathbb{E}D_k^S D_k^S = \frac{k}{k + 1} (\gamma + \psi(k + 2)) - \frac{k}{k + 2} \zeta(2, k + 2) - \frac{k}{k + 2}, \]
and
\[ \text{cov}(D_k^S, D_k^S) = -\frac{k}{k + 2} \zeta(2, k + 2) + \frac{k}{(k + 1)(k + 2)}. \]
Hence
\[ \text{var} \, D_k^P = \text{var} \, D_k^S + \sigma^2 D_k^S + 2 \text{cov} \left( D_k^S, D_k^S \right) \]
\[ = \frac{\pi^2 - 6}{3(k + 2)} - \zeta(2, k + 2) - \frac{2k}{k + 2} \left( \zeta(2, k + 2) - \beta'(k + 2) \right) + \frac{k(2k + 3)}{(k + 1)^2(k + 2)} \]
which proves (23) after using the equality
\[ \zeta(2, k + 2) - \beta'(k + 2) = \sum_{j=0}^{\infty} \frac{1 + (-1)^j}{(j + k + 2)^2} = \frac{1}{2} \zeta \left( 2, \frac{k + 2}{2} \right). \]
\[ \square \]

Corollary 3.
\[ \text{var} \, D_k^P \leq \frac{\pi^2 - 7}{3(k + 2)}. \]

Proof. Since \(-\zeta(2, k + 2) \leq -\frac{1}{k + 2}\) and \(-\zeta \left( 2, \frac{k+2}{2} \right) \leq -\frac{2}{k+2}\) then
\[ \text{var} \, D_k^P \leq \frac{\pi^2 - 7}{3(k + 2)} - \frac{(2k - 1)(k^2 - 1) + 3}{3(k + 1)^2(k + 2)^2}, \]
which proves the required assertion. \(\square\)

Remark 1. For \(k \geq 7\)
\[ \text{var} \, D_k^P \leq \frac{\pi^2 - 8}{3(k + 2)}. \]

Now we present the moments of \(D_k^G\).

**Proposition 3.** The expectation and the variance of \(D_k^G\) are given by
\[ \mathbb{E}D_k^G = \frac{k(k^2 + 5k + 10)}{(k + 2)_3}, \]
\[ \text{var} \, D_k^G = \frac{16k(k^6 + 15k^5 + 103k^4 + 435k^3 + 1282k^2 + 2700k + 3240)}{(k + 2)_3(k + 2)_7}. \]

Proof. Using (17) with \(f(x) = x(1 - x) - x^2(1 - x)^2\) write \(\mathbb{E}D_k^G = \mathbb{E}\hat{D}_k^G + \mathbb{E}D_k^G\), where
\[ \mathbb{E}\hat{D}_k^G := k(k + 1) \int_0^1 x(1 - x)^k \, dx = k(k + 1)B(2, k + 1) = \frac{k}{k + 2} \]
and
\[
\text{ED}_k^G := -k(k + 1) \int_0^1 x^2(1 - x)^{k+1} \, dx = -k(k + 1)B(3, k + 2) = -\frac{2k(k + 1)}{(k + 2)_3},
\]
which gives (25).

