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Do *Real* numbers really move? Language, thought, and gesture: the embodied cognitive foundations of mathematics

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Abstract. Robotics, artificial intelligence and, in general, any activity involving computer simulation and engineering relies, in a fundamental way, on mathematics. These fields constitute excellent examples of how mathematics can be applied to some area of investigation with enormous success. This, of course, includes embodied oriented approaches in these fields, such as Embodied Artificial Intelligence and Cognitive Robotics. In this chapter, while fully endorsing an embodied oriented approach to cognition, I will address the question of the nature of mathematics itself, that is, mathematics not as an application to some area of investigation, but as a human conceptual system with a precise inferential organization that can be investigated in detail in cognitive science. The main goal of this piece is to show, using techniques in cognitive science such as cognitive semantics and gestures studies, that concepts and human abstraction in general (as it is exemplified in a sublime form by mathematics) is ultimately embodied in nature.

1 A challenge to embodiment: The nature of Mathematics

Mathematics is a highly technical domain, developed over several millennia, and characterized by the fact that the very entities that constitute what Mathematics is are idealized mental abstractions. These entities cannot be perceived directly through the senses. Even, say, a point, which is the simplest entity in Euclidean geometry, can't be actually *perceived*. A point, as defined by Euclid is a dimensionless entity, an entity that has only location but no extension. No super-microscope will ever be able to allow us to actually perceive a point. A point, after all, with its precision and clear identity, is an idealized abstract entity. The imaginary nature of mathematics becomes more evident when the entities in question are related to *infinity* where, because of the finite nature of our bodies and brains, no direct experience can exist with the infinite itself.

Yet, infinity in mathematics is essential. It lies at the very core of many fundamental concepts such as limits, least upper bounds, topology, mathematical induction, infinite sets, points at infinity in projective geometry, to mention only a few. When studying the very nature of mathematics, the challenging and intriguing question that comes to mind is the following: if mathematics is the product of human ideas, how can we explain the nature of mathematics with its unique features such as precision, objectivity, rigor, generalizability, stability, and, of course, applicability to the real world? Such question doesn't represent a real problem for approaches inspired in platonic philosophies, which rely on the existence of transcendental worlds of ideas beyond human existence. But this view doesn't have any support based on scientific findings and doesn't provide any link to current empirical work on human ideas and conceptual systems (it may be supported, however, as a matter of faith, not of science, by many Platonist scientists and mathematicians). The question doesn't pose major problems to purely formalist philosophies either, because in that worldview mathematics is seen as a manipulation of meaningless symbols. The question of the origin of the meaning of mathematical ideas doesn't even emerge in the formalist arena. For those studying the human mind scientifically, however (e.g., cognitive scientists), the question of the nature of mathematics is indeed a real challenge, especially for those who endorse an *embodied* oriented approach to cognition. How can an embodied view of the mind give an account of an abstract, idealized, precise, sophisticated and powerful domain of ideas if direct bodily experience with the subject matter is not possible?

In *Where Mathematics Comes From*, Lakoff and Núñez (2000) give some preliminary answers to the question of the cognitive origin of mathematical ideas. Building on findings in mathematical cognition, and using mainly methods from Cognitive Linguistics, a branch of Cognitive Science, they suggest that most of the idealized abstract technical entities in Mathematics are created via human cognitive mechanisms that extend the structure of bodily experience (thermic, spatial, chromatic, etc.) while preserving the inferential organization of these domains of bodily experience. For example, linguistic expressions such as “send her my *warm* hellos” and “the teacher was very *cold* to me” are statements that refer to the somewhat abstract domain of Affection. From a purely literal point of view, however, the language used belongs to the domain of Thermic experience, not Affection. The meaning of these statements and the inferences one is able to draw from them is structured by precise mappings from the Thermic domain to the domain of Affection: Warmth is mapped onto presence of affection, Cold is mapped onto lack of affection, X is warmer than Y is mapped onto X is more affectionate than Y, and so on. The ensemble of inferences is modeled by one conceptual metaphorical mapping, which in this case is called AFFECTION IS WARMTH¹. Research in Cognitive Linguistics has shown that these phenomena are not simply about “language,” but rather they are about thought. In cognitive science

¹ Following a convention in Cognitive Linguistics, the name of a conceptual metaphorical mapping is capitalized.

the complexities of such abstract and non/literal phenomena have been studied through mechanisms such as conceptual metaphors (Lakoff & Johnson, 1980; Sweetser, 1990; Lakoff, 1993; Lakoff & Núñez, 1997; Núñez & Lakoff, in press; Núñez, 1999, 2000), conceptual blends (Fauconnier & Turner, 1998, 2002; Núñez, in press), conceptual metonymy (Lakoff & Johnson, 1980), fictive motion and dynamic schemas (Talmy, 1988, 2003), and aspectual schemas (Narayanan, 1997). Based on these findings Lakoff and Núñez (2000) analyzed many areas in mathematics, from set theory to infinitesimal calculus, to transfinite arithmetic, and showed how, via everyday human embodied mechanisms such as conceptual metaphor and conceptual blending, the inferential patterns drawn from direct bodily experience in the real world get extended in very specific and precise ways to give rise to a new emergent inferential organization in purely imaginary domains². For the remainder of this chapter we will be building on these results as well as on the corresponding empirical evidence provided by the study of human speech-gesture coordination. Let us now consider a few mathematical examples.

