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A Note on Aoki-Yoshikawa Model

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Abstract

In this paper, we explore a dynamical version of the Aoki and Yoshikawa model (AYM) for an economy driven by demand. We show that when an appropriate Markovian dynamics is taken into account, AYM has different equilibrium distributions depending on the form of transition probabilities. In the version of the dynamic AYM presented here, transition probabilities depend on a parameter c tuning the choice of a new sector for workers leaving their sector. The solution of Aoki and Yoshikawa is recovered only in the case c=0. All the other possible cases give different equilibrium probability distributions, including the Bose-Einstein distribution.

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I. INTRODUCTION

In their recent book *Reconstructing Macroeconomics*, Masanao Aoki and Hiroshi Yoshikawa present a model used to derive the amount of production factor n_i for the i-th economic sector, based on an exogenously given demand D [1, 2] and given different levels of productivity a_i for each economic sector i. In the following, this model will be called *Aoki-Yoshikawa Model* or *AYM*. More specifically, let us suppose that an economy is made up of g sectors of size n_i, where, as written before, n_i is the amount of production factor used in sector i. For the sake of simplicity, in the following, we shall interpret n_i as the number of workers active in sector i, therefore limiting the production factor to labour. In AYM, the total endowment of production factor in the economy is exogenously given and set to n:

$$\sum_{i=1}^{g} n_i = n. \tag{1}$$

Notice that Aoki and Yoshikawa claim that n is akin to population rather than workforce as it includes people who are enjoying leisure or are active in household production. In any case, the output of sector i is given by

$$Y_i = a_i n_i, \tag{2}$$

where a_i is the productivity of sector i. It is further assumed that productivities differ across sectors and can be ordered as follows:

$$a_1 < a_2 < \ldots < a_g. \tag{3}$$

The total output of the economy is given by

$$Y = \sum_{i=1}^{g} Y_i = \sum_{i=1}^{g} a_i n_i. \tag{4}$$

This quantity is the *Gross Domestic Product* or *GDP* and it is assumed to be equal to an exogenously given aggregate demand D:

$$Y = D, \tag{5}$$

yielding

$$\sum_{i=1}^{g} a_i n_i = D. \tag{6}$$

Aoki and Yoshikawa are interested in finding the probability distribution of production factors across sectors, that is the distribution of the occupation vector

$$\mathbf{n} = (n_1, n_2, \ldots, n_g) \tag{7}$$

when statistical equilibrium is reached.

The problem of AYM coincides with a well-known problem in Statistical Physics, namely finding the statistical equilibrium allocation of n particles into g energy levels ε_i so that the number of particles is conserved

$$\sum_i n_i = n \tag{8}$$

and the total energy E is conserved

$$\sum_i \varepsilon_i n_i = E. \tag{9}$$

One can immediately see that the levels of productivity a_i correspond to energy levels, whereas the demand D has the meaning of total energy E.

After a first attempt in 1868 [3], Ludwig Boltzmann solved this problem in 1877 using the *most probable occupation vector*, an approximate method [4]. One can introduce configurations $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, with $x_i \in \{1, ..., g\}$, where $x_i = j$ means that the i-th worker is active in sector j; then, the number of distinct configurations belonging to a given occupation vector is $W(\mathbf{x}|\mathbf{n})$:

$$W(\mathbf{x}|\mathbf{n}) = \frac{n!}{\Pi_{i=1}^{g} n_i!}$$

Boltzmann noticed that, when statistical equilibrium is reached, the probability $\pi(\mathbf{n})$ of an accessible occupation state is proportional to $W(\mathbf{x}|\mathbf{n})$, this means that

$$\pi(\mathbf{n}) = CW(\mathbf{x}|\mathbf{n}) = C\frac{n!}{\Pi_{i=1}^{g} n_i!}; \tag{10}$$

therefore, occupation vectors that maximize $\pi(\mathbf{n})$ must minimize $\Pi_{i=1}^{g} n_i!$ subject to the two constraints (8) and (9). For large systems, Stirling's approximation can be used for the factorial:

$$\log\left[\Pi_{i=1}^{g} n_i!\right] \simeq \sum_{i=1}^{g} n_i(\log n_i - 1), \tag{11}$$

and the bounded extremum problem can be solved using Lagrange multipliers and finding the maximum of

$$L(\mathbf{n}) = -\sum_{i=1}^{g} n_i(\log n_i - 1) + \nu\left(\sum_{i=1}^{g} n_i - N\right) - \beta\left(\sum_{i=1}^{g} a_i n_i - D\right) \tag{12}$$

with respect to n_i. This gives

$$0 = \frac{\partial L}{\partial n_i} = -\log n_i + \nu - \beta a_i \tag{13}$$

or

$$n_i^* = \mathrm{e}^{\nu}\mathrm{e}^{-\beta a_i} \tag{14}$$

where ν and β can be obtained from the constraints in equations (1) and (6). An approximate evaluation of n_i^* is possible if $a_i = ia$ with $i = 1, 2, \ldots, g$. If $g >> 1$, the sums in (1) and (6) can be accurately replaced by infinite sums of the geometric series. In this case ν and β can be derived and replaced in (14) and one gets the most probable vector in terms of known parameters:

$$n_i^* = \frac{n}{r-1}\left(\frac{r-1}{r}\right)^i, \; i = 1, 2, .., \tag{15}$$

where $r = D/na$ is the aggregate demand per agent divided by the smallest productivity. In the limit $r >> 1$, one gets

$$n_i^* \simeq \frac{n}{r}\mathrm{e}^{-i/r}. \tag{16}$$

Notice that equation (16) defines the occupation vectors that maximizes the probability given in equation (10); they are events and not random variables. However, if the economy is in the state $\mathbf{n}^*$, and if you select a worker at random, the probability of finding her/him in sector i is

$$P(i|\mathbf{n}^*) = \frac{n_i^*}{n} \simeq \frac{na}{D}\exp\left(-\frac{na}{D}i\right). \tag{17}$$

Hence the marginal probability that a worker is in sector i, given $\mathbf{n}^*$ follows the exponential distribution.

All the previous results depend on the hypothesis for which Equation (10) holds true, that is the equiprobability of all the configurations $\mathbf{x}$ compatible with the constraints (1) and (6). This is typical in classical statistical mechanics, where the uniform $\mathbf{x}$-distribution is the only one compatible with the underlying deterministic dynamics (via Liouville's theorem). For Boltzmann himself, this link was not enough, and the dynamical part of his work (Boltzmann's equation, as well as the related H-Theorem) was introduced in order to prove that the most probable $\mathbf{n}^*$ summarizing the equilibrium distribution is actually achieved as a consequence of atomic/molecular collisions. Indeed, the equilibrium distribution (if it exists) depends on the detail of the dynamics with which workers change sector. In Physics, Brillouin's ideas and a generalized Ehrenfest urn model vindicate Boltzmann's attempt which can also encompass quantum statistics (see references [5–7] for the so-called *Ehrenfest-Brillouin Model* or *EBM*, and the original paper by Paul and Tatiana Ehrenfest for the Ehrenfest urn model [8]). One cannot say that Boltzmann would be satisfied by this approach, as it is intrinsically probabilistic. In fact, he devoted an unbelievable mass of mechanical calculations to obtain his fundamental results. In any case, the *Ehrenfest-Brillouin Model* and its relationship with the AYM will be the subject of the next section.