Next using (18) with \( f(x) = x(1 - x) \)
\[
E\left(\hat{D}_k^G\right)^2 = k(k + 1) \int_0^1 x^2(1 - x)^{k+1} \, dx
+ k^2(k^2 - 1) \int_0^1 \int_0^{1-x} (1 - x - y)^{k-2}x(1 - x)y(1 - y) \, dy \, dx.
\]
Making the substitution \( y = (1 - x)t \) in the second integral we get
\[
E\left(\hat{D}_k^G\right)^2 = k(k + 1) \int_0^1 x^2(1 - x)^{k+1} \, dx + k^2(k^2 - 1)
\cdot \int_0^1 \int_0^1 x(1 - x)^{k+1}t(1 - t)^{k-2}(1 - (1 - x)t) \, dt \, dx = \frac{k(k^2 + 5k + 2)}{(k + 2)_3}.
\]
Similarly, from (18) and (19), we get
\[
E\left(D_k^G\right)^2 = \frac{4k(k^3 + 18k^2 + 59k + 18)}{(k + 3)_6} \quad \text{and} \quad E\hat{D}_k^G D_k^G = -\frac{2k(k^2 + 8k + 3)}{(k + 2)_4}.
\]
Hence
\[
\text{var} \hat{D}_k^G = \frac{4k}{(k + 2)(k + 2)_3},
\]
\[
\text{var} D_k^G = \frac{4k(k^6 - 3k^5 + 59k^4 + 147k^3 + 1714k^2 + 2520k + 864)}{(k + 2)_3(k + 2)_7}
\]
and
\[
\text{cov} \left(\hat{D}_k^G, D_k^G\right) = \frac{4k(k^2 - 7k - 6)}{(k + 2)(k + 2)_5}.
\]
Finally (26) we obtain from
\[
\text{var} D_k^G = \text{var} \hat{D}_k^G + \text{var} D_k^G + 2\text{cov} \left(\hat{D}_k^G, D_k^G\right). \quad \square
\]

**Corollary 4.** \( \text{var} D_k^G \leq \frac{16}{k^3} \)

For \( \alpha \)-entropy \( D_k^\alpha \) we get

**Proposition 4.** The expectation and the variance of \( D_k^\alpha \) are given by
\[
ED_k^\alpha = \frac{1}{2^{1-\alpha} - 1} (k(k + 1)B(\alpha + 1, k) - 1), \quad (27)
\]
\[
\text{var } D_\alpha^k = \frac{k(k+1)}{(2^{1-\alpha}-1)^2} \left( B(2\alpha+1,k)(1+\alpha k B(\alpha+1,\alpha)) - k(k+1) B^2(\alpha+1,k) \right). \tag{28}
\]

**Proof.** Using (17) and (18) we get

\[
E \sum_{j=0}^{k} Y_{j,k}^\alpha = k(k+1) \int_0^1 (1-x)^{k-1} x^\alpha \, dx
\]

and

\[
E \left( \sum_{j=0}^{k} Y_{j,k}^\alpha \right)^2 = k(k+1) \int_0^1 (1-x)^{k-1} x^{2\alpha} \, dx
\]

\[
+ k^2(k^2-1) \int_0^1 \int_0^{1-x} (1-x-y)^{k-2} x^\alpha y^\alpha \, dy \, dx,
\]

which gives (27) and (28). \qed

**Corollary 5.**

\[
ED_\alpha^k \sim \frac{1}{2^{1-\alpha} - 1} \left( \frac{\Gamma(\alpha+1)}{k^{\alpha-1}} - 1 \right), \tag{29}
\]

\[
\text{var } D_\alpha^k \sim \frac{\Gamma(2\alpha+1) - (\alpha^2 + 1) \Gamma^2(\alpha+1)}{(2^{1-\alpha}-1)^2 k^{2\alpha-1}} \quad \text{(cf. [4]).} \tag{30}
\]

**Proof.** Using the formula

\[
\frac{\Gamma(k)}{\Gamma(k+\beta)} = \frac{1}{k^\beta} - \frac{\beta(\beta-1)}{2k^{\beta+1}} + o \left( \frac{1}{k^{\beta+1}} \right), \quad \beta \geq 0, \quad k \to \infty
\]

(cf. [4], [24], p. 67, 3.31) in (27) and (28) we get the desired assertions. \qed

4. ASYMPTOTIC PROPERTIES

Let \( U_0, U_1, \ldots, U_k \) be exponential distributed random variables with mean 1. It is known that

\[
Y_{j,k} \overset{d}{=} \frac{U_j}{\sum_{i=0}^{k} U_i} \quad \text{for } 0 \leq j \leq k \tag{31}
\]