2 Limits, curves, and continuity

Through the careful analysis of technical books and articles in mathematics, we can learn a good deal about what structural organization of human everyday ideas have been used to create mathematical concepts. For example, let us consider a few statements regarding limits in infinite series, equations of curves in the Cartesian plane, and continuity of functions, taken from mathematics books such as the now classic *What is Mathematics?* by R. Courant & H. Robbins (1978).

a) Limits of infinite series

In characterizing limits of infinite series, Courant & Robbins write:

“We describe the behavior of s_n by saying that the sum s_n *approaches* the limit 1 as n *tends* to infinity, and by writing

$$1 = 1/2 + 1/2^2 + 1/2^3 + 1/2^4 + \dots”$$
 (p. 64, our emphasis)

Strictly speaking, this statement refers to a sequence of discrete and motionless partial sums of s_n (real numbers), corresponding to increasing discrete and motionless values taken by n in the expression $1/2^n$ where n is a natural number. If we examine this statement closely we can see that it describes some facts about numbers and about the result of discrete operations with numbers, but that there is *no motion* whatsoever involved. No entity is actually *approaching* or *tending* to anything. So, why then did Courant and Robbins (or mathematicians in general, for that matter) use dynamic

² The details of how conceptual metaphor and conceptual blending work go beyond the scope of this piece. For a general introduction to these concepts see Lakoff & Núñez (2000, chapters 1-3), and the references given therein.

language to express static properties of static entities? And what does it mean to say that the “sum s_n approaches,” when in fact a sum is simply a fixed number, a result of an operation of addition?

b) Equations of lines and curves in the Cartesian Plane

Regarding the study of conic sections and their treatment in analytic geometry, Courant & Robbins’ book says:

“The hyperbola *approaches* more and more nearly the two straight lines $qx \pm py = 0$ as we *go out farther and farther* from the origin, but it never actually *reaches* these lines. They are called the asymptotes of the hyperbola.” (p. 76, our emphasis).

And then the authors define hyperbola as “the locus of all points P the *difference* of whose distances to the two points $F(\sqrt{p^2 + q^2}, 0)$ and $F'(-\sqrt{p^2 + q^2}, 0)$ is $2p$.” (p. 76, original emphasis).

Strictly speaking, the definition only specifies a “locus of all points P ” satisfying certain properties based exclusively on arithmetic differences and *distances*. Again, no entities are actually moving or approaching anything. There are only statements about static differences and static distances. Besides, as Figure 1 shows, the authors provide a graph of the hyperbola in the Cartesian Plane (bottom right), which in itself is a static illustration that doesn’t have the slightest insinuation of motion (like symbols for arrows, for example). The figure illustrates the idea of *locus* very clearly, but it says nothing about motion. Moreover the hyperbola has *two* distinct and separate *loci*. Exactly which one of the two is then “the” moving agent (3rd person singular) in the authors’ statement “*the hyperbola approaches* more and more nearly the two straight lines $qx \pm py = 0$ as we *go out farther and farther* from the origin”?

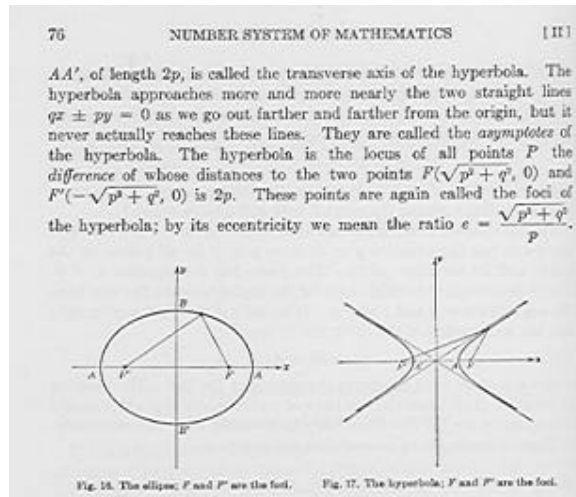


Fig. 1. Original text analyzing the hyperbola as published in the now classic book *What is Mathematics?* by R. Courant & H. Robbins (1978).

c) Continuity

Later in the book, the authors analyze cases of continuity and discontinuity of trigonometric functions in the real plane. Referring to the function $f(x) = \sin 1/x$ (whose graph is shown in Figure 2) they say: "... since the denominators of these fractions increase without limit, the values of x for which the function $\sin(1/x)$ has the values 1, -1, 0, will cluster nearer and nearer to the point $x = 0$. Between any such point and the origin there will be still an infinite number of *oscillations* of the function" (p. 283, our emphasis).

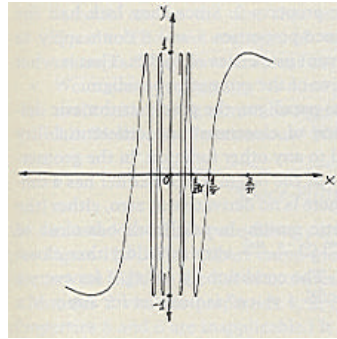


Fig. 2. The graph of the function $f(x) = \sin 1/x$.

Once again, if, strictly speaking, a function is a mapping between elements of a set (coordinate values on the x -axis) with one and only one of the elements of another set (coordinate values on the y -axis), all what we have is a static correspondence between points on the x -axis with points on the y -axis. How then can the authors (or mathematicians in general) speak of "*oscillations* of the function," let alone an infinite number of them?

These three examples show how ideas and concepts are described, defined, illustrated, and analyzed in mathematics books. You can pick your favorite mathematics books and you will see similar patterns. You will see them in topology, fractal geometry, space-filling curves, chaos theory, and so on. Here, in all three examples, static numerical structures are involved, such as partial sums, geometrical loci, and mappings between coordinates on one axis with coordinates on another. Strictly speaking, absolutely no dynamic entities are involved in the formal definitions of these terms. So, if no entities are really moving, why do authors speak of "approaching," "tending to," "going farther and father," and "oscillating"? Where is

this motion coming from? What does dynamism mean in these cases? What role is it playing (if any) in these statements about mathematics facts?

We will first look at pure mathematics to see whether we can find answers to these questions. Then, in order to get some deeper insight into them, we will turn into human language and real-time speech-gesture coordination.