II. MARKOVIAN DYNAMICS FOR AYM

We first introduce *unary* moves (or jumps). Let $\mathbf{n}$ denote the present state of the system, defined in terms of the occupation vector:

$$\mathbf{n} := (n_1, \ldots, n_g) \tag{18}$$

where, as before, n_i denotes the number of workers in the i-th sector with productivity a_i; then a unary move means that either n_i increases by one (creation) or n_i decreases by one (destruction or annihilation). We write

$$\mathbf{n}^j := (n_1, \ldots, n_j + 1, \ldots, n_g) \tag{19}$$

for creation of one unit and

$$\mathbf{n}_j := (n_1, \ldots, n_j - 1, \ldots, n_g) \tag{20}$$

for annihilation of one unit. A unary move consists in an annihilation step followed by a creation step. Thus it conserves the total number of workers, but does not fulfill the demand constraint, except for the trivial case $j = i$. The conservation of the number of workers is achieved by unary moves where a worker leaves sector i to join sector j. To fix the ideas, we assume $i < j$ and we write

$$\mathbf{n}_i^j := (n_1, \ldots, n_i - 1, \ldots, n_j + 1, \ldots, n_g) \tag{21}$$

to denote a unary move. However, as mentioned above, unary moves violate the demand constraint, if all the sectors have different productivities. Let us denote the components of the vector $\mathbf{n}_i^j$ by $(n'_1, \ldots, n'_g)$. If all the sectors have different productivities and $\sum_{k=1}^{g} a_k n_k = D$ before the move, then, for sure, one has that $\sum_{k=1}^{g} a_k n'_k \neq D$ after the move. The difference between the two sums is $a_j - a_i \neq 0$ as $a_i \neq a_j$. Under the hypothesis of different sector productivities, in order to conserve demand, one should use binary moves at least, consisting of a sequence of two annihilations and two creations so that the total production level does not change. In a generic binary move, a worker leaves sector i to join sector l and another worker leaves section j to join sector m. Let us denote the state after the binary move by $\mathbf{n}_{ij}^{mn}$, where $n'_i = n_i - 1$, $n'_j = n_j - 1$, $n'_l = n_l + 1$, and $n'_m = n_m + 1$. The difference in total product becomes $a_m + a_l - a_i - a_j$. When sector productivities are all different and incommensurable, this difference vanishes only if the two workers come back to their sectors ($l = i$ and $m = j$) or if they mutually exchange their sectors ($l = j$ and $m = i$). Indeed one has to take into account that $a_i \in \mathbb{R}$, $\forall i$ and that $n_i \in \mathbb{N}$, $\forall i$. In both cases, binary moves do not change the total number of workers per sector and any initial distribution of workers is conserved. The same applies to moves where r workers leave their sectors to join other sectors. If all the sectors have different productivities, in order to fulfill the demand constraint, workers have to rearrange so that the n_is do not vary.

A way to avoid this boring situation is to assume that $a_i = ia$ where, as usual, $i \in \{1, ..., g\}$, that is productivities are multiples of the lowest productivity a_1. In this case, binary transitions can conserve demand, but only a subset of occupation vectors can be reached from a given initial state fulfilling the demand constraint. In order to illustrate this point, let us consider a case in which there are three sectors with respective productivities $a_1 = a$, $a_2 = 2a$, and $a_3 = 3a$ and $n = 3$ workers. Suppose that the initial demand is set at the following level $D = 6a$. For instance, this situation is fulfilled by an initial state in which all the three workers are in sector 2. Therefore, the initial occupation vector is $\mathbf{n} = (0, 3, 0)$. An allowed binary move leads to state $\mathbf{n}_{22}^{13} = (1, 1, 1)$ where two workers leave sector 2 to jump to sectors 1 and 3, respectively. This state fulfills the demand constraint as $a_1 n_1 + a_2 n_2 + a_3 n_3 = 6a$.

After defining binary moves and proper constraints on accessible states, it is possible to define a dynamics on AYM using an appropriate transition probability. A possible choice is:

$$P(\mathbf{n}_{ij}^{lm}|\mathbf{n}) = A_{ij}^{lm}(\mathbf{n}) n_i n_j (1 + c n_l)(1 + c n_m), \tag{22}$$

where $A_{ij}^{lm}(\mathbf{n})$ is a suitable normalization factor and c is a model parameter, whose meaning will be explained in the following. This equation can be justified by considering a binary move as a sequence of two destructions and two creations. For the moment, let us forget the demand constraint. When a worker leaves sector i, he/she does so with probability