(cf. [6]), and the equality holds in distribution. Slud in [23] established the rate of the almost sure convergence of the sequence \( D_k^S - \log(k+1) \). Using the above representation and the law of iterated logarithm he proved that

\[
\log(k+1) - D_k^S + \gamma - 1 = O \left( (\log \log k/k)^{1/2} \right) \quad \text{a.s., } \quad k \to \infty.
\]

We prove the complete convergence of that sequence.
Theorem 1. \[ D_k^S - \log(k+1) \xrightarrow{c} \gamma - 1, \quad k \to \infty. \]

Proof. Let \( \varepsilon > 0 \). Using (31) we see that

\[
\Pr\left( |D_k^S - \log(k+1) - \gamma + 1| > \varepsilon \right) = \Pr\left( \left| \log \frac{\sum_{j=0}^{k} U_j}{k+1} + \frac{\sum_{j=0}^{k} U_j \log \frac{1}{U_j}}{\sum_{j=0}^{k} U_j} - \gamma + 1 \right| > \varepsilon \right)
\]

\[
\leq \Pr\left( \left| \log \frac{\sum_{j=0}^{k} U_j}{k+1} \right| > \frac{\varepsilon}{2} \right) + \Pr\left( \left| \sum_{j=0}^{k} U_j \log \frac{1}{U_j} \right| > \varepsilon \sum_{j=0}^{k} U_j - \gamma + 1 \right)
\]

\[
+ \Pr\left( \left| \frac{1}{k+1} \sum_{j=0}^{k} U_j - 1 \right| > \frac{\varepsilon (1 - \delta)}{4(1 - \gamma)} \right)
\]

Now let \( 0 < \delta < 1 \). Then

\[
\Pr\left( \left| \frac{1}{k+1} \sum_{j=0}^{k} U_j - 1 \right| > \frac{\varepsilon (1 - \delta)}{4(1 - \gamma)} \right) \leq \Pr\left( \left| \frac{1}{k+1} \sum_{j=0}^{k} U_j - 1 \right| \leq \delta \right)
\]

Similarly

\[
\Pr\left( \left| \frac{1}{k+1} \sum_{j=0}^{k} U_j \log \frac{1}{U_j} - \gamma + 1 \right| > \frac{\varepsilon (1 - \delta)}{4} \right) \leq \Pr\left( \left| \frac{1}{k+1} \sum_{j=0}^{k} U_j \log \frac{1}{U_j} - \gamma + 1 \right| > \delta \right)
\]

Also by the Theorem of Hsu and Robbins (cf. [5], [16])

\[
\sum_{k=1}^{\infty} \Pr\left( \left| \frac{1}{k+1} \sum_{j=0}^{k} U_j \log \frac{1}{U_j} - \gamma + 1 \right| > \frac{\varepsilon (1 - \delta)}{4} \right) < \infty,
\]

and

\[
\sum_{k=1}^{\infty} \Pr\left( \left| \frac{1}{k+1} \sum_{j=0}^{k} U_j - 1 \right| > \delta \right) < \infty.
\]
Moreover, we see that
\[
\sum_{k=1}^{\infty} \Pr \left( \left| \log \frac{\sum_{j=0}^{k} U_j}{k+1} - \varepsilon \right| > \varepsilon \right) \leq \sum_{k=1}^{\infty} \Pr \left( \left| \frac{\sum_{j=0}^{k} U_j}{k+1} - 1 \right| > \varepsilon^2 - 1 \right) + \sum_{k=1}^{\infty} \Pr \left( \left| \frac{\sum_{j=0}^{k} U_j}{k+1} - 1 \right| > 1 - e^{-\varepsilon^2} \right) < \infty,
\]
which ends the proof. \(\square\)