3 Looking at pure Mathematics

Among the most fundamental entities and properties the above examples deal with are the notion of real number and continuity. Let us look at how pure mathematics defines and provides the inferential organization of these concepts.

In pure mathematics, entities are brought to existence via formal definitions, formal proofs (theorems) or by axiomatic methods (i.e., by declaring the existence of some entity without the need of proof. For example, in set theory the axiom of infinity assures the existence of infinite sets. Without that axiom, there are no infinite sets). In the case of real numbers, ten axioms taken together, fully characterize this number system and its inferential organization (i.e., theorems about real numbers). The following are the axioms of the real numbers.

1. Commutative laws for addition and multiplication.
2. Associative laws for addition and multiplication.
3. The distributive law.
4. The existence of identity elements for both addition and multiplication.
5. The existence of additive inverses (i.e., negatives).
6. The existence of multiplicative inverses (i.e., reciprocals).
7. Total ordering.
8. If x and y are positive, so is $x + y$.
9. If x and y are positive, so is $x \cdot y$.
10. The Least Upper Bound axiom.

The first 6 axioms provide the structure of what is called a *field* for a set of numbers and two binary operations. Axioms 7 through 9, assure ordering constraints. The first nine axioms fully characterize ordered fields, such as the rational numbers with the operations of addition and multiplication. Up to here we have already a lot of structure and complexity. For instance we can characterize and prove theorems about all possible numbers that can be expressed as the division of two whole numbers (i.e., rational numbers). With the rational numbers we can describe with any given (finite) degree of precision the proportion given by the perimeter of a circle and its diameter (e.g., 3.14; 3.1415; etc.). We can also locate along a line (according to their magnitude) any two different rational numbers and be sure (via proof) that there will always be infinitely many more rational numbers between them (a property referred to as density). With the rational numbers, however, we can't "complete" the points on this line, and we can't express with infinite exactitude the magnitude of the proportion mentioned

above ($\pi = 3.14159 \dots$). For this we need the full extension of the real numbers. In axiomatic terms, this is accomplished by the tenth axiom: the Least Upper Bound axiom. All ten axioms characterize a complete ordered field.

In what concerns our original question of where is motion coming from in the above mathematical statements about infinite series and continuity, we don't find any answer in the first nine axioms of real numbers. All nine axioms simply specify the existence of static properties regarding binary operations and their results, and properties regarding ordering. There is no explicit or implicit reference to motion in these axioms. Since what makes a real number a real number (with its infinite precision) is the Least Upper Bound axiom, it is perhaps this very axiom that hides the secret of motion we are looking for. Let's see what this axiom says:

10. Least Upper Bound axiom: every nonempty set that has an upper bound has a least upper bound.

And what exactly is an upper bound and a least upper bound? This is what pure mathematics says:

Upper Bound

b is an upper bound for S if

$x \leq b$, for every x in S .

Least Upper Bound

b_0 is a least upper bound for S if

- b_0 is an upper bound for S , and
- $b_0 \leq b$ for every upper bound b of S .

Once again, all what we find are statements about motionless entities such as universal quantifiers (e.g., for every x ; for every upper bound b of S), membership relations (e.g., for every x in S), greater than relationships (e.g., $x \leq b$; $b_0 \leq b$), and so on. In other words, there is absolutely no indication of motion in the Least Upper Bound axiom, or in any of the other nine axioms. In short, the axioms of real numbers, which are supposed to completely characterize the "truths" (i.e., theorems) of real numbers don't tell us anything about a sum "approaching" a number, or a number "tending to" infinity (whatever that may mean!).

Let's try continuity. What does pure mathematics say about it?

Mathematics textbooks define continuity for functions as follows:

- A function f is continuous at a number a if the following three conditions are satisfied:
 1. f is defined on an open interval containing a ,
 2. $\lim_{x \rightarrow a} f(x)$ exists, and
 3. $\lim_{x \rightarrow a} f(x) = f(a)$.

Where by $\lim_{x \rightarrow a} f(x)$ what is meant is the following:

Let a function f be defined on an open interval containing a , except possibly at a itself, and let L be a real number. The statement

$$\lim_{x \rightarrow a} f(x) = L$$

means that $\forall \epsilon > 0, \exists \delta > 0,$

$$\text{such that if } 0 < |x - a| < \delta, \\ \text{then } |f(x) - L| < \varepsilon.$$

As we can see, pure formal mathematics defines continuity in terms of limits, and limits in terms of

- static universal and existential quantifiers predicating on static numbers (e.g., $\forall \varepsilon > 0, \exists \delta > 0$), and
- on the satisfaction of certain conditions which are described in terms of motionless arithmetic difference (e.g., $|f(x) - L|$) and static smaller than relations (e.g., $0 < |x - a| < \delta$).

That's it. Once again, these formal definitions don't tell us anything about a sum "approaching" a number, or a number "tending to" infinity, or about a function "oscillating" between values (let alone doing it infinitely many times, as in the function $f(x) = \sin 1/x$).

But this shouldn't be a surprise. Lakoff & Núñez (2000), using techniques from cognitive linguistics showed what well-known contemporary mathematicians had already pointed out in more general terms (Hersh, 1997; Henderson, 2001):

- The structure of human mathematical ideas, and its inferential organization, is richer and more detailed than the inferential structure provided by formal definitions and axiomatic methods. Formal definitions and axioms neither fully formalize nor generalize human concepts.