$$P(\mathbf{n}_i|\mathbf{n}) = \frac{n_i}{n} \tag{23}$$

proportional to the number of workers in sector i before the move. When he/she joins sector l, this happens with probability

$$P(\mathbf{n}^l|\mathbf{n}) = \frac{1 + cn_l}{g + cn}. \tag{24}$$

Remember that the probability of any creation or destruction is a function of the actual occupation number, that is the occupation number seen by the moving agent. Therefore, in general, the worker will not choose the arrival sector independently from its occupation before the move, but he/she will be likely to join more populated sectors if $c > 0$ or he/she will prefer to stay away from populated sectors if $c < 0$. Finally, he/she will be equally likely to join any sector if $c = 0$. Further notice that, if $c \geq 0$, there is no restriction in the number of workers who can occupy a sector, whereas for negative values of c, only situations in which $1/|c|$ is integer are allowed and no more than $1/|c|$ workers can be allocated in each sector [5, 6].

In principle, given equation (22), one can explicitly write the transition matrix and find the stationary (or invariant) distribution by diagonalizing it (this method is described in standard textbooks on Markov chain and summarized in [9]). However, when the number g of sectors is large, the direct method becomes numerically cumbersome. In this case, the *master equation* can be used. If $P(\mathbf{n}, t)$ denotes the probability that the economy is in state $\mathbf{n}$ at step t, one has

$$P(\mathbf{n}, t+1) - P(\mathbf{n}, t) = \sum_{\mathbf{n}' \neq \mathbf{n}} [P(\mathbf{n}|\mathbf{n}')P(\mathbf{n}', t) - P(\mathbf{n}'|\mathbf{n})P(\mathbf{n}, t)]. \tag{25}$$

If there is a probability distribution $\pi(\mathbf{n})$ that satisfies the *detailed balance* condition, defined as

$$P(\mathbf{n}|\mathbf{n}')\pi(\mathbf{n}') = P(\mathbf{n}'|\mathbf{n})\pi(\mathbf{n}) \tag{26}$$

then if $P(\mathbf{n}, t) = \pi(\mathbf{n})$ one gets

$$P(\mathbf{n}, t+1) = P(\mathbf{n}, t) = \pi(\mathbf{n}), \tag{27}$$

that is $\pi(\mathbf{n})$ is the invariant distribution of the chain, a.k.a. stationary distribution. This is a formal property, and nothing assures that it will be achieved by the chain. A Markov chain with an invariant distribution satisfying detailed balance is said *reversible* with respect to the distribution $\pi(\mathbf{n})$. However, if a Markov chain is irreducible (i.e. all possible states $\mathbf{n}$ sooner or later communicate) and it is aperiodic (all entries of the $s-$step matrix are positive for all $s > s_0$), then there exists a unique invariant distribution $\pi(\mathbf{n})$ and

$$\lim_{t->\infty} P(\mathbf{n},t|\mathbf{n}',0) = \pi(\mathbf{n}), \tag{28}$$

independent of the initial state $\mathbf{n}'$; this means that the invariant distribution coincides with the equilibrium distribution.

Turning to the chain (22), in the absence of constraints, all possible states are sooner or later reachable *via* binary moves. Notice that at the end of each move a worker can go back to the starting sector; hence binary moves cover unary ones and no periodicity is present. The presence of constraints reduces the set of accessible states, but these states can be reached from any other state by means of (22) if productivities are of the form $a_i = ia$. In the general case, if binary moves were not enough to let all states compatible with the constraint communicate, we could consider ternary moves or even n-ary moves until the ergodic property were fulfilled. These moves are governed by a straightforward extension of (22). Given that all $m-$move chains share the same equilibrium distribution, we can assume that the binary chain is irreducible and aperiodic without loss of generality. Then we can look for the invariant distribution of the binary chain, which will coincide with the equilibrium distribution.