Additional information can be obtained from Heyde’s theorem [15], who proved that
\[
\lim_{\varepsilon \to 0} \varepsilon^2 \sum_{n=1}^{\infty} \Pr(|S_n - n\mu| > n\varepsilon) = \sigma^2, \tag{32}
\]
where \(S_n\) is the sum of \(n\) i.i.d. random variables with mean \(\mu\) and variance \(\text{var}\). Here we obtain

**Corollary 6.**
\[
\lim_{\varepsilon \to 0} \varepsilon^2 \sum_{k=1}^{\infty} \Pr \left( \left| D_{k}^{S} - \log(k+1) - \gamma + 1 \right| > \varepsilon \right) \leq 18+16 \left( (1-\gamma)^2 + (2-\gamma)^2 + \frac{\pi^2}{3} \right).
\]

**Proof.** Letting \(\delta = \varepsilon\) in the inequalities of Theorem 1 we get
\[
\Pr \left( \left| D_{k}^{S} - \log(k+1) - \gamma + 1 \right| > \varepsilon \right) \leq 2 \Pr \left( \left| \frac{1}{k+1} \sum_{j=0}^{k} U_j - 1 \right| > \varepsilon \right) + \Pr \left( \left| \frac{1}{k+1} \sum_{j=0}^{k} U_j \log \frac{1}{U_j} - \gamma + 1 \right| > \varepsilon (1-\varepsilon) \right) + \Pr \left( \left| \frac{1}{k+1} \sum_{j=0}^{k} U_j - 1 \right| > e^{\varepsilon} - 1 \right) + \Pr \left( \left| \frac{1}{k+1} \sum_{j=0}^{k} U_j - 1 \right| > 1 - e^{-\varepsilon^2} \right).
\]
which by (32) gives
\[
\lim_{\varepsilon \to 0} \varepsilon^2 \sum_{k=1}^{\infty} \Pr \left( \left| D_{k}^{S} - \log(k+1) - \gamma + 1 \right| > \varepsilon \right) \leq (2 + 16(1-\gamma)^2) \text{var} U_1 + 16 \text{var} (U_1 \log U_1).
\]
and after using \(\text{var} U_1 = 1\) and \(\text{var} (U_1 \log U_1) = (2-\gamma)^2 + \frac{\pi^2}{3} + 1\) we complete the proof. \(\square\)
Theorem 2. \( D_P^k - \log(k + 1) \xrightarrow{c} \gamma, \ k \to \infty. \)

Proof. Since \( D^P_k = D^S_k + D^S_k \) then it is enough to show \( D^S_k \xrightarrow{c} 1. \) Using the inequality 
\[
(1 - x) x \leq (1 - x) \log \frac{1}{1 - x} \leq x, \ x < 1,
\]
we have
\[
\sum_{j=0}^{k} Y_{j,k} - \sum_{j=0}^{k} Y^2_{j,k} \leq -\sum_{j=0}^{k} (1 - Y_{j,k}) \log (1 - Y_{j,k}) \leq \sum_{j=0}^{k} Y_{j,k}.
\]

Hence for any given \( \varepsilon > 0 \)
\[
\sum_{k=1}^{\infty} \Pr \left( \left| D^S_k - 1 \right| > \varepsilon \right) \leq \sum_{k=1}^{\infty} \Pr \left( \sum_{j=0}^{k} Y^2_{j,k} > \varepsilon \right) \leq \frac{16}{\varepsilon^2} \sum_{k=1}^{\infty} \frac{1}{k^3} = \frac{16\zeta(3)}{\varepsilon^2} < \infty,
\]
which ends the proof.