We can see this with a relatively simple example taken from Lakoff & Núñez (2000). Consider the function $f(x) = x \sin 1/x$ whose graph is depicted in Figure 3.

$$f(x) = \begin{cases} x \sin 1/x & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

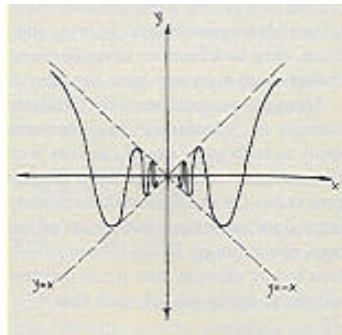


Fig. 3. The graph of the function $f(x) = x \sin 1/x$.

According to the $\varepsilon - \delta$ definition of continuity given above, this function is continuous at every point. For all x , it will always be possible to find the specified ε 's and δ 's necessary to satisfy the conditions for preservation of closeness. However,

according to the everyday notion of continuity—*natural continuity* (Núñez & Lakoff, 1998)—as it was used by great mathematicians such as Kepler, Euler, and Newton and Leibniz, the inventors of infinitesimal calculus in the 17th Century, this function is *not* continuous. According to the inferential organization of natural continuity, certain conditions have to be met. For instance, in a naturally continuous line we are supposed to be able to tell how long the line is between two points. We are also supposed to be able to describe essential components of the motion of a point along that line. With this function we can't do that. Since the function "oscillates" infinitely many times as it "approaches" the point (0, 0) we can't really tell how long the line is between two points located on the left and right sides of the plane. Moreover, as the function approaches the origin (0, 0) we can't tell, say, whether it will "cross" from the right plane to the left plane "going down" or "going up." This function violates these basic properties of natural continuity and therefore it is not continuous. The function $f(x) = x \sin 1/x$ is thus $\epsilon - \delta$ continuous but it is not naturally continuous. The point is that the formal $\epsilon - \delta$ definition of continuity doesn't capture the inferential organization of the human everyday notion of continuity, and it doesn't generalize the notion of continuity either.

The moral here is that what is characterized formally in mathematics leaves out a huge amount of inferential organization of the human ideas that constitute mathematics. As we will see, this is precisely what happens with the dynamic aspects of the expressions we saw before, such as "approaching," "tending to," "going farther and farther," "oscillating," and so on. Motion, in those examples, is a genuine and constitutive manifestation of the nature of mathematical ideas. In pure mathematics, however, motion is not captured by formalisms and axiomatic systems.

4 Embodied Cognition

It is now time to look, from the perspective of embodied cognition, at the questions we asked earlier regarding the origin of motion in the above mathematical ideas. In the case of limits of infinite series, motion in "the sum s_n approaches the limit 1 as n tends to infinity" emerges *metaphorically* from the successive values taken by n in the sequences as a whole. It is beyond the scope of this chapter to go into the details of the mappings involved in the various underlying conceptual metaphors that provide the required dynamic inferential organization (for details see Lakoff & Núñez, 2000). But we can at least point out to some of the many conceptual metaphors and metonymies³ involved.

³ A conceptual metonymy is a cognitive mechanism that allow us to conceive a part of a whole standing for the whole, as when we say *Washington and Paris have quite different views on these issues*, meaning the governments of two entire nations, namely, United States and France.

- There are conceptual metonymies in cases such as a partial sum *standing for* the entire infinite sum;
- there are conceptual metaphors in cases where we conceptualize the sequence of these metonymical sums as a *unique trajector*⁴ moving in space (as it is indicated by the use of the 3rd person singular in *the* sum s_n approaches);
- there are conceptual metaphors for conceiving infinity as a single location in space such that a metonymical n (standing for the entire sequence of values) can “tend to;”
- there are conceptual metaphors for conceiving 1 (not as a mere natural number but as an infinitely precise *real* number) as the result of the infinite sum; and so on.

Notice that none of these expressions can be *literal*. The facts described in these sentences don't exist in any real perceivable world. They are metaphorical in nature. It is important to understand that these conceptual metaphors and metonymies are not simply “noise” added on top of pre-defined formalisms. They are in fact *constitutive* of the very embodied ideas that make mathematical ideas possible. It is the inferential organization provided by our embodied understanding of “approaching” and “tending to” that is at the core of these mathematical ideas.

In the case of the hyperbola, the moving agent is one holistic object, the hyperbola in the Real plane. This object, which has two distinctive separate parts, is conceptualized as one single trajector metaphorically moving away from the origin. Via conceptual metonymies and metaphors similar to the ones we saw for the case of infinite series, the hyperbola is conceived as a trajector tracing the line, which describes the geometrical locus of the hyperbola itself. In this case, of course, because we are dealing with real numbers, the construction is done on non-countable infinite ($>\aleph_0$) discrete real values for x , which are progressively bigger in absolute terms. The direction of motion is stated as moving away from the origin of the Cartesian coordinates, and it takes place in both directions of the path schemas defined by the two branches of the hyperbola, *simultaneously*. The hyperbola not “reaching” the asymptotes is the cognitive way of characterizing the mathematically formalized fact that there are no values for x and y that satisfy equations

$$qx \pm py = 0 \text{ and } (x^2/p^2) - (y^2/q^2) = 1$$

Notice that characterizing the hyperbola as “not reaching” the asymptotes provides the same *extensionality* (i.e., it gives the same resulting cases) as saying that there is an “absence of values” satisfying the above equations. The inferential organization of these two cases, however, is cognitively very different⁵.

⁴ In cognitive linguistics, “trajector” is a technical term used to refer to the distinct entity that performs the motion traced by a trajectory. The trajector moves against a background called “landscape.”