Let us now apply detailed balance to the transition probability given in equation (22). The inverse transition move has probability

$$P(\mathbf{n}|\mathbf{n}_{ij}^{mn}) = A_{ij}^{lm}(\mathbf{n}_{ij}^{lm})(n_l+1)(n_m+1)(1+c(n_i-1))(1+c(n_j-1)). \tag{29}$$

As a consequence of equations (23) and (24) and taking into account the demand constraint, it is possible to show that $A_{ij}^{lm}(\mathbf{n}_{ij}^{lm}) = A_{ij}^{lm}(\mathbf{n})$ [6]. Then, the detailed balance condition becomes

$$\frac{\pi(\mathbf{n}_{ij}^{lm})}{\pi(\mathbf{n})} = \frac{n_i n_j (1+cn_l)(1+cn_m)}{(n_l+1)(n_m+1)(1+c(n_i-1))(1+c(n_j-1))}. \tag{30}$$

From equation (30), one can see what happens for some remarkable values of c. If $c=1$ then one gets $\pi(\mathbf{n}_{ij}^{lm})/\pi(\mathbf{n}) = 1$, meaning that $\pi(\mathbf{n})$ is uniform on the set of accessible states. If $c=-1$ then one has again $\pi(\mathbf{n}_{ij}^{lm})/\pi(\mathbf{n}) = 1$ but only if $n_i = n_j = 1$ and $n_l = n_m = 0$;

all the states satisfying an exclusion principle and the demand constraint have the same probability. If $c = 0$, the ratio in equation (30) becomes

$$\frac{\pi(\mathbf{n}_{ij}^{lm})}{\pi(\mathbf{n})} = \frac{n_i n_j}{(n_l + 1)(n_m + 1)}, \tag{31}$$

yielding an equilibrium distribution given by

$$\pi(\mathbf{n}) \propto \frac{1}{\Pi_{i=1}^{g} n_i!}, \tag{32}$$

The general solution of (30) is

$$\pi(\mathbf{n}) \propto \Pi_{i=1}^{g} \frac{c^{n_i} (1/c)^{[n_i]}}{n_i!} \tag{33}$$

where $x^{[m]} = x(x+1)...(x+m-1)$ is the Pochhammer symbol. The distribution (33) is a generalized Polya distribution (whose prize can be positive, negative or null) whose domain is just "all the states $\mathbf{n}$ compatible with the constraints" or, equally, "all the states $\mathbf{n}$ reachable from $\mathbf{n}_0$ by (22)". The values $c = 0, \pm 1$ are the only ones appearing in the applications of (22) to particles in Physics. Outside the physical realm, there is no reason to be restricted to these three possibilities and there is room for the application of the so-called parastatistics. Notice that equation (32) means that workers' configurations are uniformly distributed. As mentioned above, this is the only case considered in the book by Aoki and Yoshikawa [2]. Notice further that for all the other values of c no equilibrium probability distribution is uniform either for sector occupations (as in the cases $c = \pm 1$) or for configurations (as in the aforementioned case $c = 0$).

Finally, one can further show that the general solution of the conditional maximum problem for $\pi(\mathbf{n})$ is:

$$n_i^* = \frac{1}{\mathrm{e}^{-\nu}\mathrm{e}^{\beta a_i} - c} \tag{34}$$

which coincides with (14) in the case $c = 0$. The Bose-Einstein distribution is obtained for $c = 1$.

III. DISCUSSION AND CONCLUSIONS

In summary, we have shown that when Markovian dynamics is taken into account, AYM has different equilibrium distributions depending on the formula for transition probabilities. In our version of the dynamic AYM, transition probabilities depend on a parameter

c tuning the choice of a new sector for workers leaving their sector. The solution of Aoki and Yoshikawa is recovered only in the case $c = 0$. All the other possible cases give different equilibrium probability distributions, including the so-called Bose-Einstein distribution for $c = 1$. This shows that AYM is compatible with an infinite set of possible statistical equilibria.

In the case $c = 0$, the exponential distribution (17) is the continuous limit of the geometrical (15) distribution - the equilibrium distribution when D is not large. In Physics, the so-called Maxwell-Boltzmann distribution is attained for large values of energy, but this is not always the case is in Economics, where a decrease of the aggregate demand (the analog of energy) is always possible.