Now taking into account that
\[
\Pr \left( |D^P_k - \log(k + 1) - \gamma| > 2\varepsilon \right) = \Pr \left( |D^S_k - \log(k + 1) - \gamma + D^S_k| > 2\varepsilon \right)
\leq \Pr \left( |D^S_k - \log(k + 1) - \gamma + 1| > \varepsilon \right) + \Pr \left( |D^S_k - 1| > \varepsilon \right)
\]
we get

Corollary 7.
\[
\lim_{\varepsilon \to 0} \varepsilon^2 \sum_{k=1}^{\infty} \Pr \left( |D^P_k - \log(k + 1) - \gamma| > \varepsilon \right) \leq 72 + 64 \left( (1 - \gamma)^2 + (2 - \gamma)^2 + \frac{\pi^2}{3} + \zeta(3) \right).
\]

Theorem 3. \( D^G_k \xrightarrow{c} 1, \ k \to \infty. \)

Proof. Let \( \varepsilon > 0. \) If \( k \to \infty \) then \( ED^G_k \to 1 \) and by Chebyshev’s inequality and (26)
\[
\sum_{k=1}^{\infty} \Pr \left( |D^G_k - ED^G_k| > \varepsilon \right) \leq \sum_{k=1}^{\infty} \frac{\text{var} D^G_k}{\varepsilon^2} \leq \frac{16}{\varepsilon^2} \sum_{k=1}^{\infty} \frac{1}{k^3} = 16\zeta(3) < \infty,
\]
which implies the theorem.

Remark 2. Note that by (26)
\[
\lim_{\varepsilon \to 0} \varepsilon^2 \sum_{k=1}^{\infty} k \Pr \left( |D^G_k - ED^G_k| > \varepsilon \right) \leq \sum_{k=1}^{\infty} \frac{16}{k^2} = \frac{8\pi^2}{3}
\]

In the proof of complete convergence of \( \alpha \)-entropy we use the following theorem of Baum and Katz [1].
Theorem 4. (cf. Baum and Katz [1]) Let $\frac{1}{2} < \alpha \leq 1$ and $\{X_k, k \geq 1\}$ be the i.i.d. random variables. If $E|X_k|^2 < \infty$, $EX_k = \mu$ and $S_n = X_1 + \ldots + X_n$ then for all $\varepsilon > 0$
\[ \sum_{n=1}^{\infty} \Pr(|S_n - n\mu| > n^\alpha \varepsilon) < \infty. \]

Theorem 5. For $\alpha > \frac{1}{2}$
\[ D_\alpha^k - (k+1)^{1-\alpha} \frac{\Gamma(\alpha + 1)}{2^{1-\alpha} - 1} \varepsilon \frac{1}{1 - 2^{1-\alpha}}, \quad k \to \infty. \]

Proof. Let $\varepsilon > 0$. If $\alpha > 1$ then by (29) $ED_\alpha^k \to \frac{1}{1 - 2^{1-\alpha}}, \quad k \to \infty$. Using Chebyshev’s inequality and (30)
\[ \sum_{k=1}^{\infty} \Pr (|D_\alpha^k - ED_\alpha^k| > \varepsilon) \leq \sum_{k=1}^{\infty} \frac{\var D_\alpha^k}{\varepsilon^2} \leq \sum_{k=1}^{\infty} \frac{C}{k^{2\alpha - 1}} < \infty, \]
which implies the theorem.

Now let $\alpha \in (\frac{1}{2}, 1)$. Using (31) we see that
\[ \Pr \left( D_\alpha^k - (k+1)^{1-\alpha} \frac{\Gamma(\alpha + 1)}{2^{1-\alpha} - 1} \varepsilon \frac{1}{1 - 2^{1-\alpha}} \right) = \Pr \left( (k+1)^{1-\alpha} \left| \frac{1}{k+1} \sum_{j=0}^{k} U_j^\alpha - \Gamma(\alpha + 1) \left( \frac{1}{k+1} \sum_{j=0}^{k} U_j \right) \right| \right). \]