⁵ In order to clarify this point, consider the following two questions: (a) What Alpine European country does not belong to the European Union?, and (b) What is the country whose currency is the Swiss Franc? The extensionality provided by the answers to both questions is

Finally, in what concerns our “oscillating” function example, the moving object is again one holistic object, the trigonometric function in the Real plane, constructed metaphorically from non-countable infinite ($> \aleph_0$) discrete real values for x , which are progressively smaller in absolute terms. In this case motion takes place in a specific manner, towards the origin from two opposite sides (i.e., for negative and positive values of x) and always between the values $y = 1$ and $y = -1$. As we saw, a variation of this function, $f(x) = x \sin(x)$, reveals deep cognitive incompatibilities between the dynamic notion of continuity implicit in the example above and the static ε - δ definition of continuity coined by Weierstrass in the second half of the 19th century (based on quantifiers and discrete Real numbers) and which has been adopted ever since as “the” definition of what Continuity really is (Núñez & Lakoff, 1998; Lakoff & Núñez, 2000). These deep cognitive incompatibilities between dynamic-wholistic entities and static-discrete ones may explain important aspects underlying the difficulties encountered by students all over the world when learning the modern technical version of the notions of limits and continuity (Núñez, Edwards, and Matos, 1999).

5 Fictive Motion

Now that we are aware of the metaphorical (and metonymical nature) of the mathematical ideas mentioned above, I would like to analyze more in detail the dynamic component of these ideas. From where do these ideas get motion? What cognitive mechanism is allowing us to conceive static entities in dynamic terms? The answer is *fictive motion*.

Fictive motion is a fundamental embodied cognitive mechanism through which we unconsciously (and effortlessly) conceptualize static entities in dynamic terms, as when we say *the road goes along the coast*. The road itself doesn’t actually move anywhere. It is simply standing still. But we may conceive it as moving “along the coast.” Fictive motion was first studied by Len Talmy (1996), via the analysis of linguistic expressions taken from everyday language in which static scenes are described in dynamic terms. The following are linguistic examples of fictive motion:

- The Equator *passes through* many countries
- The boarder between Switzerland and Germany *runs along* the Rhine.
- The California coast *goes all the way down* to San Diego
- After Corvisart, line 6 *reaches* Place d’Italie.
- Right after *crossing* the Seine, line 4 *comes to* Chatêlet.
- The fence *stops* right after the tree.
- Unlike Tokyo, in Paris there is no metro line that *goes around* the city.

the same, namely, the country called “Switzerland.” This, however, doesn’t mean that we have to engage in the same cognitive activity in order to correctly answer these questions.

Motion, in all these cases, is fictive, imaginary, not real in any literal sense. Not only these expressions use verbs of action, but they also provide precise descriptions of the quality, manner, and form of motion. In all cases of fictive motion there is a *trajector* (the moving agent) and a *landscape* (the background space in which the trajector moves). Sometimes the trajector may be a real object (e.g., the *road* goes; the *fence* stops), and sometimes it is an imaginary entity (e.g., the *Equator* passes through; the *boarder* runs). In fictive motion, real world trajectors don't move but they have the potential to move or the potential to enact movement (e.g., a car moving along that road). In Mathematics proper, however, the trajector has always a metaphorical component. That is, the trajector as such can't be literally capable or incapable of enacting movement, because the very nature of the trajectory is imagined via metaphor (Núñez, 2003). For example, a point in the Cartesian Plane is an entity that has location (determined by its coordinates) but has no extension. So when we say "point *P* moves from *A* to *B*" we are ascribing motion to a metaphorical entity that only has location. First, as we saw earlier, entities which have only location (i.e., points) don't exist in the real world, so, as such, they don't have the potential to move or not to move in any literal sense. They simply don't exist in the real world. They are metaphorical entities. Second, literally speaking, point *A* and point *B* are distinct *locations*, and no point can change location while preserving its identity. That is, the trajector (point *P*, uniquely determined by its coordinates) can't preserve its identity throughout the process of motion from *A* to *B*, since that would mean that it is changing the very properties that are defining it, namely, its coordinates.

We now have a basic understanding of how conceptual metaphor and fictive motion work, so we are in a position to see the embodied cognitive mechanisms underlying the mathematical expressions like the ones we saw earlier. Here we have similar expressions:

- $\sin 1/x$ oscillates more and more as x approaches zero
- $g(x)$ never goes beyond 1
- If there exists a number L with the property that $f(x)$ gets closer and closer to L as x gets larger and larger; $\lim_{x \rightarrow \infty} f(x) = L$.

In these examples Fictive Motion operates on a network of precise *conceptual metaphors*, such as NUMBERS ARE LOCATIONS IN SPACE (which allows us to conceive numbers in terms of spatial positions), to provide the inferential structure required to conceive mathematical functions as having motion and directionality. Conceptual metaphor generates a purely imaginary entity in a metaphorical space, and fictive motion makes it a moving trajector in this metaphorical space. Thus, the progressively smaller numerical values taken by x which determine numerical values of $\sin 1/x$, are via the conceptual metaphor NUMBERS ARE LOCATIONS IN SPACE conceptualized as spatial locations. The now metaphorical spatial locus of the function (i.e., the "line" drawn on the plane) now becomes available for fictive motion to act upon. The progressively smaller numerical values taken by x (now metaphorically conceptualized as locations progressively closer to the origin) determine

corresponding metaphorical locations in space for $\sin 1/x$. In this imaginary space, via conceptual metaphor and fictive motion now $\sin 1/x$ can “oscillate” more and more as x “approaches” zero.

In a similar way the infinite precision of real numbers themselves can be conceived as limits of sequences of rational numbers, or limits of sequences of nested intervals. Because, as we saw, limits have conceptual metaphor and fictive motion built in, we can now see the fundamental role that these embodied mechanisms play in the constitution of the very nature of the real numbers themselves.