As a further general comment, one can notice that, in Physics, each energy level is degenerate and represented by g_i cells, depending on the structure of phase space. For a monoatomic gas, each energy level ε_i corresponds to $g_i \propto i^{1/2}$ single-particle states, and the most probable level occupation number N_i^* becomes $N_i^* \propto \mathrm{e}^{\nu} i^{1/2} \mathrm{e}^{-\beta\varepsilon_i}$. For this reason, the energy equilibrium distribution is a $Gamma(3/2, \beta^{-1})$, rather than the exponential distribution $Gamma(1, \beta^{-1})$, where β is the inverse temperature. If these quantities are interpreted in Economics, a factor i^{α}, $\alpha \geq 1$ can be introduced, taking into account that increasing productivities are usually accompanied by an increasing number of industrial sectors. In this case, the equilibrium distribution would become a $Gamma(\alpha + 1, r)$. Introducing a multiplicity for sectors with equal productivity, it would be interesting to study the fermionic case $c < 0$, where labour or other production factors tend to occupy less populated sectors.

The dynamics discussed in section II does satisfy both constraints in equations (1) and (6) at each transition step. However, it is possible to consider versions of the dynamic AYM where the demand constraint is satisfied only at the end of a period. In such versions, the exogenous demand could be given by a stochastic process $D(t)$ and announced at the beginning of the period. In this case, the economy would be obliged to move from the previous equilibrium $Y(t-1) = D(t-1)$ to the new one and the model would lead to a sequence of annealings. In these versions, the rate of convergence towards statistical equilibrium would be of great interest. One could further try to endogenize both demand and the distribution of productivity by designing a more complete stochastic model of a closed economy, taking into account some features of AYM. In our opinion all these directions of research are worth exploring and will be the subject of future investigations.

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[1] H. Yoshikawa, *The Role of Demand in Macroeconomics*, Japanese Economic Review, **54**, 1–27, (2003).

[2] M. Aoki and H. Yoshikawa, *Reconstructing Macroeconomics*, Cambridge University Press, Cambridge UK, 2007.

[3] L. Boltzmann, *Studien über das Gleichgewicht der lebendigen Kraft zwischen bewegten materiellen Punkten*, Wiener Berichte **58**, 517–560, (1868), reprinted in Wissenschaftliche Abhandlungen, Vol. I (Hasenoehrl ed.) 49–96, (1909).

[4] L. Boltzmann, *Über die Beziehung zwischen dem zweiten Hauptsatze der mechanischen Wärmetheorie und der Wahrscheinlichkeitsrechnung, respektive den Sätzen uber das Wärmegleichgewicht*, Wiener Berichte **76**, 373–435 (1877), reprinted in Wissenschaftliche Abhandlungen, Vol. II (Hasenoehrl ed.) 164–223, (1909).

[5] L. Brillouin, *Comparaison des différentes statistiques appliquées aux problèmes de quanta*, Annales de Physique, Paris, **VII**, 315–331 (1927).

[6] D. Costantini and U. Garibaldi, *A Purely Probabilistic Representation for the Dynamics of a Gas of Particles*, Foundations of Physics, **30**, 81–99, (2000).

[7] D. Costantini and U. Garibaldi, *The Ehrenfest Fleas: From Model to Theory*, Synthese, **139**, 107–142, (2004).

[8] P. Ehrenfest and T. Ehrenfest, *Begriffliche Grundlagen der statistischen Auffassung in der Mechanik*, in Encyklopädie Der Mathematischen Wissenschaften, **IV:2:II**, B.G. Teubner, Leipzig, Germany, 1912. English translation: P. Ehrenfest and T. Ehrenfest, The Conceptual Foundations of the Statistical Approach in Mechanics, Dover, Toronto, 1990.

[9] E. Scalas, U. Garibaldi and S.Donadio, *Statistical equilibrium in simple exchange games I-Methods of solution and application to the Bennati-Dragulescu-Yakovenko (BDY) game*, Eur. Phys. J. B, **53**, 267-272, (2006).