Now let $0 < \delta < 1$ and $\varepsilon_1 = \varepsilon (2^{1-\alpha} - 1)$. Then
\[ \Pr \left( (k+1)^{1-\alpha} \left| \frac{1}{k+1} \sum_{j=0}^{k} U_j^\alpha - \Gamma(\alpha + 1) \left( \frac{1}{k+1} \sum_{j=0}^{k} U_j \right) \right| > \varepsilon_1 \left( \frac{1}{k+1} \sum_{j=0}^{k} U_j \right) \right) \leq \Pr \left( \left| \sum_{j=0}^{k} (U_j^\alpha - \Gamma(\alpha + 1)) \right| > \varepsilon_1 \frac{1-\delta}{2}(k+1) \right) + \Pr \left( \left( \frac{1}{k+1} \sum_{j=0}^{k} U_j \right) - 1 > \delta \right) + \Pr \left( (k+1)^{1-\alpha} \left| \left( \frac{1}{k+1} \sum_{j=0}^{k} U_j \right) \right| > \varepsilon_1 \right) \]
\[ + \Pr \left( (k+1)^{1-\alpha} \left| \left( \frac{1}{k+1} \sum_{j=0}^{k} U_j \right) - 1 \right| > \varepsilon_1 \frac{1-\delta}{2\Gamma(\alpha + 1)} \right). \]

Using Theorem 4 we see that
\[ \sum_{k=1}^{\infty} \Pr \left( \left| \sum_{j=0}^{k} (U_j^\alpha - \Gamma(\alpha + 1)) \right| > \varepsilon_1 \frac{1-\delta}{2}(k+1)^{\alpha} \right) < \infty, \]
and by Theorem of Hsu and Robbins (cf. [5, 16])

\[
\sum_{k=1}^{\infty} \Pr \left( \left| \left( \frac{1}{k+1} \sum_{j=0}^{k} U_j \right)^{\alpha} - 1 \right| > \delta \right) \leq \sum_{k=1}^{\infty} \Pr \left( \left| \frac{1}{k+1} \sum_{j=0}^{k} U_j - 1 \right| > (\delta + 1)^{\frac{1}{\alpha}} - 1 \right)
\]

\[+ \sum_{k=1}^{\infty} \Pr \left( \left| \frac{1}{k+1} \sum_{j=0}^{k} U_j - 1 \right| > 1 - (1 - \delta)^{\frac{1}{\alpha}} \right) < \infty.\]

Now let \( \varepsilon_2 = \frac{\varepsilon_1 (1 - \delta)}{2^{\Gamma(\alpha+1)}} \). Then

\[
\Pr \left( \left( k + 1 \right)^{1-\alpha} \left| \left( \frac{1}{k+1} \sum_{j=0}^{k} U_j \right)^{\alpha} - 1 \right| > \varepsilon_2 \right) \leq \Pr \left( \sum_{j=0}^{k} (U_j - 1) > (k + 1) \left( 1 - \left( 1 - \left( k + 1 \right)^{\frac{\varepsilon_2}{\alpha}} \right) \right) \right).
\]

Since \( \left( 1 + \left( k + 1 \right)^{\frac{\varepsilon_2}{\alpha}} \right)^{\frac{1}{\alpha}} - 1 \sim \frac{\varepsilon_2}{\alpha} (k+1)^{\alpha-1} \) and \( 1 - \left( 1 - \left( k + 1 \right)^{\frac{\varepsilon_2}{\alpha}} \right) \sim \frac{\varepsilon_2}{\alpha} (k+1)^{\alpha-1} \) then

\[
\sum_{k=1}^{\infty} \Pr \left( \left( k + 1 \right)^{1-\alpha} \left| \left( \frac{1}{k+1} \sum_{j=0}^{k} U_j \right)^{\alpha} - 1 \right| > \varepsilon_2 \right)
\]

\[
\leq \sum_{k=1}^{\infty} \Pr \left( \sum_{j=0}^{k} (U_j - 1) > \frac{\varepsilon_2}{\alpha} (k+1)^{\alpha} \right) + \sum_{k=1}^{\infty} \Pr \left( \sum_{j=0}^{k} (U_j - 1) > \frac{\varepsilon_2}{\alpha} (k+1)^{\alpha} \right) < \infty,
\]

by Theorem 4. The proof is complete. \( \square \)