5 Dead Metaphors?

Up to now, we have analyzed some mathematical ideas through methods in cognitive linguistics, such as conceptual metaphor, conceptual metonymy, and fictive motion. We have studied the inferential organization modeling *linguistic expressions*. But so far no much has been said of actual people speaking, writing, explaining, learning, or gesturing in real-time when involved in mathematical activities. The analysis so far has been almost exclusively at the level of written and oral linguistic expressions. We must know whether there is any psychological (and presumably neurological) reality underlying these linguistic expressions. The remaining task now is to show that all these cases are not, as some scholars have suggested, mere instances of so-called *dead metaphors*, that is, expressions that once in the past had a metaphorical dimension but that now, after centuries of usage, have lost their metaphorical component becoming “dead.” Dead metaphorical expressions are those that have lost their psychological (and cognitive semantic) original reality, becoming simply new “lexical items.” Perhaps in the cases we have seen in mathematics, what once was a metaphorical expression has now become a *literal* expression whose meaningful origin speakers of English don’t know anymore (very much like so many English words whose Latin or Greek etymology may have been known by speakers at a certain point in history, but whose original meaning is no longer evoked by speakers today). Is this what is happening to cases such as “approaching” limits, “oscillating” functions, or hyperbolae not “reaching” the asymptotes? Maybe, after all, all what we have in the mathematical expressions we have examined, is simply a story of dead metaphors, with no psychological (or neurological) reality whatsoever. As we will see, however, the study of human *gesture* provides embodied convergent evidence showing that this is not the case at all. Gesture studies, via a detailed investigation of real-time cognitive and linguistic production, bodily motion (mainly hands and arms), and voice inflection, show that the conceptual metaphors and fictive motion involved in the mathematical ideas analyzed above, far from being dead, do have a very embodied psychological (and presumably neurological) reality.

6 Gesture as Cognition

Human beings from all cultures around the world gesture when they speak. The philosophical and scientific study of human language and thought has largely ignored this simple but fundamental fact. Human gesture constitutes the forgotten dimension of thought and language. Chomskian linguistics, for instance, overemphasizing syntax, saw language mainly in terms of abstract grammar, formalisms, and combinatorics, you could study by looking at written statements. In such a view there was simply no room for meaningful (semantic) “bodily production” such as gesture. In mainstream experimental psychology gestures were left out, among others, because being produced in a spontaneous manner, it was very difficult to operationalize them, making rigorous experimental observation on them extremely difficult. In mainstream cognitive science, which in its origins was heavily influenced by classic artificial intelligence, there was simply no room for gestures either. Cognitive science and artificial intelligence were heavily influenced by the information-processing paradigm and what was taken to be essential in any cognitive activity was a set of body-less abstract rules and the manipulation of physical symbols governing the processing of information. In all these cases, gestures were completely ignored and left out of the picture that defined what constituted genuine subject matters for the study of the mind. At best, gestures were considered as a kind of epiphenomenon, secondary to other more important and better-defined phenomena.

But in the last decade or so, this scenario has changed in a radical way with the pioneering work of A. Kendon (1980), D. McNeill (1992), S. Goldin-Meadow & C. Mylander (1984), and many others. Research in a large variety of areas, from child development, to neuropsychology, to linguistics, and to anthropology, has shown the intimate link between oral and gestural production. Finding after finding has shown, for instance, that gestures are produced in astonishing synchronicity with speech, that in children they develop in close relation with speech, and that brain injuries affecting speech production also affect gesture production. The following is a (very summarized) list of nine excellent sources of evidence supporting (1) the view that speech and gesture are in reality two facets of the same cognitive linguistic reality, and (2) an embodied approach for understanding language, conceptual systems, and high-level cognition:

- 1) Speech accompanying gesture is universal. This phenomenon is manifested in all cultures around the world. Gestures then provide a remarkable “back door” to linguistic cognition (McNeill, 1992; Iverson & Thelen 1999; Núñez & Sweetser, 2001).
- 2) Gestures are less monitored than speech, and they are, to a great extent, unconscious. Speakers are often unaware that they are gesturing at all (McNeill, 1992)
- 3) Gestures show an astonishing synchronicity with speech. They are manifested in a millisecond-precise synchronicity, in patterns which are specific to a given language (McNeill, 1992).

- 4) Gestures can be produced without the presence of interlocutors. Studies of people gesturing while talking on the telephone, or in monologues, and studies of conversations among congenitally blind subjects have shown that there is no need of visible interlocutors for people to gesture (Iverson & Goldin-Meadow, 1998).
- 5) Gestures are co-processed with speech. Studies show that stutterers stutter in gesture too, and that impeding hand gestures interrupts speech production (Mayberry and Jaques, 2000).
- 6) Hand signs are affected by the same neurological damage as speech. Studies in neurobiology of sign language show that left hemisphere damaged signers manifest similar phonological and morphological errors as those observed in speech aphasia (Hickok, Bellugi, and Klima, 1998).
- 7) Gesture and speech develop closely linked. Studies in language acquisition and child development show that speech and gesture develop in parallel (Iverson & Thelen 1999; Bates & Dick, 2002).
- 8) Gesture provide complementary content to speech content. Studies show that speakers synthesize and subsequently cannot distinguish information taken from the two channels (Kendon, 2000).
- 9) Gestures are co-produced with abstract metaphorical thinking. Linguistic metaphorical mappings are paralleled systematically in gesture (McNeill, 1992; Cienki, 1998; Sweetser, 1998; Núñez & Sweetser, 2001).

In all these studies, a careful analysis of important parameters of gestures such as handshapes, hand and arm positions, palm orientation, type of movements, trajectories, manner, and speed, as well as a careful examination of timing, indexing, preservation of semantics, and the coupling with environmental features, give deep insight into human thought⁶. An important feature of gestures is that they have three well-defined phases called preparation, stroke, and retraction (McNeill, 1992). The stroke is in general the fastest part of the gesture's motion, and it tends to be highly synchronized with speech accentuation and semantic content. The preparation phase is the motion that precedes the stroke (usually slower), and the retraction phase is the motion observed after the stroke has been produced (usually slower as well), when the hand goes back to a resting position or to whatever activity it was engaged in.