Moreover for \( 0 < \alpha \leq \frac{1}{2} \) we get the following statement

**Theorem 6.** If \( 0 < \alpha \leq \frac{1}{2} \) then

\[
k^{\alpha-1} D_k^{\alpha} \xrightarrow{\text{a.s.}} \frac{\Gamma(\alpha+1)}{2^{1-\alpha} - 1}, \; k \to \infty.
\]

**Proof.** We see that \( k^{\alpha-1} E D_k^{\alpha} \sim \frac{\Gamma(\alpha+1)}{2^{1-\alpha} - 1} \). Now by Chebyshev’s inequality and (30)

\[
\Pr \left( |k^{\alpha-1} D_k^{\alpha} - k^{\alpha-1} E D_k^{\alpha}| > \varepsilon \right) \leq \frac{\text{var} \left( D_k^{\alpha} h^{2\alpha-2} \right)}{\varepsilon^2} \leq \frac{C}{k}.
\]
But for every $k$ there exists an integer $m = m(k)$ with $m^2 < k \leq (m + 1)^2$. Hence $0 < k - m^2 \leq 2m$ and $k \to \infty$ implies $m \to \infty$. Moreover

$$
\sum_{m=1}^{\infty} \Pr \left( \left| m^{2(\alpha-1)} D_{m^2}^{\alpha} - m^{2(\alpha-1)} E D_{m^2}^{\alpha} \right| > \varepsilon \right) \leq \sum_{m=1}^{\infty} C \frac{m}{m^2} < \infty,
$$

which gives

$$
m^{2(\alpha-1)} D_{m^2}^{\alpha} \overset{\text{a.s.}}{\to} \frac{\Gamma(\alpha + 1)}{2^{1-\alpha} - 1}, \quad m \to \infty. \quad (34)
$$

Since under the constraint $\sum_{j=1}^{k} x_i = 1, x_i \geq 0$, we have $\sum_{j=1}^{k} x_i^\alpha \leq \frac{1}{k^{\alpha-1}}$, then

$$
\left| k^{\alpha-1} D_k^{\alpha} - m^{2(\alpha-1)} D_{m^2}^{\alpha} \right| = \left| \left( k^{\alpha-1} - m^{2(\alpha-1)} \right) \sum_{j=0}^{m^2} Y_{j,k}^{\alpha} - k^{\alpha-1} \sum_{j=m^2+1}^{k} Y_{j,k}^{\alpha} \right|
\leq \frac{k^{1-\alpha} - m^{2(1-\alpha)}}{k^{1-\alpha}} + \frac{(k - m^2)^{1-\alpha}}{k^{1-\alpha}} \leq \left( 1 + \frac{1}{m} \right) 2^{(1-\alpha)} - 1 + \left( \frac{2}{m} \right)^{1-\alpha} \text{ a.s.}
$$

Therefore we get

$$
\left| k^{\alpha-1} D_k^{\alpha} - \frac{\Gamma(\alpha + 1)}{2^{1-\alpha} - 1} \right| \leq \left| k^{\alpha-1} D_k^{\alpha} - m^{2(\alpha-1)} D_{m^2}^{\alpha} \right| + \left| m^{2(\alpha-1)} D_{m^2}^{\alpha} - \frac{\Gamma(\alpha + 1)}{2^{1-\alpha} - 1} \right|
\leq \left( 1 + \frac{1}{m} \right) 2^{(1-\alpha)} - 1 + \left( \frac{2}{m} \right)^{1-\alpha} + \left| m^{2(\alpha-1)} D_{m^2}^{\alpha} - \frac{\Gamma(\alpha + 1)}{2^{1-\alpha} - 1} \right|,
$$

which by (34) implies (33). \qed

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