With these tools from gesture studies and cognition, we can now analyze mathematical expressions like the ones we saw before, but this time focusing on the gesture production of the speaker. For the purposes of this chapter, an important distinction we need to make concerns the gestures that refer to real objects in the real world, and gestures that refer to some abstract idea that in itself doesn't exist in the real world. An example of the first group is shown in Figure 4, which shows renowned physicist Professor Richard Feynman giving a lecture on physics of particles at Cornell University many years ago. In this sequence he is talking about particles moving in all directions at very high speeds (Figure 4, a through e), and a few

⁶ An analysis of the various dimensions and methodological issues regarding the scientific study of gestures studies is beyond the scope of this chapter. For details see references mentioned above.

milliseconds later he completes his utterance by saying “once in a while hit” (Figure 4f). The action shown in the first five pictures correspond to the gesture characterizing the random movements of particles at high speeds. The precise finger pointing shown in figure 4f occurs when he says “once in a while hit” (the stroke of the gesture). The particle being indexed by the gesture is quite abstract and idealized, in the sense that it doesn’t preserve some properties of the real referent, such as the extremely high speed at which particles move, for instance. But the point here is that although Prof. Feynman’s talk was about a very abstract domain (i.e., particle physics), it is still the case that with his finger he is *indexing* a “particle,” an object with location, extension, and mass, which does exist in the real world. The trajector in this dynamic scene is, an extremely small and fast object, but nonetheless a real entity in the real world.

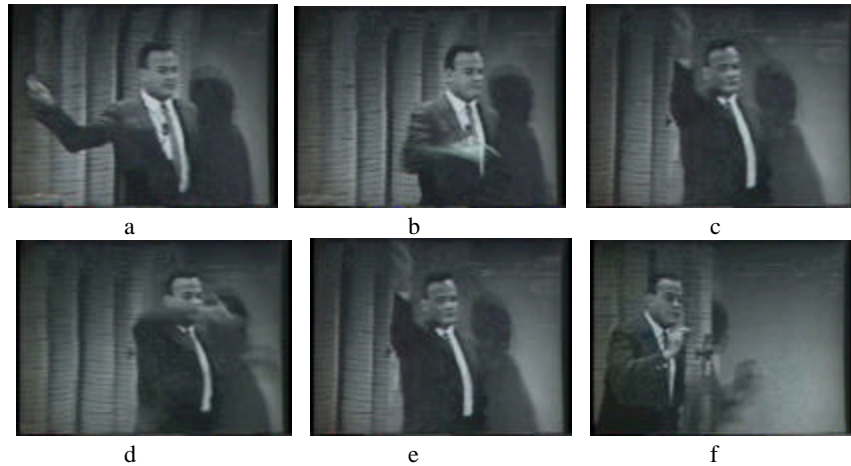


Fig. 4. Professor Richard Feynman giving a lecture on physics. He is talking about particles moving in all directions at very high speeds (a through e), which “once in a while hit” (f).

Now, the gestures we are about to analyze below are similar in many respects, but they are even more abstract. In these cases the entities that are indexed with the various handshapes are purely imaginary entities, like points and numbers in mathematics. Figure 5, for instance shows a professor of mathematics lecturing on convergent sequences in a university level class. In this particular situation, he is talking about a case in which the real values of an infinite sequence do not get closer and closer to a single real value as n increases, but “oscillate” between two fixed values. His right hand, with the palm towards his left, has a handshape called *baby O* in American Sign Language and in gesture studies, where the index finger and the thumb are touching and are slightly bent while the other three fingers are fully bent. In this gesture the touching tip of the index and the thumb are metaphorically indexing a metonymical value standing for the values in the sequence as n increases (it is almost

as if the subject is carefully holding a very tiny object with those two fingers). Holding that fixed handshape, he moves his right arm horizontally back and forth while he says “oscillating.”

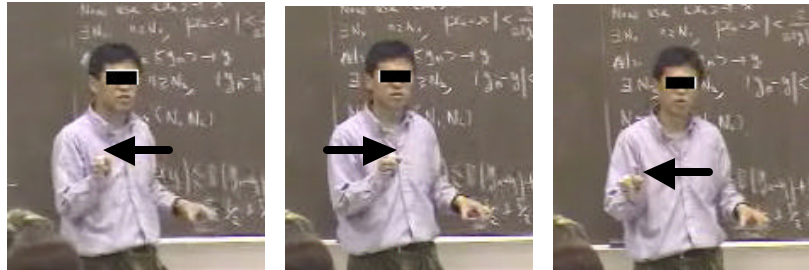


Fig. 5. A professor of mathematics lecturing on convergent sequences in a university level class. Here he is referring to a case in which the real values of a sequence “oscillate” (horizontally).

Hands and arms are essential body parts involved in gesturing. But often it is also the entire body that participates in enacting the inferential structure of an idea. In the following example (Figure 6) a professor of mathematics is lecturing on some important notions of calculus at a university level course. In this scene he is talking about a particular theorem regarding monotone sequences.

As he is talking about an unbounded monotone sequence, he is referring to the important property of “going in one direction.” As he says this he is producing frontwards iterative unfolding circles with his right hand, and at the same time he is walking frontally, accelerating at each step (Figure 6a through 6e). His right hand, with the palm toward his chest, displays a shape called *tapered O* (Thumb relatively extended and touching the upper part of his extended index finger bent in right angle, like the other fingers), which he keeps in a relatively fixed position while doing the iterative circular movement. A few milliseconds later he completes the sentence by saying “it takes off to infinity” at the very moment when his right arm is fully extended and his hand shape has shifted to an extended shape called *B spread* with a fully (almost over) extension, and the tips of the fingers pointing frontwards at eye-level.

It is important to notice that in both cases the blackboard is full of mathematical expressions containing formalisms like the ones we saw earlier (e.g., existential and universal quantifiers \exists and \forall): formalisms, which have no indication of, or reference to, motion. The gestures (and the linguistic expressions used), however, tell us a very different conceptual story. In both cases, these mathematicians are referring to fundamental dynamic aspects of the mathematical ideas they are talking about. In the first example, the oscillating gesture matches, and it is produced synchronically with, the linguistic expressions used. In the second example, the iterative frontally-unfolding circular gesture matches the inferential structure of the description of the iteration involved in the increasing monotone sequence, where even the entire body moves

forwards as the sequence unfolds. Since the sequence is unbounded, it “takes off to infinity,” idea which is precisely characterized in a synchronous way with the full frontal extension of the arm and the hand.

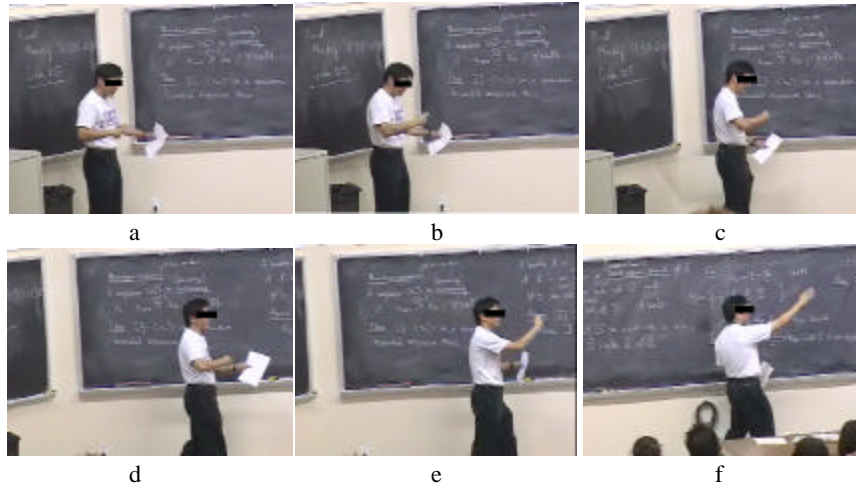


Fig. 6. A professor of mathematics at a university level class talking about an unbounded monotone sequence “going in one direction” (a through e), which “takes off to infinity” (f).

The moral we can get from these gesture examples is two fold.

- First, gestures provide converging evidence for the psychological and embodied reality of the linguistic expressions analyzed with classic techniques in cognitive linguistics, such as metaphor and blending analysis. In these cases gesture analyses show that the metaphorical expressions we saw earlier are not cases of dead metaphors. The above gestures show, in real time, that the dynamism involved in these ideas have full psychological and cognitive reality.
- Second, these gestures show that the fundamental dynamic contents involving infinite sequences, limits, continuity, and so on, are in fact *constitutive* of the inferential organization of these ideas. Formal language in mathematics, however, is not as rich as everyday language and cannot capture the full complexity of the inferential organization of mathematical ideas. It is the job of embodied cognitive science to characterize the full richness of mathematical ideas.

7 Conclusion

We can now go back to the original question asked in the title of this chapter: Do real numbers really move? Since fictive motion is a real cognitive mechanism, constitutive of the very notion of a real number, the answer is yes. Real numbers are metaphorical entities (with a very sophisticated inferential organization), and they do move, metaphorically. But, of course, this was not the main point of this chapter. The main point was to show that even the most abstract conceptual system we can think of, mathematics(!), is ultimately embodied in the nature of our bodies, language, and cognition. It follows from this that if mathematics is embodied in nature, then *any* abstract conceptual system is embodied.

Conceptual metaphor and fictive motion, being a manifestation of extremely fast, highly efficient, and effortless cognitive mechanisms that preserve inferences, play a fundamental role in bringing many mathematical concepts into being. We analyzed several cases involving dynamic language in mathematics, in domains in which, according to formal definitions and axioms in mathematics, no motion was supposed to exist at all. Via the study of gestures, we were able to see that the metaphors involved in the linguistic metaphorical expressions were not simply cases of “dead” linguistic expressions. Gesture studies provide real-time convergent evidence supporting the psychological and cognitive reality of the embodiment of mathematical ideas, and their inferential organization. Building on gesture studies we were able to tell that the above mathematics professors, not only were using metaphorical linguistic expressions, but that they were in fact, in real time, thinking dynamically!

For many, mathematics is a timeless set of truths about the universe, transcending our human existence. For others, mathematics *is* what is characterized by formal definitions and axiomatic systems. From the perspective of our work in the cognitive science *of* mathematics (itself), however, a very different view emerges: Mathematics doesn't exist outside of human cognition. Formal definitions and axioms in mathematics are themselves created by human ideas (although they constitute a very small and specific fraction of human cognition), and they only capture very limited aspects of the richness of mathematical ideas. Moreover, definitions and axioms often neither formalize nor generalize human everyday concepts. A clear example is provided by the modern definitions of limits and continuity, which were coined after the work by Cauchy, Weierstrass, Dedekind, and others in the 19th century. These definitions are at odds with the inferential organization of natural continuity provided by cognitive mechanisms such as fictive and metaphorical motion. Anyone who has taught calculus to new students can tell how counter-intuitive and hard to understand the epsilon-delta definitions of limits and continuity are (and this is an extremely well-documented fact in the mathematics education literature). The reason is (cognitively) simple. Static epsilon-delta formalisms neither formalize nor generalize the rich human dynamic concepts underlying continuity and the “approaching” of locations.

By finding out that real numbers “really move,” we can see that even the most abstract, precise, and useful concepts human beings have ever created are ultimately *embodied*.

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